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1 Algebra

1.1 Monoids

1.1.1 Definitions and Basic Properties

Definition 1.1.1. For a subcategory $\mathcal{C}$ of Set, a *$\mathcal{C}$-monoid* is an object M in $\mathcal{C}$ together with morphisms $\circ : M \times M \to M$, $e : \{*\} \to M$ such that

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mathbb{1}_M \times \circ} & M \times M \\ {\scriptstyle \circ \times \mathbb{1}_M}\downarrow & & \downarrow{\scriptstyle \circ} \\ M \times M & \xrightarrow{\quad \circ \quad} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} M & \xrightarrow{(\mathbb{1}_M, e)} & M \times M & \xleftarrow{(e, \mathbb{1}_M)} & M \\ & {\scriptstyle \mathbb{1}_M} \searrow & \downarrow{\scriptstyle \circ} & \swarrow {\scriptstyle \mathbb{1}_M} & \\ & & M & & \end{array}$$

commute. When $\mathcal{C}$ is Set, we simply refer to such an object as a *monoid*.

Remark 1.1.2.

1. For a pair of elements $m_1, m_2 \in M$, we use the notational convention $m_1 m_2 := m_1 \circ m_2 := m_1 m_2$.
2. The first diagram is a compact way of saying that $\circ$ is an *associative binary operation* on M.
3. We may identify e with the element $e(*) \in M$. This element has the property that for any $m \in M$,

$$em = \circ e \times \mathbb{1}_M(e, m) = \pi_1(e, m) = m = \pi_2(m, e) = \circ \mathbb{1}_m \times e(m, e) = me$$

   we call this the *neutral element with respect to* $\circ$
4. As we saw in Example D.0.34 BM := $(\{M\}, M)$ is a one object category, where composition is given by evaluation of the binary operation on M and the neutral element is the identity morphism. As an immediate consequence, e is unique.
5. For an element $m \in M$ and a non-negative integer n we define

$$m^n = \underbrace{m \cdots m}_{n}$$

   with the convention that $m^0 = e$.

The data specifying a monoid is often written as the tuple $(M, \circ)$ or $(M, \circ, e)$.

Definition 1.1.3. A *commutative monoid* is a monoid M such that for every $m_1, m_2 \in M$,

$$m_1 m_2 = m_2 m_1.$$

Definition 1.1.4. Let $(M, \circ)$ be a monoid. A subset $N \subset M$ is called a *submonoid (of M)* if

1. $e \in N$
2. For every $n_1, n_2 \in N$,
$$n_1 n_2 \in N.$$

Remark 1.1.5. $(N, \circ|_N)$ is a monoid. Indeed, Since $n_1 n_2 \in N$ for every $n_1, n_2 \in N$, the operation $\circ\ |_{N\times N}\colon N \times N \to N$ is well-defined. The operation $\circ$ is associative on N since it is associative on M. By the definition of a submonoid $e \in N$ and again clearly the property of being the neutral element with respect to $\circ$ on N is inherited by e being so with respect to $\circ$ on M.

Example 1.1.6.
1. The non-negative integers $\mathbb{N}$ is a monoid with respect to addition and multiplication.
2. Once constructed, $(\mathbb{Z}, +), (\mathbb{Z}, \cdot), (\mathbb{Q}, +), (\mathbb{Q}, \cdot), (\mathbb{R}, +), (\mathbb{R}, \cdot), (\mathbb{C}, +), (\mathbb{C}, \cdot)$ are monoids.
3. Let A be a set. Consider $\text{Set}(A, A) := \{f : A \to A\}$. This a monoid under function composition.
4. Given a non-empty set X and a monoid M the set
$$\text{Set}(X, M)$$
with the binary operation
$$\text{Set}(X, M) \times \text{Set}(X, M) \to \text{Set}(X, M)$$
$$(f, g) \mapsto fg$$
where for $f, g \in \text{Set}(X, M)$
$$fg : X \to M$$
$$x \mapsto f(x)g(x)$$
Indeed, for each $x \in X$ the diagram
$$\begin{array}{ccccc} \text{Set}(X,M)\times\text{Set}(X,M)\times\text{Set}(X,M) & \xrightarrow{\text{ev}_x\times\text{ev}_x\times\text{ev}_x} & M\times M\times M & \xrightarrow{\mathbb{1}_M\times\circ} & M\times M \\ & & \downarrow{\scriptstyle \circ\times\mathbb{1}_M} & & \downarrow{\scriptstyle \circ} \\ & & M\times M & \xrightarrow{\quad\circ\quad} & M \end{array}$$
commutes Moreover, for the function $e : X \to M$, mapping every element in X to e_M we have that
$$ef(x) = e(x)f(x) = e_M f(x) = f(x) \text{ and } fe(x) = f(x)e(x) = f(x)e_M = f(x).$$

5. Let M be a monoid. Then $M \subset M$ is a submonoid.

6. Let M be a monoid. Then $\{e\} \subset M$ is a submonoid.

7. Let M be a monoid and $L \subset N \subset M$ be submonoids of M. Then L is a submonoid of N. Similarly if $N \subset M$ is a submonoid and $L \subset N$ is a submonoid, then $L \subset M$ is a submonoid.

1.1.2 Morphisms of Monoids

Definition 1.1.7. Let M, N be monoids. A *monoid homomorphism/map of monoids/morphism of monoids* is a map $\rho : M \to N$ such that

1. For every $m_1, m_2 \in M$
$$\rho(m_1 m_2) = \rho(m_1)\rho(m_2).$$

2.
$$\rho(e_M) = e_N.$$

Remark 1.1.8. The collection of monoids and the collection of monoid homomorphisms form a subcategory of Set denoted Mon. Indeed, given monoid homomorphisms $f : A \to B$, $g : B \to C$, then for each $x, y \in A$,

$$gf(xy) = g(f(x)f(y)) = gf(x)gf(y)$$

and

$$gf(e_A) = g(e_B) = e_C,$$

hence gf is a monoid homomorphism.

Example 1.1.9. By a prior example (cf. Example 1.1.6) we have seen that $\text{Set}(M, N)$ is a monoid. $\text{Mon}(M, N) \subset \text{Set}(M, N)$ is a submonoid if N is commutative. Indeed, for $f, g \in \text{Mon}(M, N)$ and $x, y \in M$. Then

$$fg(xy) = f(xy)g(xy) = f(x)f(y)g(x)g(y) = f(x)g(x)f(y)g(y) = fg(x)fg(y),$$

hence $fg \in \text{Mon}(M, N)$ Furthermore, we have

$$e(xy) = e_M(xy) = xy = (e_N x)(e_N y) = e(x)e(y) \implies e \in \text{Mon}(M, N).$$

Lemma 1.1.10. *Let $\rho : M \to N$ be a monoid homomorphism and $L \subset M$ a submonoid. Then $\rho(L) \subset N$ is a submonoid.*

Proof. Let $\rho(l_1), \rho(l_2) \in \rho(L)$. Then since $l_1 l_2 \in L$,

$$\rho(l_1)\rho(l_2) = \rho(l_1 l_2) \in \rho(L).$$

Clearly $e_N = \rho(e_M) \in \rho(L)$. □

Corollary 1.1.11. *The image of a monoid homomorphism $\rho : M \to N$ is a submonoid of N.*

Proof. This follows from the above lemma (cf. Example 1.1.6). □

Definition 1.1.12. Let $\rho : M \to N$ be a monoid homomorphism. We define *the kernel of* ρ to be the set

$$\ker \rho := \rho^{-1}(e_N) = \{m \in M : \rho(m) = e_N\} \subset M$$

Lemma 1.1.13. *Let $\rho : M \to N$ be a monoid homomorphism, and $L \subset N$ a submonoid. Then $\rho^{-1}(L) \subset M$ is a submonoid.*

Proof. Let $m_1, m_2 \in \rho^{-1}(L)$. Then since $\rho(m_1), \rho(m_2) \in L$,

$$\rho(m_1 m_2) = \rho(m_1)\rho(m_2) \in L,$$

hence $m_1 m_2 \in \rho^{-1}(L)$. Since $\rho(e_M) = e_N \in L$, it follows that $\rho^{-1}(L)$ is a submonoid of N. □

Corollary 1.1.14. *The kernel of a monoid homomorphism $\rho : M \to N$ is a submonoid of M.*

Proof. Since $\{e_N\}$ is a submonoid of N it follows by the above lemma that $\ker \rho = \rho^{-1}(\{e_N\})$ (cf. Example 1.1.6) is a submonoid. □

Lemma 1.1.15. *Let M be a commutative monoid and $N \subset M$ a submonoid. Then N is a commutative monoid.*

Proof. By Lemma 1.1.5 N is a monoid. Let $n_1, n_2 \in N$, then since $n_1, n_2 \in M$, $n_1 n_2 = n_2 n_1$. □

Lemma 1.1.16. *Let $\rho : M \to N$ be a monoid homomorphism. Let $L \subset M$ be a submonoid. If M is commutative, then $\rho(L) \subset N$ is a commutative monoid.*

Proof. Since $\rho(L) \subset N$ is a submonoid, it is a monoid. Let $\rho(l_1), \rho(l_2) \in \rho(L)$. Then since L is a commutative by Lemma 1.1.15 it follows that

$$\rho(l_1)\rho(l_2) = \rho(l_1 l_2) = \rho(l_2 l_1) = \rho(l_2)\rho(l_1).$$

□

1.1.3 Product Monoids & Finitely Supported Product of Monoids

Theorem 1.1.17. *Let A be a set and $\{M_\alpha\}_{\alpha\in A}$ a family of monoids. We define a binary operation on the product $\prod_{\alpha\in A} M_\alpha$ by $(m_\alpha)(m'_\alpha) = (m_\alpha m'_\alpha)$ for $(m_\alpha), (m'_\alpha) \in \prod_{\alpha\in A} M_\alpha$. With this operation $\prod_{\alpha\in A} M_\alpha$ becomes a monoid. If M_α is commutative for every $\alpha \in A$ so is $\prod_{\alpha\in A} M_\alpha$.*

Proof. We define $e := (e) := (e_\alpha)$ where e_α is the neutral element in M_α for each α. Let $(m_\alpha), (m'_\alpha), (m''_\alpha) \in \prod_{\alpha\in A} M_\alpha$. We then have that

$$\begin{aligned}(m_\alpha)\left((m'_\alpha)(m''_\alpha)\right) &= (m_\alpha)(m'_\alpha m''_\alpha) = (m_\alpha(m'_\alpha m''_\alpha)) = ((m_\alpha m'_\alpha)m''_\alpha) = (m_\alpha m'_\alpha)(m''_\alpha)\\ &= \left((m_\alpha)(m'_\alpha)\right)(m''_\alpha)\end{aligned}$$

and that

$$e(m_\alpha) = (e)(m_\alpha) = (e_\alpha m_\alpha) = (m_\alpha) \text{ and } (m_\alpha)e = (m_\alpha)(e) = (m_\alpha e_\alpha) = (m_\alpha),$$

hence $\left(\prod_{\alpha\in A} M_\alpha, \cdot\right)$ is a monoid. Suppose M_α is commutative for each $\alpha \in A$. Then

$$(m_\alpha)(m'_\alpha) = (m_\alpha m'_\alpha) = (m'_\alpha m_\alpha) = (m_\alpha)(m'_\alpha).$$

□

Lemma 1.1.18. *Let A be a set and $\{M_\alpha\}_{\alpha\in A}$ a family of monoids. Then*

$$\pi_\beta : \prod_{\alpha\in A} M_\alpha \to M_\beta$$

is monoid homomorphism. Given a monoid N and a family monoid homomorphisms $\{f_\alpha : N \to M_\alpha\}_{\alpha\in A}$ then the unique map $f : N \to \prod_{\alpha\in A}$ (cf. Proposition D.3.17) such that $\pi_\alpha \circ f = f_\alpha$ for every $\alpha \in A$ is monoid homomorphism. Hence the cartesian product is the product in the category of monoids.

Proof. Let $(m_\alpha), (m'_\alpha) \in \prod_{\alpha\in A} M_\alpha$ and fix $\beta \in A$. Then

$$\begin{aligned}&\pi_\beta((m_\alpha)(m'_\alpha))\\ &\qquad = \pi_\beta((m_\alpha m'_\alpha)) = m_\beta m'_\beta = \pi_\beta((m_\alpha))\pi_\beta((m'_\alpha)).\end{aligned}$$

Lastly

$$\pi_\beta(e) = \pi_\beta((e_\alpha)) = e_\beta.$$

Let $n, n' \in N$. Then

$$f(nn') = (f_\alpha(nn')) = (f_\alpha(n)f_\alpha(n')) = (f_\alpha(n))(f_\alpha(n')) = f(n)f(n'),$$

and

$$f(e_N) = (f_\alpha(e_N)) = (e_\alpha) = e$$

□

Proposition 1.1.19. *Let A be a set and $\{M_\alpha\}_{\alpha\in A}$ a family of monoids. Consider a family of sets $\{N_\alpha\}_{\alpha\in A}$ such that $N_\alpha \subset M_\alpha$ is a submonoid for each $\alpha \in A$. Then*

$$\prod_{\alpha\in A} N_\alpha \subset \prod_{\alpha\in A} M_\alpha$$

is a submonoid.

Proof. Since $e_\alpha \in N_\alpha$ for each $\alpha \in A$. Then $e = (e_\alpha) \in \prod_{\alpha\in A} N_\alpha$. Let $(n_\alpha), (n'_\alpha) \in \prod_{\alpha\in A} N_\alpha$. Then since $n_\alpha n'_\alpha \in N_\alpha$ for each $\alpha \in A$. This implies that $(n_\alpha)(n'_\alpha) = (n_\alpha n'_\alpha) \in \prod_{\alpha\in A} N_\alpha$. □

Example 1.1.20. Not every submonoid of a product monoid arises in this fashion. For instance consider $N = \{(n, n) \in \mathbb{N} \times \mathbb{N}\}$ which is a proper submonoid of $(\mathbb{N} \times \mathbb{N}, +)$. Indeed, $0_{\mathbb{N}\times\mathbb{N}} = (0, 0) \in N$ and if $(n_1, n_1), (n_2, n_2) \in N$, then $(n_1, n_1) + (n_2, n_2) = (n_1 + n_2, n_1 + n_2) \in N$. Let any two submonoids $A, B \subset \mathbb{N}$ be given and suppose neither are trivial. Take any pair of elements $a \in A$, $b \in B$. It is clear that we can add a to itself to the point where the outcome is not equal to b. I.e. there is an $n \in \mathbb{N}$ such that $na \neq b$. Then $(na, b) \notin N$, hence $N \neq A \times B$. This means N cannot be the product monoid of submonoids of $\mathbb{N}$.

Definition 1.1.21. Let A be a set and $\{M_\alpha\}_{\alpha\in A}$ a family of monoids. We define the *finitely supported direct product* of M_α over A as the set

$$\prod_{\alpha\in A,\text{fin}} M_\alpha := \left\{ (m_\alpha) \in \prod_{\alpha\in A} M_\alpha : m_\alpha = e_\alpha \text{ for all but finitely many } \alpha \in A \right\}$$

Lemma 1.1.22. *Let A be a set and $\{M_\alpha\}_{\alpha\in A}$ a family of monoids. $\prod_{\alpha\in A,\text{fin}} M_\alpha$ is a submonoid of $\prod_{\alpha\in A} M_\alpha$.*

Proof. Let $(m_\alpha), (m'_\alpha) \in \prod_{\alpha\in A,\text{fin}} M_\alpha$. For some distinct $\alpha_1, \ldots, \alpha_r \in A$ and $\beta_1, \ldots, \beta_p \in A$, $m_\alpha = e_\alpha$ for every $\alpha \in A \setminus \{\alpha_1, \ldots, \alpha_r\}$ and $m'_\alpha = e_\alpha$ for every $\alpha \in A \setminus \{\beta_1, \ldots, \beta_p\}$. Then $m_\alpha m'_\alpha = e_\alpha$ for every $\alpha \in A\setminus\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_p\}$ hence $(m_\alpha)(m'_\alpha) = (m_\alpha m'_\alpha) \in \prod_{\alpha\in A,\text{fin}} M_\alpha$. Clearly $e = (e_\alpha) \in \prod_{\alpha\in A,\text{fin}} M_\alpha$. □

Here is a note on notation: Given a finite set of elements $E \subset M$ in a monoid (M, e), we define when E is empty,

$$\prod_{m\in E} m := e.$$

When $E = \{m_1, \ldots, m_n\}$ we define

$$\prod_{x \in E} := \left(\prod_{m \in E \setminus \{m_n\}} m \right) m_n$$

In the non empty case we often use the notation

$$\prod_1^n m_i := \prod_{m \in E} m.$$

When $(M, +)$ is commutative we opt to write

$$\sum_{m \in E} m$$

instead of

$$\prod_{m \in E} m$$

Definition 1.1.23. Given a monoid M and a set X, for a function $f : X \to M$, we define *the support of* f to be the set

$$\text{supp } f := \{x \in X : f(x) \neq e\}$$

Lemma 1.1.24. *Given a monoid M and a set X, for functions $f, g : X \to M$,*

$$\text{supp } fg \subset \text{supp } f \cup \text{supp } g$$

Proof. If $x \in X$ is such that $f(x) = e = g(x)$, then $fg(x) = e$. □

Lemma 1.1.25. *Given a family of submonoids $\{M_\alpha\}_{\alpha \in A}$ of a commutative monoid $(M, +)$, there is a monoid map*

$$\prod_{\alpha \in A, \text{fin}} M_\alpha \to M$$
$$(m_\alpha) \mapsto \sum_{\alpha \in A} m_\alpha := \sum_{\alpha \in \text{supp } f} m_\alpha$$

We denote the image of this monoid map by

$$\sum_\alpha M_\alpha.$$

Proof. Indeed,

$$(m_\alpha) + (n_\alpha) \mapsto \sum_{\alpha \in \text{supp } (m_\alpha + n_\alpha)} m_\alpha + n_\alpha = \sum_{\alpha \in (\text{supp } (m_\alpha)) \cup (\text{supp } (n_\alpha))} m_\alpha + n_\alpha$$
$$= \sum_{\alpha \in \text{supp } (m_\alpha)} m_\alpha + \sum_{\alpha \in \text{supp } (n_\alpha)} n_\alpha,$$

using Lemma 1.1.24 in the first equality. It is clear that (0) is mapped to 0. □

Definition 1.1.26. Let X be a set and M a monoid. We define *the monoid over M generated by X* to be the set

$$M[X] := \{(m_x) \in \text{Set}(X, M) : \text{supp } (m_x) \text{ is finite}\}.$$

When $(M, +, 0)$ is a commutative monoid, we denote an element of $M[X]$ by

$$\sum_{x \in X} m_x x$$

In this case, we call $M[X]$ *the free commutative monoid generated by X over M*

Remark 1.1.27. From Lemma 1.1.24 it follows that $M[X]$ is submonoid of $\text{Set}(X, M)$. Alternatively, note that $\text{Set}(X, M) = \prod_{x \in X} M$ and that $M[X]$ just the finitely supported product.

1.1.4 The Free Monoid Generated by a Set

Definition 1.1.28. Given a set X, we define *the free monoid generated by X* to be the set

$$X_{\mathbb{N}} := \bigsqcup_{n \in \mathbb{N}} X^n$$

with the convention that X^0 is a singleton set. We call the element of this set *the empty string* in $\mathbb{N}[X]$. Given

$$x = (x_1, \ldots, x_n) \in X^n \text{ and } y = (y_1, \ldots, y_m) \in X^m,$$

we define *concatenation*

$$xy := (x_1, \ldots, x_n, y_1, \ldots, y_m) \in X^{n+m}$$

Remark 1.1.29. Note that any element is $X_{\mathbb{N}}$ is of the form

$$x_1 \cdots x_n$$

for $x_1, \ldots, x_n \in X$

Lemma 1.1.30. *The free monoid generated by a set X with concatenation is a monoid with the neutral element being the empty string.*

Proof. the operation of concatenation is clearly associative and the empty string is also clearly seen to be neutral element with respect to concatenation. □

Definition 1.1.31. Consider a pair of monoids M and N. Consider the set

$$(M \cup N)_{\mathbb{N}}.$$

An element in this monoid is the empty string or can be written as

$$a_1 \cdots a_n$$

where $a_{2k+1} M_{\mathbb{N}}$ and $a_{2k} \in N_{\mathbb{N}}$ or vice versa. Such an element, $a_1 \cdots a_n$ is *reduced* if it is the empty string or if each $a_i \in (M \setminus \{e_M\}) \cup (N \setminus \{e_N\})$. We denote the set of reduced elements by $M * N$. This is called the *free product of M and N*

For any monoid A there is a function

$$\alpha_A : A_{\mathbb{N}} \to A \cup \{*\}$$

defined recursively, where

$$\alpha_A(a) = \begin{cases} \{*\} & \text{if } a = e \\ a & \text{otherwise} \end{cases}$$

and

$$\alpha_A(a_1 \cdots a_{n+1}) = \begin{cases} \alpha_A(a_1 \cdots a_n) & \text{if } a_{n+1} = e \\ \alpha_A(a_1 \cdots a_n) a_{n+1} & \text{otherwise} \end{cases}$$

for $a_1, \ldots, a_{n+1} \in A$ Let M and N be monoids. There is a function

$$\text{red} : (M \cup N)_{\mathbb{N}} \to \text{red}(M, N)$$
$$a_1 \cdots a_n \mapsto \alpha(a_1)\alpha(a_2) \cdots \alpha(a_n)$$

where α denotes the appropriate choice of function in $\{\alpha_M, \alpha_N\}$. The image under this operation is called *reduction*. Note that this function restricts to the identity on $M * N$. We concatenate elements in this set in the following way,

$$(a_1 \cdots a_n)(b_1 \cdots b_m) := \text{red}(a_1 \cdots a_n b_1 \cdots b_m)$$

Lemma 1.1.32. *The above operation of concatenation is well-defined and with it $M * N$ is a monoid with neutral element being the empty string.*

Proof. To check associativity, the following is sufficient. For monoids $(A, \cdot)$ and $(B, \cdot)$,

$$\text{red}(\text{red}(ab)b') = \text{red}(a\text{red}(bb'))$$

for $a \in A$ and $b, b' \in B$ and

$$\text{red}(\text{red}(ab)a') = \text{red}(a\text{b}a')$$

for $a, a' \in A$ and $b \in B$. This is just a matter of case checking. Clearly the empty string is the neutral element. □

1.1.5 The Opposite Monoid

Definition 1.1.33. Consider a monoid $(M, \circ)$. The opposite category is then $M^{\text{op}} := \text{End}(\text{B}M^{\text{op}})$.

Remark 1.1.34. In other words, M^{op} is the monoid $(M, \star)$, where for $x, y \in M$

$$x \star y := y \circ x.$$

The upshot of having such a construction is in many cases we are interested in statements that involve some sort of action from the left and a notion of an action from the right. The statement about the right action can be reduced to a statement about the left action in the opposite monoid, allowing us prove only one statement.
Note that in the case of a commutative monoid M,

$$(M^{\text{op}}, \star) = (M, \circ)$$

1.2 Groups

1.2.1 Definition & Basic Properties

Definition 1.2.1. A *group* is a monoid $(G, \circ)$ where for every $g \in G$ there is an element $g^{-1} \in G$ such that

$$gg^{-1} = g^{-1}g = e.$$

For $g \in G$ we refer to g^{-1} as the *inverse of* g. The data specifying a group is also often written as the tuple $(G, \circ, e)$ or $(G, \circ)$.

Remark 1.2.2. Not that for $g \in G$ the inverse g^{-1} is the mutually inverse morphism of g, when we interpret G as the groupoid $\text{B}G = (\{G\}, G)$. For the same reason we have that

$$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$$

for each $g_1, g_2 \in G$ and that

$$\left(g^{-1}\right)^{-1} = g$$

Remark 1.2.3. For an element $g \in G$ and a non-negative integer n, we define

$$g^{-n} := \left(g^{-1}\right)^n .$$

It is easy to check that $(g^n)^{-1} = g^{-n}$.

Definition 1.2.4. A group $(G, +)$ is called *abelian* or *additive*, if it is also a commutative monoid with respect to $+$. We denote the inverse of $g \in G$ with respect to addition by $-g$, and for $g_1, g_2 \in G$ we define

$$g_1 - g_2 := g_1 + (-g_2).$$

and $ng_1 = \underbrace{g_1 + \ldots + g_1}_{n \text{ times}}$

Lemma 1.2.5. *Let $(G, \circ)$ be a group. Let $g, g', a \in G$. If $ag = ag'$, then $g = g'$. Similary, if $ga = g'a$, then $g = g'$.*

Proof. This follows from each $g \in G$ being an isomorphism in BG, which in particular implies that they are epimorphisms and monomorphisms. □

Definition 1.2.6. Let G be a group. A subset $H \subset G$ is called a *subgroup* if it is a submonoid of G and for every $h \in H$ we have that $h^{-1} \in H$.

Remark 1.2.7. $(H, \circ)$ is a group. Indeed, $(H, \circ)$ is a monoid since $H \subset G$ is a submonoid. Since $h^{-1} \in H$ for every $h \in H$, it follows that every element in H has in inverse in H, hence H is a group.

Example 1.2.8.

1. For $n \in \mathbb{Z}$, the set $n\mathbb{Z} := \{nm : m \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.
2. $\mathbb{R} \subset \mathbb{C}$ is a subgroup of $(\mathbb{C}, +)$. $\mathbb{R} \setminus \{0\} \subset \mathbb{C} \setminus \{0\}$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \cdot)$
3. Let G be a group. Then G and $\{e\}$ are subgroups of G.
4. Given a set A. The set of invertible maps $A \to A$, denoted Aut(A) forms a submonoid of Set(A, A) under composition, furthermore it is a group, when picking the inverse elements to be inverse maps and the neutral to be the identity on A.
5. Given a non-empty set X and a group G the monoid Set(X, G) forms a group. Indeed for $f \in$ Set(X, G), define $f^{-1} : X \to G$ by $f^{-1}(x) = (f(x))^{-1}$. Then

$$ff^{-1}(x) = f(x)\,(f(x))^{-1} = e_N = e(x).$$

6. Let G be a group and consider $I \subset H \subset G$. Then $I, H \subset G$ are subgroups if and only if $I \subset H$ and $H \subset G$ are subgroups.

Definition 1.2.9. Let G be a group and $S \subset G$. Then we define *the subgroup generated by* S to be the set

$$\langle S \rangle = \left\{ s_1^{v_1} \cdots s_n^{v_n} \in G : n \geq 1, s_1, \ldots, s_n \in S, v_1, \ldots, v_n \in \{\pm 1\} \right\}.$$

Our convention will be that $\langle \emptyset \rangle = \{e\}$

Remark 1.2.10. Disallowing negative exponents in $\langle S \rangle$ gives the definition of *the submonoid generated by* S. From the following, we can derive that this is a submonoid even if we allow G to just be a monoid. If S is empty, it is clearly a subgroup. So suppose S is non-empty. Let $s_1^{v_1} \cdots s_n^{v_n}, t_1^{w_1} \cdots t_m^{w_m} \in \langle S \rangle$. Then if we define $s_i = t_{i-n}$ and $v_i = w_{i-n}$ for $i \in \{n+1, \ldots, n+m\}$. Then

$$s_1^{v_1} \cdots s_n^{v_n} t_1 \ldots^{w_1} \cdots t_m^{w_m} = s_1^{v_1} \cdots s_{n+m}^{v_{n+m}} \in \langle S \rangle$$

Clearly $e \in \langle S \rangle$. We also have

$$\left(s_1^{v_1} \cdots s_n^{v_n}\right)^{-1} = s_n^{-v_n} \cdots s_1^{-v_1} \in \langle S \rangle.$$

It follows that $\langle S \rangle$ is a subgroup. Let $H \subset G$ be a subgroup containing S. Then clearly $s_1^{v_1} \cdots s_n^{v_n} \in H$. Thus $\langle S \rangle$ is the smallest subgroup containing S.
$S, T \subset G$ such that $S \subset T \subset G$. Then $\langle S \rangle \subset \langle T \rangle$. If $H \subset G$ is a subgroup, then $\langle H \rangle = H$, since H is the smallest subgroup containing H.

Definition 1.2.11. A group G is *finitely generated* if $G = \langle g_1, \ldots, g_n \rangle$ for some $g_1, \ldots, g_n \in G$. If G is generated by one element it is called *cyclic*.

Lemma 1.2.12. *Let G be a cyclic group. Then any subgroup of G is cyclic.*

Proof. Let $H \subset G$ be a subgroup. If $H = \{e\}$ we are done, so suppose it is not. We have that $G = \langle g \rangle$ for some g. The set $\{n > 0 : g^n \in H\}$ is non-empty and thus have a minimum by the well-ordering of the natural numbers. Call this number m. We claim that $\langle g^m \rangle = H$. The first inclusion is trivial. Let $h \in H$. Then $h = g^l$ for some $l \in \mathbb{Z}$. It is sufficient to check that $h \in \langle g^m \rangle$ the case where $l > 0$, so we assume this. By minimality $l \geq m$. Then $h = g^l = g^{qm+r}$ for some $q, r \geq 0$ and $r < m$. Then $g^r = g^{qm+r-qm} = q^{qm+r} q^{-qm} \in H$, hence by minimality $r = 0$, hence $h = g^{qm} \in \langle g^m \rangle$ □

1.2.2 Morphisms of groups

Definition 1.2.13. Let G, H be groups. A map $\rho : G \to H$ is called a *group homomorphism/map of groups/morphism of groups*, if for every $g_1, g_2 \in G$,

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

Remark 1.2.14. 1. Denote the neutral elements of G and H by e_G and e_H respectively. Then $\rho(e_G) = e_H$. Indeed,

$$\rho(e_G)e_H = \rho(e_G e_G) = \rho(e_G)\rho(e_G),$$

hence by Lemma 1.2.5, $e_H = \rho(e_G)$. Thus a group homomorphism is a monoid homomorphism. It follows that the collection of all groups together with group homomorphisms form a subcategory of Mon, denoted Grp.
We also have that $\rho\left(g^{-1}\right) = \rho(g)^{-1}$. Indeed,

$$\rho(g)\rho(g)^{-1} = e_H = \rho(e_G) = \rho\left(gg^{-1}\right) = \rho(g)\rho\left(g^{-1}\right),$$

hence by uniqueness of inverse elements $\rho\left(g^{-1}\right) = \rho(g)^{-1}$.

2. The class of all groups together with group homomorphisms being morphisms is a subcategory of Set, denoted Grp. As a subcategory of Grp, we have the category of abelian groups, denoted ab. The zero object in these categories is the trivial group.

3. Suppose H is commutative. Let $\rho \in \text{Grp}(G, H)$ and $x \in M$. Then

$$\rho(x)\left(\rho(x)\right)^{-1} = e_N = e(x),$$

implying that $\text{Grp}(G, H) \subset \text{Set}(G, H)$ is a subgroup. Let $f \in \text{Grp}(G, H)$ and $x, y \in G$, then

$$f^{-1}(xy) = \left(f(xy)\right)^{-1} = \left(f(x)f(y)\right)^{-1} = \left(f(x)\right)^{-1}\left(f(y)\right)^{-1} = f^{-1}(x)f^{-1}(y)$$
$$\implies f^{-1} \in \text{Grp}(G, H)$$

The following lemma follows directly from Lemmas 1.1.10 and 1.1.13

Lemma 1.2.15. *Let $\rho : G \to H$ be a group homomorphism and $I \subset G$, $J \subset H$ be subgroups. Then $\rho(I) \subset H$ and $\rho^{-1}(J) \subset G$ are subgroups. In particular, the kernel and image of a ρ are subgroups of G and H respectively.*

Proof. By Lemma 1.1.10 and Lemma 1.1.13 both sets in question are submonoids of H and G respectively. Let $g \in \rho^{-1}(J)$. Then by Remark 1.2.14,

$$\rho\left(g^{-1}\right) = \rho(g)^{-1} \in J \implies g^{-1} \in \rho^{-1}(J),$$

hence $\rho^{-1}(J)$ is a subgroup of G. Let $\rho(i) \in \rho(I)$. Then by Remark 1.2.14

$$\rho(i)^{-1} = \rho\left(i^{-1}\right) \in \rho(I),$$

hence $\rho(I)$ is a subgroup of H. □

Proposition 1.2.16. *Let G be an additive group and $H \subset G$ a subgroup. Then H is an additive subgroup.*

Proof. This follows from Lemma 1.2.7 and Lemma 1.1.15. □

1.2.3 Product Groups, Direct Sums & and Other Enumerated Constructions

Lemma 1.2.17. *Given a family of groups $\{G_\alpha\}$. The product monoid*

$$\prod_\alpha G_\alpha$$

is a group where the inverse of an element (g_α) is given by (g_α^{-1}).

Proof. Indeed, given $(g_\alpha) \in \prod_\alpha G_\alpha$ we have that

$$(g_\alpha)(g_\alpha)^{-1} = (g_\alpha)(g_\alpha^{-1}) = (g_\alpha g_\alpha^{-1}) = (e_\alpha) = e$$

proving that $(g_\alpha)^{-1}$ is the left inverse of (g_α) follows from a dual argument. □

Definition 1.2.18. Let A be a set and $\{G_\alpha\}$ a family of additive groups. We define the *direct sum of G_α over A* denoted

$$\bigoplus_{\alpha\in A} G_\alpha$$

to be the finitely supported product monoid of $\{G_\alpha\}$.

One readily verifies that if an element (g_α) is finitely supported then $(g_\alpha)^{-1}$ is finitely supported, so the above construction defines an additive group.

Remark 1.2.19. Let $\{G_\alpha\}$ a family of subgroups of an additive group G. Then

$$\sum_{\alpha\in A} G_\alpha \in G,$$

Lemma 1.1.25 defines a subgroup of G, since $(g_\alpha) \mapsto \sum_\alpha g_\alpha$ defines a group homomorphism, $\bigoplus_\alpha G_\alpha \to G$.

Remark 1.2.20.

1. The kernel of s is contained in

$$\left\{ (h_\alpha) \in \bigoplus_{\alpha\in A} : h\alpha \in H_\alpha \cap \sum_{\beta\in A\setminus\{\alpha\}} H_\beta \text{ for each } \alpha \in A \right\}.$$

   Indeed, let $(h_\alpha) \in \ker\ s$. Then $\sum_1^n h_{\alpha_i} = \sum_\alpha h_\alpha = 0$. Let $\alpha \in A$. If $h_\alpha = 0$, then it is trivially in $H_\alpha \cap \sum_{\beta\in A\setminus\{\alpha\}} H_\beta$. Otherwise $\alpha = \alpha_i$ for some $i \in \{1, \dots, n\}$. Then $h_{\alpha_i} = \sum_{\beta\in A\setminus\{\alpha_i\}} h_\alpha \in \sum_{\beta\in A\setminus\{\alpha_i\}} H_\beta$, hence $h_{\alpha_i} \in H_{\alpha_i} \cap \sum_{\beta\in A\setminus\{\alpha_i\}} H_\beta$.

2. One should note that

$$\sum_{\alpha\in A} H_\alpha = \left\langle \bigcup_{\alpha\in A} H_\alpha \right\rangle.$$

Indeed, $h_\alpha \in \bigcup_{\alpha\in A} H_\alpha$ for every $\alpha \in A$, hence

$$\sum_{\alpha\in A} h_\alpha = \sum_{1}^{n} h_\alpha \in \left\langle \bigcup_{\alpha\in A} H_\alpha \right\rangle.$$

Let $\sum_1^n m_i h_{\alpha_i} \in \left\langle \bigcup_{\alpha\in A} H_\alpha \right\rangle$ where $m_i \geq 0$, $\alpha_i \in A$, $h_{\alpha_i} \in H_{\alpha_i}$. For each i we then have that

$$m_i h_{\alpha_i} = \sum_{j=1}^{m_i} h_{\alpha_i} \in H_{\alpha_i},$$

Hence upon putting $h'_\alpha = 0$ for $\alpha \in A \setminus \{\alpha_1, \ldots, \alpha_n\}$ and $h'_{\alpha_i} = m_i h_{\alpha_i}$, implying

$$\sum_{1}^{n} m_i h_{\alpha_i} = \sum_{\alpha\in A} h'_\alpha \in \sum_{\alpha\in A} H_\alpha.$$

Proposition 1.2.21. *Let A be a set and $\{G_\alpha\}_{\alpha\in A}$ a family of groups.Then*

$$\pi_\beta : \prod_{\alpha\in A} G_\alpha \to G_\beta$$

is group homomorphism. Given a group H and a family of group homomorphisms $\{f_\alpha : H \to G_\alpha\}_{\alpha\in A}$ then the unique group homomorphism $f : H \to \prod_{\alpha\in A} G_\alpha$ (cf. Lemma 1.1.18) such that $\pi_\alpha \circ f = f_\alpha$ for every $\alpha \in A$ is group homomorphism.

Proof. π_β and f being monoid homomorphism they are automatically group homomorphisms. □

Proposition 1.2.22. *Let A be a set and $\{G_\alpha\}_{\alpha\in A}$ a family of groups. Consider a family of sets $\{H_\alpha\}_{\alpha\in A}$ such that $H_\alpha \subset G_\alpha$ is a subgroup for each $\alpha \in A$. Then*

$$\prod_{\alpha\in A} H_\alpha \subset \prod_{\alpha\in A} G_\alpha$$

is a subgroup.

Proof. By Proposition 1.1.19 it follows that $\prod_{\alpha\in A} H_\alpha$ is a submonoid. It thus suffices to check that it is closed under inversion of elements. Let $(h_\alpha) \in \prod_{\alpha\in A} H_\alpha$. Then $h_\alpha^{-1} \in H_\alpha$ for each $\alpha \in A$, hence $(h_\alpha)^{-1} = (h_\alpha^{-1}) \in \prod_{\alpha\in A} H_\alpha$. □

Example 1.2.23. Not all subgroups of $\prod_\alpha G_\alpha$ arise in this fashion. For instance $\{(n, n) \in \mathbb{Z}\times\mathbb{Z} : n \in \mathbb{Z}\}$, which is a subgroup of $(\mathbb{Z}\times\mathbb{Z}, +)$, is not the product of pair of subgroups of $\mathbb{Z}$.

Lemma 1.2.24. *Let $\rho : G \to H$ be a group homomorphism where G is an abelian group and $J \subset G$ be a subgroup. Then $\rho(J)$ is an abelian group*

Proof. Since $\rho(J) \subset H$ is a subgroup it is a group. It remains to check that $\rho(J)$ is abelian. Let $\rho(g_1)\rho(g_2) \in \rho(J)$. Then

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2) = \rho(g_2 g_1) = \rho(g_2)\rho(g_1).$$

□

Proposition 1.2.25. *Let A be a set, G a group and $\{H_\alpha\}_{\alpha \in A}$ be a family of subgroups of G. Then*

$$\bigcap_{\alpha \in A} H_\alpha$$

is a subgroup of G

Proof. Clearly $e \in H_\alpha$ for each $\alpha \in A$, hence $e \in \bigcap_{\alpha \in A} H_\alpha$. Fix a $\beta \in A$. Let $h, h' \in \bigcap_{\alpha \in A} H_\alpha$. Then $hh', h^{-1} \in H_\alpha$ for each $\alpha \in A$, hence $hh', h^{-1} \in \bigcap_{\alpha \in A} H_\alpha$. □

1.2.4 Quotient Groups

Definition 1.2.26. Let G be a group and $X, Y \subset G$ we then define

$$XY := \circ(X \times Y) = \{xy \in G : (x, y) \in X \times Y\},$$

and

$$X^{-1} := i(X) = \left\{x^{-1} \in G : x \in X\right\}.$$

Remark 1.2.27. One easily sees for $X, Y, Z \subset G$ that $X(YZ) = (XY)Z$ and that $(XY)^{-1} = Y^{-1}X^{-1}$.

Definition 1.2.28. Let G be a group, $H \subset G$ be a subgroup and $g \in G$. The *left coset of H with respect to g* is defined to be the set

$$gH := \{g\}H = \{gh : h \in H\}$$

The *right coset of H with respect to g* is defined to be the set

$$Hg := H\{g\} = \{hg : h \in H\}.$$

Remark 1.2.29. Note that clearly $hH = H = Hh$ for any $h \in H$. If $gH \neq H$, then $gh \notin H$ for some $h \in H$, meaning $g \notin H$. Since $eg = g = ge$, it also follows that $g \in gH$ and $g \in Hg$ for every $g \in G$. It is also easy to check that $H^{-1} = H$.

Proposition 1.2.30. *Let G be a group and $H \subset G$ a subgroup. Then the sets*

$$\sim_l := \{(g_1, g_2) \in G \times G : g_1 H = g_2 H\} \text{ and } \sim_r := \{(g_1, g_2) \in G \times G : H g_1 = H g_2\}$$

define equivalence relations. We define $G/H := G/\sim_l$ and $G \setminus H := G/\sim_r$.

Proof. We only check the left case, since the right case is dual. Let $g_1, g_2, g_3 \in G$. Obviously $g_1 H = g_1 H$, hence $g_1 \sim_l g_1$. Suppose $g_1 \sim_l g_2$. Then $g_1 H = g_2 H$, hence $g_2 H = g_1 H$, meaning $g_2 \sim_l g_1$. Suppose $g_1 \sim_l g_2$ and $g_2 \sim_l g_3$. Then $g_1 H = g_2 H = g_3 H$, hence $g_1 \sim_l g_3$. □

Lemma 1.2.31. *Let G be a group and $H \subset G$. Then for $g, g' \in G$*

$$g \sim_l g' \iff g^{-1} g' \in H.$$

Proof. Indeed,

$$g \sim_l g' \iff gH = g'H \iff g^{-1}(gH) = g^{-1}(g'H) \iff H = \left(g^{-1} g\right) H = \left(g^{-1} g'\right) H$$
$$\iff g^{-1} g' \in H,$$

where the last bi-implication follows from Remark 1.2.29. □

Proposition 1.2.32. *A group G with a subgroup $H \subset$ is the disjoint union of the elements of G/H respectively $G \setminus H$.*

Proof. We check the left case. The right case is dual. Since $g \in gH$ for every $g \in G$, it follows that

$$G = \bigcup_{gH \in G/H} gH.$$

Let $g_1, g_2 \in G$. Suppose $g_1 H \cap g_2 H \neq \emptyset$. Then there is an element $x \in g_1 H \cap g_2 H$, hence $g_1 h_1 = x = g_2 h_2$ for suitable $h_1, h_2 \in H$. This implies that

$$g_1^{-1} g_2 = h_1 h_2^{-1} \in H \implies g_1 H = g_2 H.$$

Thus if $g_1 H \neq g_1 H$, then $g_1 H \cap g_2 H = \emptyset$. □

Definition 1.2.33. A subgroup $H \subset G$ is *normal* if

$$gNg^{-1} = N$$

for every $g \in G$.

Remark 1.2.34. Note that

$$gNg^{-1} = N \iff gN = gN(g^{-1}g) = (gNg^{-1})g = Ng,$$

Hence any subgroup of an abelian group is normal. Furthermore $\sim_l$ and $\sim_r$ define the same set. Thus we may define $\sim$ to be $\sim_l$. Let $X \subset G$. Then $XN = NX$. Indeed, if $xn \in XN$, then $xn \in xN = Nx$, hence $XN \subset NX$. The other inclusion is shown in a similar way.

Lemma 1.2.35. *The kernel of a group homomorphism $\rho : G \to H$ is a normal subgroup of G.*

Proof. Let $g \in G$ and $k \in \ker\ \rho$. Then

$$\rho\left(gkg^{-1}\right) = \rho(g)\rho(k)\rho(g)^{-1} = \rho(g)e\rho(g)^{-1} = e,$$

hence $g(\ker\ \rho)g^{-1} \subset \ker\ \rho$. Conversely $k \in g(\ker\ \rho)g^{-1}$ since $k = g\left(g^{-1}kg\right)g^{-1}$ and $g^{-1}kg \in \ker\ \rho$ by the above computation. □

It may be that a subgroup is not normal, but we may construct a normal subgroup containing it.

Lemma 1.2.36. *The intersection of a collection of normal subgroups of some group G is a normal subgroup of G.*

Proof. Consider a collection of normal subgroups G, $\{N_\alpha\}$. Let $g \in G$. Let $ghg^{-1} \in g\left(\bigcap_{alpha} N\right)g^{-1}$. Then $ghg^{-1} \in N_\alpha$ for each α, hence $ghg^{-1} \in N_\alpha$ for each α. This means $ghg^{-1} \in \bigcap_\alpha N_\alpha$. □

Lemma 1.2.37. *Let $H \subset G$ be a subgroup of a group G. We define the normal closure of H to be the intersection of all normal subgroups containing H. This is the smallest normal subgroup containing H, which we denote $\overline{H}$.*

Proof. By the prior lemma $\overline{H}$ is a normal subgroup of G and it trivially contains H. If N is a normal subgroup of G containing H, then trivially it contains the intersection of all normal subgroups of G containing H. □

Proposition 1.2.38. *Let G be a group and $H, N \subset G$ be subgroup, where N is normal. Then $HN, NH \subset G$ are subgroups of G.*

Proof. Let $h_1n_1, h_2n_2 \in HN$. Then

$$h_1n_1h_2n_2 \in (HN)(HN) = (NH)(HN) = (N(HH))N = (NH)N = (HN)N = H(NN) = HN.$$

Since $e \in H$ and $e \in N$, $ee \in HN$. Thus HN is a subgroup of G. Since $HN = NH$, it follows that NH is also a subgroup of G. □

Proposition 1.2.39. *Let G be a group and $H \subset G$ a normal subgroup. Then the operation*

$$\cdot : G/H \times G/H \to G/H$$

given by $(g_1H)(g_2H) := g_1g_2H := (g_1g_2)H$ for $g_1H, g_2H \in G/H$ is well-defined and $(G/H, \cdot)$ is a group.

Proof. We first need to check that the group operation is well-defined. Let $g_1, g_2 \in G$. We need to check that $(g_1H)(g_2H) = (g_1g_2)H$. Let $g_1h_1g_2h_2 \in g_1Hg_2H$. Then $h_1g_2 \in Hg_2 = g_2H$, hence $h_1g_2h_2 \in g_2Hh = g_2H$, hence $g_1(h_1g_2h_2) = g_1(g_2H) = (g_1g_2)H$. If $g_1g_2h \in g_1g_2H$, then $g_1g_2h = g_1eg_2h \in (g_1H)(g_2H)$. Thus the operation is well-defined, since if $(g_1H, g_2H) = (g_1'H, g_2'H)$, then trivially

$$(g_1g_2)H = (g_1H)(g_2H) = (g_1'H)(g_2'H) = (g_1'g_2')H.$$

For $g_1H, g_2H, g_3H \in G/H$, it follows by Remark 1.2.27 that

$$(g_1H)((g_2H)(g_3H)) = (g_1Hg_2H)(g_3H).$$

Define the neutral element with respect to $\cdot$ to be eH. Indeed

$$(eH)(gH) = (eg)H = gH,$$

for every $gH \in G/H$. We inverse element of $gH \in G/H$ to be $(g^{-1}H) = H^{-1}g^{-1} = Hg^{-1} = g^{-1}H$. Indeed

$$(g^{-1}H)(gH) = (g^{-1}g)H = eH.$$

□

Corollary 1.2.40. *If $(G, +)$ is an additive group and $H \subset G$ is a subgroup, then $(G/H, +)$ (where $+$ is defined as in the above proposition) is an additive group*

Proof. By Remark 1.2.34 H is normal and by the above proposition G/H is a group. Let $g_1 + H, g_2 + H \in G/H$ be given. Then

$$(g_1 + H) + (g_2 + H) = (g_1 + g_2) + H = (g_2 + g_1) + H = (g_2 + H) + (g_1 + H).$$

□

Lemma 1.2.41. *Let G be a group and N a normal subgroup. Let $N \subset H \subset G$ a subgroup. Then $N \subset H$ is normal and $H/N \subset G/N$ is a subgroup.*

Proof. For every $h \in H$, $hNh^{-1} = N$, since $h \in G$. Let $h_1N, h_2N \in H/N$. Then since $h_1h_2 \in H$,

$$(h_1N)(h_2N) = h_1h_2N \in H/N.$$

Furthermore since $e \in H$, $eN \in H/N$. □

Proposition 1.2.42. *Let G be a group and N a normal subgroup. Then $S = \{H \subset G : H \text{ is a subgroup of } G, N \subset H\}$ is in one-to-one correspondence with the set $S' = \{K \subset G/N : K \text{ a subgroup of } G/N\}$. Any subgroup $K \subset G/N$ is of the form H/N for some $H \in S$.*

Proof. We show that $u : S \to S'$, $H \mapsto H/N$ is a bijection. This map is well-defined by the above lemma. For $K \in S'$, let $H(K) = \{g \in G : gN \in K\}$. We check that $H(K)$ is a subgroup G containing N. Let $h_1, h_2 \in H(K)$. Then $h_1N, h_2N \in K$, hence $h_1h_2N \in K$, implying $h_1h_2 \in H(K)$. Clearly $eN \in K$, hence $e \in H(K)$. Let $n \in N$. Then $nN = eN \in K$, hence $n \in H(K)$. Then the map $u' : S' \to S$, $K \to H(K)$ is well-defined. We check that u and u' are mutual inverses. Let $K \in S'$. We need to check that $uu'(K) = H(K)/N = K$. Let $k \in K$, then $k = gN$ for some $g \in G$, then $g \in H(K)$, hence $k = gN \in H(K)/N$. Let $hN \in H(K)/N$. Then by definition $hN \in K$. Let $H \in S$. Then we need to check that $u'u(H) = H(H/N) = H$. Let $h \in H(H/N)$, then $hN \in H/N$, hence $h \in H$. Let $h \in H$. Then $hN \in H/N$, hence $h \in H(H/N)$. □

Proposition 1.2.43. *Let G be a group and $N \subset G$ a normal subgroup. The surjection $\pi : G \to G/N, g \mapsto gN$ is a group map with kernel N.*

Proof. Let $g_1, g_2 \in G$. Then $\pi(g_1g_2) = g_1g_2N = g_1Ng_2N = \pi(g_1)\pi(g_2)$. Suppose $\pi(g) = eN$. Then $g \in N$ hence ker $\pi = N$. □

Corollary 1.2.44. *Since the kernel of a group map is a normal subgroup, it follows that every normal subgroup arises as the kernel of a group map*

Theorem 1.2.45. *Given a normal subgroup N of a group G, consider the diagram*

$$N \underset{\iota}{\overset{0}{\rightrightarrows}} G$$

G/N with $\pi : G \to G/N$ is the coequaliser of this diagram

Proof. Let $\alpha : G \to H$ be a group map such that $\alpha\iota = 0$. The second condition is equivalent to ker $\alpha \subset N$. We define

$$\beta : G/N \to H$$
$$gN \mapsto \alpha(g)$$

If $gN = g'N$, then $g(g')^{-1} \in N$, hence

$$\alpha(g)\alpha(g')^{-1} = \alpha(gg'^{-1}) = e,$$

hence

$$\beta(gN) = \alpha(g) = \alpha(g') = \beta(g'N).$$

It is a group map. Indeed,

$$\beta(gNg'N) = \beta(gg'N) = \alpha(gg') = \alpha(g)\alpha(g') = \beta(gN)\beta(g'N)$$

Clearly the coequaliser diagram

$$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\alpha} & \\ G/N & \xrightarrow{\beta} & H \end{array}$$

commutes. If $\gamma : G/N \to H$ is a group map such that $\alpha = \gamma\pi$, then for each $gN \in G/N$,

$$\gamma(gN) = \gamma\pi(g) = \alpha(g) = \beta\pi(g) = \beta(gN),$$

hence $\gamma = \beta$. □

Corollary 1.2.46. *Let $\alpha : G \to H$ be a group homomorphism and $I \subset G$ $J \subset H$ be normal subgroups. Then*

$$\beta : G/I \to H/J$$
$$gI \mapsto \alpha(g)J$$

is a well-defined group homomorphism if and only if $\alpha(I) \subset J$. If so β is the unique group map such that the square

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_J} \\ G/I & \dashrightarrow & G/J \end{array}$$

commutes

Proof. " $\Longrightarrow$ ": Suppose β is a well-defined group homomorphism. Let $\alpha(g) \in \alpha(I)$. Then since $gI = eI$,

$$\alpha(g)J = \beta(gI) = \beta(eI) = \alpha(e)J = eJ,$$

hence $\alpha(g) \in J$.
" $\Longleftarrow$ ": Suppose $\alpha(I) \subset J$. Then for the canonical surjection $\pi_J : H \to H/J$,

$$I \subset \ker\ \pi_J\alpha,$$

hence the result follows from G/I being the coequaliser of

$$I \underset{\iota}{\overset{0}{\rightrightarrows}} G$$

Uniqueness when β is well-defined likewise follows from this fact. □

Definition 1.2.47. Let $\alpha : G \to H$ be a group map. We define *the cokernel of* α, denoted coker α, to be the group $H/\overline{\text{im } \alpha}$

Corollary 1.2.48. *The cokernel of a group map $\alpha : G \to H$ is the coequaliser of the diagram*

$$G \underset{\alpha}{\overset{0}{\rightrightarrows}} H$$

Proof. Note that $\pi : H \to$ coker α is a map such that $\pi\alpha = 0$, since im $\alpha \subset \overline{\text{im } \alpha}$. Let $\beta : G \to K$ be a group map with $\beta\alpha = 0$. Note that then $\beta\iota = 0$ where $\iota : \overline{im\ \alpha} \hookrightarrow H$ is the canonical embedding. Since coker α is the coequaliser of

$$\overline{\text{im } \alpha} \underset{\iota}{\overset{0}{\rightrightarrows}} H$$

the result follows. □

1.2.5 The First Isomorphism Theorem of Groups

Proposition 1.2.49. *Given a group map $\alpha : G \to H$, the equaliser of the diagram*

$$G \underset{\alpha}{\overset{0}{\rightrightarrows}} H$$

is ther kernel of α

Proof. Let $\beta : K \to G$ be a group map such that $\alpha\beta = 0$. Then im $\beta \subset$ ker α. Then

$$\beta' : K \to \text{ker } \alpha$$
$$k \mapsto \beta(k)$$

is a well-defined group map such that

$$\begin{array}{ccc} K & & \\ \vdots & \searrow^{\beta} & \\ \downarrow & & \\ \text{ker } \alpha & \xrightarrow{\iota} & G \end{array}$$

commutes. Uniqueness follows from ι being a monomorphism. □

Lemma 1.2.50. *A group map $f : G \to H$ is an isomorphism of groups if and only if it is a bijective group map.*

Proof. One implication is trivial. If f is a bijective group map it suffices to show that its inverse is a group. Let $h, h' \in H$ be given. Then

$$f^{-1}(hh') = f^{-1}(f(f^{-1}(h))f(f^{-1}(h'))) = f^{-1}(f(f^{-1}(h)f^{-1}(h'))) = f^{-1}(h)f^{-1}(h').$$

□

Lemma 1.2.51. *A group map $\alpha : G \to H$ is injective if and only if* $\ker\ \alpha = \{e\}$

Proof. If $\ker\ \alpha = \{e\}$, then if $h_1, h_2 : K \to G$, such that $\alpha h_1 = \alpha h_2$. By the universal property of the kernel, we obtain maps $h_1', h_2' : K \to \ker\ \alpha$ such that

$$h_1 = 0h_1' = 0 = 0h_2'h_2,$$

hence α is mono and therefor injective.
If α is injective, then given $g \in \ker\ \alpha$

$$\alpha(g) = e = \alpha(e) \implies g = e.$$

□

Theorem 1.2.52. *(First Isomorphism Theorem of Groups) Given a group map $\alpha : G \to H$, we get a group isomorphism*

$$\widetilde{\alpha} : G/\ker\ \alpha \to \operatorname{im} \alpha$$
$$g(\ker\ \alpha) \mapsto \alpha(g)$$

Proof. By prior results presented above $\widetilde{\alpha}$ is a well-defined group map. Note that every $\alpha(g) \in \operatorname{im} \alpha$ is equal to $\widetilde{\alpha}(g(\ker\ \alpha))$. Thus $\widetilde{\alpha}$ is surjective. Let $g(\ker\ \alpha) \in \ker\ \widetilde{\alpha}$. Then $\alpha(g) = \widetilde{\alpha}(g(\ker\ \alpha)) = e$, hence $g \in \ker\ \alpha$, which implies $g(\ker\ \alpha) = e(\ker\ \alpha)$. □

Corollary 1.2.53. *If $\alpha : G \to H$ is a surjective group map, $G/\ker\ \alpha \simeq H$.*

1.3 Rings

1.3.1 Definition & Basic Properties

Definition 1.3.1. A *ring (with unity)* is a set R with operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$ called *multiplication* such that $(R, +)$ is an additive group, $(R, \cdot)$ is a monoid and for $r_1, r_2, r_3 \in R$

$$r_1(r_2 + r_3) = r_1r_2 + r_1r_3 \ \&\ (r_1 + r_2)r_3 = r_1r_3 + r_2r_3.$$

We denote the neutral element with respect to multiplication by 1. The data specifying a ring is often written $(R, +, \cdot)$.

Lemma 1.3.2. *Let R be a ring and $r \in R$. The following identities are true for rings*

1. $0 \cdot r = 0$.

2. $(-1)r = -r$, $r(-1) = -r$.

Proof. 1. follows from the following computation.

$$0r = 0r + 0 = 0r + r - r = 0r + 1r - r = (0 + 1)r - r = 1r - r = r - r = 0$$

2. follows from the following computation.

$$(-1)r = (-1)r + 0 = (-1)r + r - r = (-1)r + 1r - r = (-1 + 1)r - r = 0r - r = 0 - r = -r$$

the other statement is proven similarly. □

Definition 1.3.3. Let R be a ring. If $(R, \cdot)$ is a commutative monoid, then R is called a *commutative ring*.

Definition 1.3.4. Let R be a ring. A *subring* is a subset $S \subset R$ such that S is a subgroup of $(R, +)$ and a submonoid of $(R, \cdot)$.

Remark 1.3.5. $(S, +, \cdot)$ is a ring. Indeed, clearly $(S, +)$ is an additive group since $S \subset R$ is a subgroup and $(S, \cdot)$ is a monoid since $S \subset R$ is a submonoid. Let $r_1, r_2, r_3 \in S$, then since $S \subset R$,

$$r_1(r_2 + r_3) = r_1r_2 + r_1r_2 \ \& \ (r_1 + r_2)r_3 = r_1r_3 + r_2r_3.$$

One should also note that R is commutative then S is commutative

Example 1.3.6. 1. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are rings.

2. Given a ring $(R, +, \cdot)$, we obtain the *opposite ring of* R, denoted R^{op} to be $(R, +, \star)$ where $(R, \star)$ is the opposite monoid of $(R, \cdot)$. Since $\cdot$ distributes over $+$, it follows that the same obviously true for $\star$ over $+$.

3. For a non-empty set X and a ring R, the set $\text{Set}(X, R)$ is a monoid and an additive group with respect to multiplication and addition defined earlier. One easily verifies that it is also a ring.

1.3.2 Morphisms of Rings

Definition 1.3.7. Let R, S be rings. A map $\sigma : R \to S$ is called a *ring homomorphism/map of rings/morphism of rings* if σ is a group homomorphism between $(R, +)$ and $(S, +)$ and a monoid homomorphism between $(R, \cdot)$ and $(S, \cdot)$. The set of ring homomorphisms between R to S is denoted $\mathrm{Hom}^{\mathrm{Ring}}(R, S)$.

Remark 1.3.8. The class of all rings together with the class of all ring homomorphisms form a subcategory of Set denoted Ring. We denote the category of commutative rings by ComRing.

Example 1.3.9. For rings R, S, if $\mathrm{Ring}(R, S) \neq \emptyset$ is never a subring of $\mathrm{Set}(R, S)$. Indeed, note that the zero map is never a ring homomorphism since it maps 1 to 0. The terminal object is the zero ring. We shall see that the initial object is $\mathbb{Z}$.

Lemma 1.3.10. *Let $\sigma : R \to S$ be a ring homomorphism and $T \subset R$, $U \subset S$ be subrings. Then $\sigma(T) \subset S$ and $\sigma^{-1}(U)$ are subrings. If T is commutative then so is $\sigma(T)$.*

Proof. Prior lemmas ensure that these sets are appropriate additive subgroups and submonoids. □

1.3.3 Product Rings

Proposition 1.3.11. *Let A be a set, $\{R_\alpha\}_{\alpha \in A}$ a family of rings. Then $\left(\prod_{\alpha \in A} R_\alpha, \cdot\right)$ is a monoid and $\left(\prod_{\alpha \in A} R_\alpha, +\right)$ an additive group. In addition $\left(\prod_{\alpha \in A} R_\alpha, +, \cdot\right)$ is a ring.*

Proof. It remains to check that multiplication distributes over addition. Let $(r_\alpha), (r'_\alpha), (r''_\alpha) \in \prod_{\alpha \in A} R_\alpha$. Then

$$\begin{aligned}(r_\alpha)\left((r'_\alpha) + (r''_\alpha)\right) &= (r_\alpha)(r'_\alpha + r''_\alpha) = (r_\alpha(r'_\alpha + r''_\alpha)) = (r_\alpha r'_\alpha + r_\alpha r''_\alpha) = (r_\alpha r'_\alpha) + (r_\alpha r''_\alpha) \\ &= (r_\alpha)(r'_\alpha) + (r_\alpha)(r''_\alpha).\end{aligned}$$

□

Remark 1.3.12. The above ring is called *the product ring (of $\{R_\alpha\}_{\alpha \in A}$ over A).*

Corollary 1.3.13. *The direct sum of rings is a subring of the direct product.*

Proposition 1.3.14. *Let A be a set and $\{R_\alpha\}_{\alpha \in A}$ a family of rings. Then*

$$\pi_\beta : \prod_{\alpha \in A} R_\alpha \to R_\beta$$

is ring homomorphism. Given a ring S and a family of ring homomorphisms $\{f_\alpha : S \to R_\alpha\}_{\alpha \in A}$ then the unique group and monoid homomorphism $f : S \to \prod_{\alpha \in A} R_\alpha$ (cf. Lemma 1.1.18 and Proposition 1.2.21) such that $\pi_\alpha \circ f = f_\alpha$ for every $\alpha \in A$ is a ring homomorphism.

Proof. This follows immediately from the fact both π_β and f are both group and monoid homomorphisms. □

1.3.4 The Ring of Integers: $\mathbb{Z}$

Definition 1.3.15. For $(a, b), (c, d) \in \mathbb{N}$ we define $(a, b) \sim (c, d)$ if $a + d = b + c$. One easily checks that this is an equivalence relation. We define

$$\mathbb{Z} := \mathbb{N}^2 / \sim .$$

Proposition 1.3.16. *On $\mathbb{Z}$ we define*

$$[(a, b)] + [(c, d)] := [(a + c, b + d)]$$

and

$$[(a, b)][(c, d)] := [(ac + bd, ad + bc)].$$

Moreover we define $0 := [(0, 0)]$, $-[(a, b)] := [(b, a)]$ and $1 := [(1, 0)]$. With these definitions, $\mathbb{Z}$ becomes a commutative ring. The set $\{[(a, 0)] \in \mathbb{Z} : a \in \mathbb{N}\}$ is a sub semi-ring isomorphic to $\mathbb{N}$.

Proof. Suppose first that $([(a, b)], [(c, d)]) = ([(x, y)], [(v, w)])$. Then

$$a + y = b + x, \quad c + w = d + v$$

hence

$$(a + c) + (y + w) = (a + y) + (c + w) = (b + x) + (d + v) = (b + d) + (x + v).$$

This implies

$$[(a + c, b + d)] = [(x + v, y + w)].$$

So addition is well-defined. We check that $\mathbb{Z}$ is a group. Associativity of addition on $\mathbb{Z}$ readily follows from associativity of addition on $\mathbb{N}$. Let $[(a, b)], [(c, d)] \in \mathbb{Z}$ be arbitrary. Then

$$[(a, b)] + [(0, 0)] = [(a + 0, b + 0)] = [(a, b)]$$

and

$$[(a,b)] + (-[(a,b)]) = [(a,b)] + [(b,a)] = [(a+b, a+b)] = [(0,0)] = 0$$

and

$$[(a,b)] + [(c,d)] = [(a+c, b+d)] = [(c+a, d+b)] = [(c,d)] + [(a,b)]$$

hence $\mathbb{Z}$ is an additive group. It is easy to check that $\{[(a,0)] \in \mathbb{Z} : a \in \mathbb{N}\}$ is a submonoid isomorphic to $\mathbb{N}$. Suppose again $([(a,b)],[(c,d)]) = ([(x,y)],[(v,w)])$. We find that

$$\begin{aligned} ac + bd + xw + yv + yc + xd + xc + yd = & \\ &= c(a+y) + d(b+x) + x(c+w) + y(d+v) \\ &= c(b+x) + d(a+y) + x(d+v) + y(c+w) \\ &= ad + bc + xv + yw + yc + xd + xc + yd. \end{aligned}$$

implying that, using a fact from group theory,

$$ac + bd + xw + yv = ad + bc + xv + yw \implies [(ac+bd, ad+bc)] = [(xv+yw, xw+yv)].$$

We now find that

$$\begin{aligned} ([(a,b)][(c,d)])[(e,f)] &= [(ac+bd, ad+bc)][(e,f)] \\ &= [(ace + bde + adf + bcf, acf + bdf + ade + bce)] \\ &= [(a(ce+df) + b(cf+de), a(cf+de) + b(ce+df)] \\ &= [(a,b)][(ce+df, cf+de)] = [(a,b)]([(c,d)][(e,f)]). \end{aligned}$$

and

$$[(a,b)][(1,0)] = [(a + b\cdot 0, a\cdot 0 + b)] = [(a,b)]$$

and easily we check that

$$[(a,b)][(c,d)] = [(c,d)][(a,b)].$$

Lastly,

$$\begin{aligned} [(a,b)]([(c,d)] + [(e,f)]) = [(a,b)][(c+e, d+f)] &= [(ac + ae + bd + bf, ad + af + bc + be)] \\ &= [(ac+bd, ad+bc)] + [(ae+bf, af+be)] \\ &= [(a,b)][(c,d)] + [(a,b)][(e,f)], \end{aligned}$$

making $\mathbb{Z}$ a commutative ring. One readily verifies that $\{[(a,0)] \in \mathbb{Z} : a \in \mathbb{N}\}$ is a submonoid of $\mathbb{Z}$ with respect to multiplication. The isomorphism from $\mathbb{N}$ is given by $a \mapsto [(a,0)]$. □

One important fact about the ring of integers is that it is the initial object in the category of rings.

Lemma 1.3.17. *$\mathbb{Z}$ is the initial object in the category of rings.*

Proof. Let R be any ring. We define a function

$$f_R : \mathbb{Z} \to R$$
$$n \mapsto \sum_1^n 1$$

One readily verifies that this is a ring homomorphism. Moreover, given any ring map $f : \mathbb{Z} \to R$, we have for each $n \in \mathbb{Z}$

$$f(n) = f\left(\sum_1^n 1\right) = \sum_1^n f(1) = \sum_1^n 1 = f_R(n).$$

Thus we conclude $f = f_R$. □

1.4 Modules

1.4.1 Initial Definitions, Basic Properties & Constructions

Definition 1.4.1. Let R be a ring. A *left R-module* is an additive group $(M, +)$ with a *left scalar multiplication* $\cdot : R \times M \to M$, where $rm := r \cdot m := \cdot(r, m)$ for $(r, m) \in R \times M$ satisfying the following axioms

1. For every $r \in R$, $m, m' \in M$,
$$r(m + m') = rm + rm'.$$
2. For every $r, r' \in R$, $m \in M$,
$$(r + r')m = rm + r'm.$$
3. For every $r, r' \in R$, $x \in M$,
$$(rr')m = r(r'm).$$
4. For every $m \in M$,
$$1m = m.$$

To emphasise that a module M is a left R-module, we may write ${}_RM := M$.
A *right R-module* is an additive group $(M, +)$ with a *right scalar multiplication* $\cdot : M \times R \to M$, where $mr := m \cdot r := \cdot(m, r)$, satisfying axioms dual to ones for left scalar multiplication. To be precise a right R-module is a left R^{op}-module. To emphasise that a module M is a right R-module, we may write $M_R := M$
Let S be a ring. An (R, S)-bimodule is an additive group $(M, +)$, that is a left R-module and a right S-module satisfying

$$(rm)s = r(ms),$$

for every $r \in R$,$s \in S$, $m \in M$. To emphasise that a module M is an (R, S)-bimodule, we may write ${}_RM_S := M$.
When R is a commutative ring, a left (or right) R-module is simply called an *R-module*.

Since we can view any right module as a left module it is also sufficient to prove a result about left modules in order to obtain a dual statement on right modules. We therefore only talk about left module results.

Definition 1.4.2. Let M be a left R-module. A left R-submodule is a subset $N \subset M$ such that N is a subgroup of $(M, +)$ and for every $r \in R$, $n \in N$ we have that $rn \in N$ resp. $nr \in N$.

Remark 1.4.3. A left R-submodule $N \subset M$ is a left R-module. Indeed $(N, +)$ is group since it is a subgroup of $(M, +)$. N being closed under left scalar multiplication ensures that $\cdot := \cdot|_{R\times N}$: $R\times N \to N$ respectively $\cdot := \cdot|_{N\times R}$: $N\times R \to N$ are well-defined and M being a left R-module, these left actions respect the axioms for left scalar multiplication. If M is a (R, S)-bimodule and N is a left R-submodule and a right S-submodule of M then it is also an (R, S)-bimodule. Indeed, if $r \in R$, $n \in N$ and $s \in S$, then since $n \in M$, $r(ns) = (rn)s$.

Example 1.4.4. 1. Let $(G, +)$ be an additive group. Then G is a $\mathbb{Z}$-module under the left scalar multiplication

$$ng = \sum_1^n g \;\&\; gn = \sum_1^n g.$$

Under this definition $gn = ng$ hence if G is a left $\mathbb{Z}$-module, it is automatically a

$\mathbb{Z}$-module. Let $n, m \in \mathbb{Z}$ and $g, g' \in G$. Then

1. $n(g + g') = \sum_{1}^{n}(g + g') = \sum_{1}^{n} g + \sum_{1}^{n} g' = ng + ng'$
2. $(n + m)g = \sum_{1}^{n+m} g = \sum_{1}^{n} g + \sum_{n+1}^{n+m} g = ng + \sum_{1}^{m} g = ng + mg$
3. $n(mg) = n\sum_{1}^{m} g = \sum_{1}^{n}\sum_{1}^{m} g = \sum_{1}^{nm} g = (nm)g$
4. $1g = \sum_{1}^{1} g = g$

To prove the second to last equality 3. one should really use induction in m. Note that the induction start uses 4.

2. Let $(R, +, \cdot)$ be a ring. Then $(R, +)$ is an additive group, which becomes an (R, R)-bimodule under the action $rx := r \cdot x$ and $xs := x \cdot s$ for $r, s, x \in R$.

3. A left/right R-submodule of $I \subset R$ is called a *left ideal in* R. If I is an (R, R)-bimodule, it is called a *both-sided ideal in* R. If R is commutative a left ideal is simply referred to as an *ideal in* R.

4. Let A be a set and R a ring, then $\prod_{\alpha\in A} R$ is an R-module, under the left given by $a(r_\alpha) := (ar_\alpha) = (r_\alpha a)$ for $a \in R$ and $(r_\alpha) \in \prod_{\alpha\in A} R$. This is easily checked using 2. together with the fact that $(a)(r_\alpha) = a(r_\alpha)$ for every $a \in R$, $(r_\alpha) \in \prod_{\alpha\in A} R$.

5. Let S be a ring and R a subring of S. Then $R \times S \ni (r, s) \mapsto rs \in S$ defines a left scalar multiplication of R on S, hence S is a left R-module.

Lemma 1.4.5. *Let R be a ring and M a left R-module.*

1. $0m = 0$ *for every* $m \in M$.
2. $(-1)m = -m$ *for every* $m \in M$.
3. $r(-m) = -rm$ *for every* $r \in R$, $m \in M$.
4. $r0 = 0$ *for every* $r \in R$.

Proof. 1. Really this is just a generalization of Lemma 1.3.2 1. Indeed,

$$0m = 0m + m - m = 0m + 1m - m = (0 + 1)m - m = 1m - m = m - m = 0.$$

2. Really this is just a generalization of Lemma 1.3.2 2. Indeed,

$$(-1)m = (-1)m+m-m = (-1)m+1m-m = (-1+1)m-m = 0m-m = 0-m = -m$$

3. Indeed,

$$r(-m) = r((-1)m) = (r(-1))m = ((-1)r)m = -rm.$$

4. Indeed,

$$r0 = r(0-0) = r0 + r(-0) = r0 - r0 = (r-r)0 = 0 \cdot 0 = 0.$$

□

Definition 1.4.6. Let R be a ring. An element m of a left/right R-module M is called a torsion *torsion element* if there is an $r \in R$ that is not a left/right zero-divisor satisfying $rm = 0$ (resp. $mr = 0$). A module is a torsion module if every element is a torsion element and *torsion free* if the only torsion element is 0.

Remark 1.4.7. An example of a torsion free module is a domain R.

Definition 1.4.8. Let ${}_RM, {}_RN$ be left R-modules. A map $\rho : M \to N$ is a *left R-module homomorphism/map of left R-modules/morphism of left R-modules* if for every $r \in R$, $m, m' \in M$,

$$\rho(rm + m') = r\rho(m) + \rho(m').$$

Remark 1.4.9. A left/right/bimodule module homomorphisms $(M, +) \to (N, +)$ are automatically group homomorphisms. Similarly, for every $r \in R$ and $m \in M$,

$$\rho(rm) = r\rho(m),$$

which is seen by setting $m' = 0$ when $\rho : M \to N$ is a left R-module. This is also true from the right if ρ was a right R-module map.

Remark 1.4.10. Given a ring R, the class of all left R-modules together with the class of all left R-module maps is a subcategory of $mathrmab$ denoted ${}_R$Mod. There are analogous categories of right R-modules, Mod$_R$ and of (R, S)-bimodules ${}_S$Mod$_R$. The zero object in this category is 0.

Lemma 1.4.11. *If $\rho : M \to N$ is a left R-module homomorphism, then for a left R-submodule $L \subset M$, $\sigma(L) \subset N$ is a left R-submodule. If M and N are (R, S)-bimodules and ρ is an (R, S)-bimodule homomorphism, then $\sigma(L)$ is an (R, S)-subbimodule.*

Proof. We already know that $\sigma(L)$ is a subgroup. Let $r \in R$ and $\sigma(l) \in \sigma(L)$. Then

$$r\sigma(l) = \sigma(rl) \in \sigma(L).$$

Let $r \in R$, $\sigma(l) \in \sigma(L)$, $s \in S$. Then

$$r(\sigma(l)s) = r\sigma(ls) = \sigma(r(ls)) = \sigma((rl)s) = \sigma(rl)s = (r\sigma(l))s.$$

□

Lemma 1.4.12. *Let $\rho : M \to N$ be a left R-module map. Then for any left R-submodule $K \subset N$, $\rho^{-1}(K)$ is a left R-submodule of M. In particular,* ker *ρ is a left R-submodule of M.*

Proof. It is sufficient to check $\rho^{-1}(K)$ is closed under left scalar multiplication, since we have already verified it to be a subgroup in an earlier result. Let $r \in R$ and $m \in \rho^{-1}(K)$. Then

$$\rho(rm) = r\rho(m) \in K,$$

hence $rm \in \rho^{-1}(K)$. □

Lemma 1.4.13. *Let R be a ringz, A a set and $\{M_\alpha\}_{\alpha \in A}$ be a family of left R-modules indexed by A. Then*

$$\prod_{\alpha \in A} M_\alpha$$

is a left R-module.

Proof. Note that by Proposition 1.2.22 the direct product of a family of left modules is an additive group. We check the left case. Let $r_1, r_2 \in R$, $(m_\alpha), (m'_\alpha) \in \prod_{\alpha \in A} M_\alpha$. Then

1. $(r_1 + r_2)(m_\alpha) = ((r_1 + r_2)m_\alpha) = (r_1 m_\alpha + r_2 m_\alpha) = (r_1 m_\alpha) + (r_2 m_\alpha)$
$= r_1(m_\alpha) + r_2(m_\alpha)$.
2. $r_1((m_\alpha) + (m'_\alpha)) = r_1(m_\alpha + m'_\alpha) = (r_1 m_\alpha + r_1 m'_\alpha) = (r_1 m_\alpha) + (r_1 m'_\alpha)$
$= r_1(m_\alpha) + r_1(m'_\alpha)$.
3. $1(m_\alpha) = (1m_\alpha) = (m_\alpha)$.
4. $(r_1 r_2)(m_\alpha) = ((r_1 r_2)m_\alpha) = (r_1(r_2 m_\alpha)) = r_1(r_2 m_\alpha) = r_1(r_2(m_\alpha))$.

□

Remark 1.4.14. Suppose $\{M_\alpha\}_{\alpha \in A}$ is a family of (R, S)-bimodules and let $r \in R$, $s \in S$. Then

$$r((m_\alpha)s) = r(m_\alpha s) = (r(m_\alpha s)) = ((rm_\alpha)s) = (rm_\alpha)s = (r(m_\alpha))s.$$

hence a product of (R, S)-modules is again also an (R, S)-bimodule.

Proposition 1.4.15. *Let A be a set, R a ring and $\{M_\alpha\}_{\alpha\in A}$ a family of left modules. Then*

$$\pi_\beta : \prod_{\alpha\in A} M_\alpha \to M_\beta$$

is a left R-module homomorphism. Given a left R-module N and a family of left R-module homomorphisms $\{f_\alpha : N \to M_\alpha\}_{\alpha\in A}$ then the unique group homomorphism $f : N \to \prod_{\alpha\in A} M_\alpha$ (cf. Proposition 1.2.21) such that $\pi_\alpha \circ f = f_\alpha$ for every $\alpha \in A$ is a left R-module homomorphism. Thus we conclude that the direct product is the product in the category of left R-modules

Proof. Let $r \in R$ and $(m_\alpha) \in \prod_{\alpha\in A} M_\alpha$, $n \in N$. Then for $\beta \in A$,

$$\pi_\beta(r(m_\alpha)) = \pi_\beta((rm_\alpha)) = rm_\beta = r\pi_\beta((m_\alpha)).$$

Furthermore, we have that

$$f(rn) = (f_\alpha(rn)) = (rf_\alpha(n)) = r(f_\alpha(n)) = rf(n).$$

□

Lemma 1.4.16. *Let R be a ring, M a left R-module, A a set and $\{M_\alpha\}_{\alpha\in A}$ a family of left R-submodules. Then*

$$\bigcap_{\alpha\in A} M_\alpha$$

is a left R-submodule.

Proof. From Proposition 1.2.25 we already know that $\bigcap_{\alpha\in A} M_\alpha$ is an additive subgroup. Let $r \in R$ and $m \in \bigcap_{\alpha\in A} M_\alpha$ then $rm \in M_\alpha$ for every $\alpha \in A$, meaning $rm \in \bigcap_{\alpha\in A} M_\alpha$.
□

Proposition 1.4.17. *Let A be a set, R a ring and $\{M_\alpha\}_{\alpha\in A}$ a family of left R-modules. Then*

$$\bigoplus_{\alpha\in A} M_\alpha$$

is a submodule of $\prod_{\alpha\in A} M_\alpha$.

Proof. We already know it to be an additive subgroup. Let $r \in R$ and $(m_\alpha) \in \bigoplus_{\alpha\in A} M_\alpha$. Then for some finite subset $B \subset A$, $m_\alpha = 0$ for every $\alpha \in A \setminus B$. Hence $rm_\alpha = 0$ for every $\alpha \in A \setminus B$. It thus follows that $r(m_\alpha) = (rm_\alpha) \in \bigoplus_{\alpha\in A} M_\alpha$. □

Lemma 1.4.18. *Let R be a ring M be a left R-module. Let $N \subset M$ be a left R-submodule. Then M/N is a left submodule under the left scalar multiplication $r(m + N) := rm + N$ resp. $(m + N)r := mr + N$. If M is an (R, S)-bimodule for some ring S and N is a left R-submodule and a right S-submodule. Then M/N is an (R, S)-bimodule.*

Proof. Let $r_1, r_2 \in R$, $m + N, m' + N \in M/N$. Then

1. $(r_1 + r_2)(m + N) = (r_1 + r_2)m + N = (r_1 m + r_2 m) + N$
   $= (r_1 m + N) + (r_2 m + N) = r_1(m + N) + r_2(m + N)$.
2. $r_1((m + m') + N) = r_1(m + m') + N = (r_1 m + r_1 m') + N$
   $= (r_1 m + N) + (r_1 m' + N) = r_1(m + N) + r_1(m' + N)$.
3. $1(m + N) = 1m + N = m + N$
4. $(rr')(m + N) = (rr')m + N = r(r'm) + N = r(r'm + N) = r(r'(m + N))$.

Suppose M is an (R, S)-bimodule. Let $r \in R$, $s \in S$. Then

$$r((m + N)s) = r(ms + N) = r(ms) + N = (rm)s + N = (rm + N)s$$
$$= (r(m + N))s.$$

□

Corollary 1.4.19. *The canonical surjective group map $\pi : M \to M/N$ is a left module map*

Proof. Let $r \in R$, $m \in M$. Then $\pi(rm) = rm + N = r(m + N) = r\pi(m)$. □

Lemma 1.4.20. *Let R be a ring and M a left R-module and $N \subset M$ a left R-submodule. Then there is one-to-one correspondence between the sets*

$$U = \{L \subset M : L \text{ is an } R\text{-submodule of } M \text{ containing } N\}$$

and

$$U' = \{K \subset M/N : K \text{ is an } R\text{-submodule of } M/N\}.$$

Proof. This is a corollary of Proposition 1.2.42. Note that by Lemma 1.4.18, $u : U \to U'.L \mapsto L/N$ is well-defined- Note that $U \subset S$ and $U' \subset S'$. Let $K \in U'$, we check that $L(K) := \{m \in M : m + N \in K\}$ is an R-submodule. Let $r \in R$, $m \in L(K)$. Then $(rm + N) = r(m + N) \in M/N$, hence $rm \in L(K)$. Then $u' : U' \to U, K \mapsto L(K)$ is well-defined. One easily verifies that u and u' are mutual inverses. □

Lemma 1.4.21. *Let R be a ring, M a left R-module, A a set and $\{M_\alpha\}_{\alpha \in A}$ a family of left R-submodules of M. Then*

$$s : \bigoplus_{\alpha \in A} M_\alpha \to M, (m_\alpha) \mapsto \sum_{\alpha \in A} m_\alpha$$

hence

$$\sum_{\alpha \in A} M_\alpha$$

is a left R-submodule.

Proof. Indeed for $r \in R$, $(m_\alpha) \in \bigoplus_{\alpha \in A} M_\alpha$,

$$s(r(m_\alpha)) = s((rm_\alpha)) = \sum_{\alpha \in A} rm_\alpha = \sum_{1}^{n} rm_{\alpha_i} = r \sum_{1}^{n} m_{\alpha_i} = r \sum_{\alpha \in A} m_\alpha = rs((m_\alpha)),$$

where $\alpha_1, \ldots, \alpha_n \in A$ are chosen suitably. □

Lemma 1.4.22. *Let R be a ring, $I, J \subset R$ be a left resp. right ideal, M a left/right R-module and $m \in M$. Then*

$$Im := \{rm : r \in I\} \;\&\; mJ := \{mr : r \in J\}$$

is a left resp. right R-submodule of M. Let $X \subset M$. Then

$$IX := \sum_{x \in X} Rx \;\&\; XJ := \sum_{x \in X} xR$$

is a left resp. right R-submodule of M.

Proof. Indeed for the first statement let $a, b \in I$ and $r \in R$. Then $ra \in I$, hence

$$r(am) = (ra)m \in Im.$$

Furthermore, since $a + b \in I$, hence

$$am + bm = (a + b)m \in Im.$$

The right case follows from J being a left $R^{(\mathrm{op})}$-module hence mJ is a left $R^{(\mathrm{op})}$-module, hence mJ is a right R-module. IX, XJ being left/right modules follows from the first statement and Lemma 1.4.21. □

Theorem 1.4.23. *Let $N \subset M$ be a left R-submodule of a left R-module M. Then the coequaliser of*

$$N \underset{0}{\overset{\iota}{\rightrightarrows}} M$$

is M/N.

Proof. Given any left R-module map $\alpha : M \to K$. The universal property of M/N in the category of groups tells that $\beta : M/N \to K, m + N \mapsto \alpha(m)$ is the unique group map such that $\alpha = \beta\pi$. It remains to verifiy that β is a left R-module map and this is trivial. □

Immediately we get the following corollaries:

Corollary 1.4.24. *Let $\alpha : M \to N$ be a left R-module homomorphism and $I \subset M$ $J \subset N$ be left R-submodules. Then*

$$\beta : M/I \to N/J$$
$$m + I \mapsto \alpha(m) + J$$

is a well-defined left R-module map if and only if $\alpha(I) \subset J$. If so β is the unique left R-module map such that the square

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_J} \\ M/I & \dashrightarrow & N/J \end{array}$$

commutes

Corollary 1.4.25. *The cokernel of a left R-module map $\alpha : M \to N$ is the coequaliser of the diagram*

$$M \underset{\alpha}{\overset{0}{\rightrightarrows}} N$$

Corollary 1.4.26. *(First Isomorhism Theorem for Modules)*
For any left R-module map $\alpha : M \to N$

$$\beta : M/\ker\ \alpha \to \text{im}\ \alpha$$
$$m + \ker\ \alpha \mapsto \alpha(m)$$

is an isomorphism.

Another result that one could note that is that

Lemma 1.4.27. *The kernel of a left R-module map $\alpha : M \to N$ is the equaliser of $\alpha, 0 : M \to N$.*

Definition 1.4.28. Let R be a ring and M a left R-module. Then M is *generated over by a set $X \subset M$ over R* if

$$M = \sum_{x \in X} Rx.$$

M is said to be *finitely generated over R* if there is a finite sequence $m_1, \ldots, m_n \in M$ such that ${}_RM = \sum_1^n Rm_i$ resp. $M_R = \sum m_i R$.

Definition 1.4.29. Let A be a set, R a ring and $\{M_\alpha\}_{\alpha \in A}$ a family of left R-modules. We say that $\sum_{\alpha \in A} M_\alpha$ is *direct*, if for every $\beta \in A$,

$$M_\beta \cap \sum_{\alpha \in A \setminus \{\beta\}} M_\alpha = 0.$$

Lemma 1.4.30. *Let A be a set, R a ring and $\{M_\alpha\}_{\alpha\in A}$ a family of left R-modules such that $\sum_{\alpha\in A} M_\alpha$ is direct. Then*

$$\sum_{\alpha\in A} M_\alpha \simeq \bigoplus_{\alpha\in A} M_\alpha.$$

Proof. We define the map

$$\begin{aligned}\rho : \bigoplus_{\alpha\in A} M_\alpha &\to \sum_{\alpha\in A} M_\alpha \\ (m_\alpha) &\mapsto \sum_{\alpha\in A} m_\alpha,\end{aligned}$$

where $\sum_{\alpha\in A} m_\alpha$ is defined to be the sum of non-zero entries of (m_α). Let $\sum_1^n m_{\alpha_i}$, where $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in A$. One easily finds that this is a module homomorphism. For $\alpha \in A$, we then define $m_\alpha = m_{\alpha_i}$ if $\alpha = \alpha_i$ for some i and $m_\alpha = 0$ if not. Then Clearly

$$\sum_1^n m_{\alpha_i} = \sum_{\alpha\in A} m_\alpha = \rho((m_\alpha)),$$

which means ρ is surjective. Suppose $(m_\alpha) \in \ker\ \rho$. Then

$$0 = \rho((m_\alpha)) = \sum_{\alpha\in A} m_\alpha = \sum_1^n m_{\alpha_1},$$

for some distinct $\alpha_1, \ldots, \alpha_n \in A$. Let $j \in \{1, \ldots, n\}$. Then

$$-m_{\alpha_j} = \sum_{i\in\{1,\ldots,n\}\setminus\{j\}} m_{\alpha_i} \in \sum_{\alpha\in A\setminus\{\alpha_j\}} M_\alpha.$$

This implies $m_{\alpha_j} \in M_{\alpha_j} \cap \sum_{\alpha\in A\setminus\{\alpha_j\}} M_\alpha = 0$, hence $m_{\alpha_j} = 0$, which means $m_\alpha = 0$ for each $\alpha \in A$ and so $(m_\alpha) = 0$. By the 1st Isomorphism Theorem for modules it follows that $\sum_{\alpha\in A} M_\alpha \simeq \bigoplus_{\alpha\in A} M_\alpha$. □

Definition 1.4.31. Let R be a ring and M a left R-module. A subset $X \subset M$ is said to be *left linearly independent over R* (or if R is commutative just *linearly independent*), if for every finite sequence of distinct $m_1, \ldots, m_n \in X$ and every finite sequence $r_1, \ldots, r_n \in R$,

$$\sum_1^n r_i m_i = 0 \iff r_i = 0\ \forall i \in \{1, \ldots, m\}$$

Remark 1.4.32. One should note that $0 \notin X$, since $1 \cdot 0 = 0$ and $1 \neq 0$.

Proposition 1.4.33. *Let R be a ring, M a left R-module and $X \subset M$ a subset. Then if X is left linearly independent over R, $\sum_{x\in X} Rx$ is direct.*

Proof. When X is empty the statement is trivial, hence suppose $X \neq \emptyset$.
Let $y \in X$ and let $m \in Ry \cap \sum_{x \in X \setminus \{y\}} Rx$. Then

$$r_{n+1}y = m = \sum_{1}^{n} r_i x_i,$$

for suitable $x_1, \ldots, x_n \in X \setminus \{y\}$ and $r_1, \ldots, r_{n+1} \in R$. Thus, we have that

$$r_{n+1}y + \sum_{1}^{n} r_i x_i = 0 \implies 0 = r_1 = r_2 = \cdots = r_{n+1} \implies m = 0.$$

We then conclude that $Ry \cap \sum_{x \in X \setminus \{y\}} Rx = 0$. □

Definition 1.4.34. Let R be a ring, M a left R-module. A subset $X \subset M$ is called a *basis of M over R* if X is linearly independent over R and $M = RX$ respectively $M = XR$. If X is finite and a basis of M over R it is called a *finite basis*.

Lemma 1.4.35. *Consider a ring R and a non-empty set X. Then*

$$\bigoplus_{x \in X} R$$

has basis

$$D := \left\{ (\delta_{yx}) \in \bigoplus_{x \in X} R \;\middle|\; x \in X \right\}$$

where

$$\delta_{yx} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

as an (R, R)-bimodule.

Proof. Let $(r_x) \in \bigoplus_x R$. Then for each $x \in X$,

$$(r_x)(x) = r_x = \sum_{y \in \text{supp } (r_x)} r_y \delta_{yx} = \left(\sum_{y \in \text{supp } (r_x)} r_x(\delta_{zy})(x) \right)$$
$$= \left(\sum_{y \in \text{supp } (r_x)} r_x(\delta_{zy}) \right)(x),$$

hence

$$(r_x) = \sum_{x \in X} r_x(\delta_{yx}) \in \sum_{x \in X} R(\delta_{yx})$$

Therefore D generates $\bigoplus_{x \in X} R$. Suppose

$$\sum_{1}^{m} a_i(\delta_{yx_i}) = \sum_{1}^{m} b_i(\delta_{yx_i}).$$

Then for each j,

$$a_j = \left(\sum_1^m a_i(\delta_{yx_i})\right)(x_j) = \left(\sum_1^m b_i(\delta_{yx_i})\right)(x_j) = b_j.$$

So D is linearly independent over R. □

The following gives an argument for why we get to define a module map by only defining a map on a basis.

Lemma 1.4.36. *Let M and N be left R-modules with a left basis $\mathcal{B}$ of M over R. Consider a function $\rho : \mathcal{B} \to N$. Then there is a unique extension of ρ to a left R-module map $\rho : M \to N$.*

Proof. We define

$$\rho : \sum_{v\in\mathcal{B}} a_v v = M \to N$$
$$\sum_v a_v v \mapsto \sum_v a_v \rho(v)$$

Note that the left linear independence of $\mathcal{B}$ over R, ensures that the representation of v in terms of $\mathcal{B}$ is unique, so this is well-defined. Let $a, b \in R$ and $\sum_v a_v v, \sum_v b_v v \in M$. Then

$$\begin{aligned}\rho\left(\left[a\sum_v a_v\right] + \left[b\sum_v b_v v\right]\right) &= \rho\left(\sum_v (aa_v + bb_v)v\right)\\ &= \sum_v (aa_v + bb_v)\rho(v)\\ &= a\sum_v a_v\rho(v) + b\sum_v b_v\rho(v)\\ &= a\rho\left(\sum_v a_v v\right) + b\rho\left(\sum_v b_v v\right).\end{aligned}$$

so this is indeed a linear map. Suppose $\rho' : M \to N$ is a linear map that restricts to $\rho : \mathcal{B} \to N$. Then

$$\rho'\left(\sum_v a_v v\right) = \sum_v a_v \rho'(v) = \sum_v a_v \rho(v) = \rho\left(\sum_v a_v v\right),$$

which proves the uniqueness of ρ. □

We denote category of left R-modules with finite left bases by

$$\mathrm{mod}_R^{\mathrm{fb}}$$

Lemma 1.4.37. *Fix a ring R and consider the set*

$$\{R^n : n \in \mathbb{N}\}$$

This together with left R-linear maps restricted to this set, define the skeleton of $\mathrm{mod}_R^{\mathrm{fb}}$.

Proof. Let M be a left R-module with basis $\mathcal{V} := \{v_1, \ldots, v_n\}$. Consider the left R-linear map

$$R^n \to M$$
$$(a_1, \ldots, a_n) \mapsto \sum_1^n a_i v_i$$

This is surjective since $\mathcal{V}$ generates M over R and injective since $\mathcal{V}$ is left linearly independent over R. □

Proposition 1.4.38. *Let S be a ring and $R \subset S$ a subring. Consider $(M, \cdot, +)$, a left S-module, then $rm := r \cdot m$, for $r \in R$, defines a structure of left R-modules. If in addition Q is a subring of a ring T and M is an (S, T)-bimodule, then M is an (R, Q)-bimodule.*

Proof. Let $m_1, m_2 \in M$, $r_1, r_2 \in R$. Then

1. $r_1(m_1 + m_2) = r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2 = r_1 m_1 + r_1 m_2$,
2. $(r_1 + r_2)m_1 = (r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1 = r_1 m_1 + r_2 m_1$,
3. $(r_1 r_2)m_1 = (r_1 r_2) \cdot m_1 = r_1(r_2 \cdot m_1) = r_1(r_2 m_1)$,
4. $1m_1 = 1 \cdot m_1 = m_1$.

Let $r \in R$, $q \in Q$, $m \in M$. Then

$$r(mq) = r \cdot (m \cdot q) = (r \cdot m) \cdot q = (rm)q.$$

□

1.4.2 Ideals

Definition 1.4.39. Recall that a *left/right ideal* in a ring R, is a left/right R-submodule of R. If it is an (R, R)-module it is called a *both-sided ideal*. If R is a commutative a left/right ideal is simply refered to as an ideal.

Definition 1.4.40. Let $\sigma : R \to S$ be a ring homomorphism. When we refer to the kernel of σ, we refer to the kernel of σ when seen as a group homomorphism between $(R, +)$ and $(S, +)$, i.e. $\ker \sigma := \sigma^{-1}(0)$.

Lemma 1.4.41. *Let $\sigma : R \to S$ be a ring homomorphism and $I \subset S$ be a left ideal. Then $\sigma^{-1}(I) \subset R$ is a left ideal.*

Proof. By Lemma 1.2.15 it follows that $\sigma^{-1}(I) \subset R$ is an additive subgroup. Let $r \in R$ and $a \in \sigma^{-1}(I)$. Then

$$\sigma(ra) = \sigma(r)\sigma(a) \in I,$$

hence $ra \in \sigma^{-1}(I)$. □

Corollary 1.4.42. *The kernel of a ring homomorphism $\sigma : R \to S$ is an ideal in R*

Lemma 1.4.43. *Let $\sigma : R \to S$ be a surjective ring homomorphism and $I \subset R$ be a left ideal. Then $\sigma(I) \subset S$ is a left ideal.*

Proof. By Lemma 1.2.15 $\sigma(I)$ is an additive subgroup of S. Let $s \in S$ and $\sigma(a) \in \sigma(I)$. Then for some $r \in R$, $s = \sigma(r)$. It follows that

$$s\sigma(a) = \sigma(r)\sigma(a) = \sigma(ra) \in \sigma(I).$$

□

Remark 1.4.44. We call RX and XR *the left ideal generated by X. The ideal generated by X* is the ideal $\langle X \rangle := R(XR) = (RX)R$.
Suppose R is commutative. For $M \subset R$, one can easily check that $RM = MR$, and hence the left/right ideal generated by M over R is a two-sided ideal. Thus $\langle M \rangle = RM = MR$.

Example 1.4.45. One may note that quite clearly $R = R1 = 1R$ and hence that $R = \langle 1 \rangle$.

Lemma 1.4.46. *Let R be a ring and $I \subset R$ a left ideal. Then*

$$1 \in I \iff I = R.$$

Proof. We need only work with the assumption that I is a left ideal.
" $\Longrightarrow$ ": Let $r \in R$. Then $r = r1 \in I$, hence $R \subset I \implies R = I$.
" $\Longleftarrow$ ": This is trivial, since $1 \in R = I$. □

Definition 1.4.47. An ideal in a ring generated by only one element is called a *principal ideal*. A ring in which every ideal is principal is called a *principal ideal domain* or a *PID*

An example of a PID is $\mathbb{Z}$. Indeed $(\mathbb{Z}, +)$ is a cyclic group generated by 1, thus every subgroup of the form $\langle n \rangle = n\mathbb{Z}$ for some $n \geq 0$, in particular every ideal is of this form.

Proposition 1.4.48. *Let S be a ring and $R \subset S$ a subring. Consider a left ideal $I \subset S$. Then $I \cap R \subset R$ is an ideal.*

Proof. R and I are both left R-submodules of S, hence so is $R \cap I$. □

Lemma 1.4.49. *Let R be a commutative ring and $I, J \subset R$ ideals. Then*

$$IJ \subset I \cap J.$$

Proof. Let $ij \in IJ$. Then since $j \in J$, $ij \in J$ and since $i \in I$, $ij \in I$, hence $ij \in I \cap J$. □

1.4.3 Quotient Rings

Proposition 1.4.50. *Let R be a ring and $I \subset R$ a two-sided ideal. View I as a subgroup of the additive group $(R, +)$. Then $(R/I, +)$ is an additive group by Corollary 1.2.40. Define*

$$\cdot : R/I \times R/I \to R/I$$

by $(r + I)(r' + I) := (rr' + I)$. This is a well-defined operation and $(R/I, +, \cdot)$ is a ring. It is also commutative if R is commutative.

Proof. Let $(r_1 + I, r_2 + I) = (r_1' + I, r_2' + I) \in R/I \times R/I$. Then

$$r_1 r_2 - r_1' r_2' = r_1 r_2 - r_1 r_2' + r_1 r_2' - r_1' r_2' = r_1(r_2 - r_2') + (r_1 - r_1') r_2' \in I,$$

hence $r_1 r_2 + I = r_1' r_2' + I$ and it follows that $\cdot$ is well-defined. Let $r_1 + I, r_2 + I, r_3 + I \in R/I$. Then

$$\begin{aligned}(r_1 + I)((r_2 + I)(r_3 + I)) &= (r_1 + I)(r_2 r_3 + I) = (r_1(r_2 r_3)) + I = (r_1 r_2) r_3 + I \\ &= (r_1 r_2 + I)(r_3 + I) = ((r_1 + I)(r_2 + I))(r_3 + I).\end{aligned}$$

We also have that

$$(1 + I)(r + I) = (1r + I) = (r1 + I) = r + I.$$

Furthermore

$$\begin{aligned}(r_1 + I)((r_2 + I) + (r_3 + I)) &= (r_1 + I)((r_2 + r_3) + I) = (r_1(r_2 + r_3)) + I = (r_1 r_2 + r_1 r_3) + I \\ &= (r_1 r_2 + I) + (r_1 r_3 + I) = (r_1 + I)(r_2 + I) + (r_1 + I)(r_3 + I).\end{aligned}$$

Suppose R is commutative. Then

$$(r_1 + I)(r_2 + I) = (r_1 r_2) + I = (r_2 r_1) + I = (r_2 + I)(r_1 + I).$$

□

Corollary 1.4.51. *The canonical surjective group map $\pi : R \mapsto R/I$ is a ring map.*

Proof. Let $r_1, r_2 \in R$. Then $\pi(r_1 r_2) = r_1 r_2 + I = (r_1 + I)(r_2 + I) = \pi(r_1)\pi(r_2)$. Lastly $\pi(1) = 1 + I$. □

Lemma 1.4.52. *Consider a ring map $\sigma : R \to S$. $(S, +)$ is an (R, R)-bimodule with the scalar multiplication*

$$rs := \sigma(r)s \quad (r \in R, s \in S)$$

Proof. This follows from S being an $(\text{im } \sigma, \text{im } \sigma)$-bimodule. □

Theorem 1.4.53. *Let R be a ring and I a left ideal of R. Then R/I is the coequaliser of the left R-module maps $0, \iota : I \to R$. Given a ring map $\sigma : R \to S$, considered as a (R, R)-bimodule map in the canonical way (cf. the prior lemma), the (R, R)-bimodule map*

$$\tau : R/I \to S$$
$$r + I \mapsto \sigma(r)$$

is the unique map such that $\tau\pi = \sigma$. The map τ is then the unique ring map with this property.

Proof. This follows readily from σ being a ring map □

Corollary 1.4.54. *(First Isomorphism Theorem of Rings)*
For a ring map $\sigma : R \to S$, the ring map

$$\tau : R/\ker\ \sigma \to \text{im } \sigma$$

is an isomorphism of rings.

1.4.4 Noetherian and Artinian Modules

Definition 1.4.55. Let M be a left R-module and

$$\mathcal{M} := \left\{ N \in 2^M : N \text{ is a left/right submodule of } M \right\}$$

be a chain. We say that M is *left noetherian* if every ascending chain stabilizes and *left artinian* if every descending chain stabilizes. A ring S is left noetherian/artinian, if it is noetherian/artinian as a left S-module or simply noetherian/artinian if is both left and right noetherian/artinian.

Definition 1.4.56. A simple left/right R-module is one whose only submodules are 0 and M.

Example 1.4.57. Any simple left R-module M is left noetherian/artinian. A family of submodules of M is of the form $\{0, M\}, \{M\}$ or $\{0\}$. These are all finite sets, hence any chain C is finite and thus has an upper/lower bound in C.

Assuming the axiom of choice we get the following result.

Proposition 1.4.58. *If M is a left R-module and X is a non-empty set of submodules of M, and M is noetherian/artinian then X has maximal/minimal element*

Proof. If M is noetherian/artinian, then every chain C in X has an upper/lower bound in C by Lemma C.2.31. Using Zorn's Lemma, this implies that X has a maximal/minimal element. □

Theorem 1.4.59. *Let M be a left/right R module. Then M is left/right noetherian if ad only if every submodule of M is finitely generated over R.*

Proof. " $\Longrightarrow$ ": Suppose M is not finitely generated. Let $m_1 \in M$. Then $M_1 := Rm_1 \subsetneq M$ since M is not finitely generated. We then recursively define $M_{n+1} = Rm_{n+1} + M_n$ where $m_{n+1} \in M \setminus M_n$ which we can do since every M_n is finitely generated and thus by assumption a proper submodule of M. We thus obtain an strictly ascending chain of submodules

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \dots$$

Hence this is an ascending chain that does not stabilize.
" $\Longleftarrow$ ": Suppose every submodule of M is finitely generated over R. Let an ascending chain of submodules, say

$$M_1 \subset M_2 \subset \dots$$

be given. Then by Lemma 1.6.16 $N := \bigcup_1^\infty M_n$ is a submodule. By assumption $N = \sum_1^m Rn_k$ for some $n_1, \dots, n_m \in N$. For each $k \in \{1, \dots, m\}$ there is a $j(k) \geq 0$ such that $n_k \in M_{j(k)}$. Let $p = \max\{j(1), \dots, j(k)\}$. Thus we have that $n_1, \dots, n_m \in M_p$. Hence, $M_p = \sum_1^m Rn_k = N$. Hence for $d \geq 0$, $N = M_p \subset M_{p+d}$, implying $M_p = M_{p+d}$. In other words, every ascending chain of submodules of M stabilizes. □

Corollary 1.4.60. *A PID is a Noetherian ring.*

Lemma 1.4.61. *Let a left/right R-module M and a submodule $N \subset M$ be given. Then M is left/right noetherian/artinian if and only if N and M/N is left/right noetherian/artinian.*

Proof. We prove the left noetherian version.
" $\Longrightarrow$ ": An ascending chain of submodules of N, is in particular an ascending chain of submodules of M, which by assumption stabilizes. A chain of submodules in M/N is of the form

$$L_1/N \subset L_2/N \subset L_3/N \subset \dots$$

where $L_i \subset M$ is a submodule of M containing N by Lemma 1.4.20. For some $n \geq 0$, $L_n = L_{n+d}$ for every $d \geq 0$, hence $L_n/N = L_{n+d}/N$ for every $d \geq 0$.
" $\Longleftarrow$ ": Let $M_1 \subset M_2 \subset M_3 \subset \dots$. Then

$$M_1 + N \subset M_2 + N \subset \dots$$

is an ascending chain of submodules of M containing N. Hence

$$(M_1 + N)/N \subset (M_2 + N)/N \subset \dots$$

is an ascending chain of submodules of M/N. Hence for some $n \geq 1$, $M_n + N = M_{n+d} + N$ for every $d \geq 0$. Then by Lemma 1.4.20 $M_n + N = M_{n+d} + N$ for every $d \geq 0$. We also have that

$$M_1 \cap N \subset M_2 \cap N \subset \dots$$

is an ascending chain of submodules of N. Thus for some $m \geq 1$, $M_m \cap N = M_{m+d} \cap N$ for every $d \geq 0$. Put $k = n + m$ and let $d \geq 0$ be given. Let $x \in M_{k+d}$. Then $x \in M_{k+d} + N = M_k + N$. Hence $x = y + z$ for some $y \in M_k$ and $z \in N$. It thus follows that $z = x - y \in N \cap M_{k+d} = N \cap M_k$. In particular, $z \in M_k$, hence $x = z + y \in M_k$, hence $M_k = M_{k+d}$. Thus $M_1 \subset M_2 \subset \dots$ stabilizes
We proceed to prove the left artinian case. " $\Longrightarrow$ " is dual to the noetherian case.
" $\Longleftarrow$ ": Consider a descending chain of submodules of M,

$$M_1 \supset M_2 \supset \dots$$

Similarly as above $M_1+N \supset M_2+N \supset \dots$ and $M_1 \cap N \supset M_2 \cap N \supset \dots$ give descending chain stabilizing some positive n respectively m. Put $k = n + m$ and let $d \geq 0$. Consider $x \in M_k$. Then in particular $x \in M_k + N = M_{k+d} + N$. Thus $x = y + z$ for some $y \in M_{k+d}$ and $z \in N$. Then $z = x - y \in M_k \cap N = M_{k+d} \cap N$, hence $x = z + y \in M_{k+d}$, hence $M_k = M_{k+d}$, meaning $M_1 \subset M_2 \subset \dots$ stabilizes. □

Definition 1.4.62. Consider a left R-module M. A *maximal left R-submodule of M* is a maximal element of the family of all proper left R-submodules

1.4.5 A First Look at Algebras over Rings

Definition 1.4.63. By a *ring extension S over R* we mean a ring S containing a ring R as a subring. Such a pair is denoted $S \supset R$. Such a pair is a *field extension* if S is a field and R is a subfield.

Remark 1.4.64. This defines a partial order. Indeed any ring is a ring extension of itself. If $S \supset R$ and $R \supset S$ then $R = S$. Lastly, if $T \supset S$ and $S \supset R$ are ring extensions, then $T \supset R$ and R is a subring of T.

Definition 1.4.65. Let R be a ring. We define *the center of* R to be the set

$$Z(R) := \{x \in R : xy = yx \text{ for every } y \in R\}$$

Remark 1.4.66. The center R is a subring of R. Indeed, let $x, x' \in Z(R)$. Then given $y \in R$,

$$y(x + x') = yx + yx' = xy + x'y = (x + x')y \implies x + x' \in Z(R).$$

We also have that $y0 = 0 = 0y$ for every $y \in R$, hence $0 \in Z(R)$. In addition,

$$y(-x) = -(yx) = -xy \implies -x \in Z(R).$$

This means $Z(R)$ is a subgroup of R. Furthermore,

$$y(xx') = (yx)x' = (xy)x' = x(yx') = x(x'y) = (xx')y \implies xx' \in Z(R)$$

and $y1 = y = 1y$, hence $1 \in Z(R)$. We thus see that $Z(R)$ is the largest commutative subring of R

Definition 1.4.67. Let R be a commutative ring. A ring A is an R-algebra, if $(A, +)$ is an R-module such that

$$r(a_1a_2) = (ra_1)a_2$$

for every $r \in R$, $a_1, a_2 \in A$.

Remark 1.4.68. An equivalent definition is that an algebra is a ring A together with ring homomorphism $R \to A$ whose image is contained in $Z(A)$.

Definition 1.4.69. Let A be an R-algebra. A subset B of A is *an R-subalgebra of A* if it is a subring of A and an R-submodule of A.

Remark 1.4.70. One easily checks that indeed an R-subalgebra is itself an R-algebra.

Definition 1.4.71. Let S be an algebra over commutative ring R and $s_1, \ldots, s_n \in Z(S)$. We define the *algebra over R generated by $s_1, \ldots, s_n$* to be the set

$$R[s_1, \ldots, s_n] := \left\{ \sum_{v=(v_1,\ldots,v_n)\in\mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n} : (a_v) \in \bigoplus_{v\in\mathbb{N}^n} R \right\}.$$

If $S \supset R$, then the above set is the *ring extension over R generated by $s_1 \ldots, s_n$*

Lemma 1.4.72. *Let S be an algebra over a commutative ring R and $s_1, \ldots, s_n \in Z(S)$. $R[s_1, \ldots, s_n]$ is an R-subalgebra of S. Furthermore, $R[s_1, \ldots, s_n]$ is the smallest subalgebra over R containing $s_1, \ldots, s_n$.*

Proof. Clearly $1, 0 \in R[s_1, \ldots, s_n]$. Let $\sum_{v\in\mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n}, \sum_{v\in\mathbb{N}^n} b_v s_1^{v_1} \cdots s_n^{v_n} \in R[s_1, \ldots, s_n]$. Then

$$\begin{aligned}\sum_{v\in\mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n} + \sum_{v\in\mathbb{N}^n} b_v s_1^{v_1} \cdots s_n^{v_n} &= \sum_{v\in\mathbb{N}^n} (a_v s_1^{v_1} \cdots s_n^{v_n} + b_v s_1^{v_1} \cdots s_n^{v_n}) \\ &= \sum_{v\in\mathbb{N}^n} (a_v + b_v) s_1^{v_1} \cdots s_n^{v_n} \in R[s_1, \ldots, s_n].\end{aligned}$$

and

$$\begin{aligned}\left(\sum_{v\in\mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n}\right)\left(\sum_{w\in\mathbb{N}^n} b_w s_1^{w_1} \cdots s_n^{w_n}\right) &= \sum_{v\in\mathbb{N}^n}\sum_{w\in\mathbb{N}^n} a_v b_w s_1^{v_1+w_1} \cdots s_n^{v_n+w_n} \\ &= \sum_{u\in\mathbb{N}^n} \left(\sum_{v,w\in\mathbb{N}^n : v+w=u} a_v b_w\right) s_1^{u_1} \cdots s_n^{u_n} \in R[s_1, \ldots, s_n]\end{aligned}$$

Let $r \in R$. Then

$$r \sum_{v\in\mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n} = \sum_{v\in\mathbb{N}^n} (r a_v) s_1^{v_1} \cdots s_n^{v_n} \in R[s_1, \ldots, s_n].$$

□

Lemma 1.4.73. *Let $S \supset R$ be a ring extension of commutative rings and $s_1, \ldots, s_n, t_1, \ldots, t_m \in S$. Then $R[s_1, \ldots, s_n, t_1, \ldots, t_m] = R[s_1, \ldots, s_n][t_1, \ldots, t_m]$.*

Proof. We already have that $R[s_1, \ldots, s_n, t_1, \ldots, t_m] \supset R[s_1, \ldots, s_n]$, hence $R[s_1, \ldots, s_n, t_1, \ldots, t_m] \supset R[s_1, \ldots, s_n][t_1, \ldots, t_m]$. Let $\sum_{(v,w)\in\mathbb{N}^{n+m}} a_v s_1^{v_1} \cdots s_n^{v_n} t_1^{w_1} \cdots t_m^{w_m} \in R[s_1, \ldots, s_n, t_1, \ldots, t_m]$. We then see that

$$\sum_{(v,w)\in\mathbb{N}^{n+m}} a_{vw} s_1^{v_1} \cdots s_n^{v_n} t_1^{w_1} \cdots t_m^{w_m} = \sum_{w\in\mathbb{N}^m} \left[\sum_{v\in\mathbb{N}^n} a_{vw} s_1^{v_1} \cdots s_n^{v_n}\right] t_1^{w_1} \cdots t_m^{w_m} \in R[s_1, \ldots, s_n][t_1, \ldots, t_m].$$

□

Definition 1.4.74. Let R be a ring and consider R-algebras A, B. A map $\sigma : A \to B$ is an *R-algebra homomorphism* if it is both a ring homomorphism and an R-module homomorphism.

Lemma 1.4.75. *Let R be a ring, consider R-algebras A, B and a map $\sigma : A \to B$. Then σ is an R-algebra homomorphism if and only if σ is a multiplicative, additive map fixing R, i.e. for every $a_1, a_2 \in A$, $\sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2)$ and for every $r \in R$, $\sigma(r) = r$. Here $r := r1$.*

Proof. If σ is an R-algebra homomorphism, then it is by definition multiplicative. Moreover, since σ is an R-module map,

$$\sigma(r) = r\sigma(1) = r1.$$

If σ is a multiplicative, additive map fixing R, then

$$\sigma(ra) = \sigma(r)\sigma(a) = r\sigma(a).$$

□

1.5 Diagram Lemmas

1.5.1 Exact Sequences

Definition 1.5.1. *A sequence of left/right R-modules*, is a collection of pairs

$$\{(M_i, \rho_i) : i \in \mathbb{Z}, M_i \text{ is a left/right } R\text{-module }, \rho_i \in \text{Hom}(M_i, M_{i+1})\}.$$

A sequence is finite if $M_i = 0$ for every but finitely many i. A sequence is in general denoted

$$\cdots \xrightarrow{\rho_{i-1}} M_i \xrightarrow{\rho_i} M_{i+1} \xrightarrow{\rho_{i+1}} \cdots$$

When the maps are obvious we opt to not explicitly denote them. When a sequence is finite we of often opt to denote it

$$0 \longrightarrow M_s \longrightarrow M_{s+1} \longrightarrow \cdots \longrightarrow M_{b-1} \longrightarrow M_b \longrightarrow 0$$

Where s, b are respectively the largest and smallest index for which $M_s \neq 0$ and $M_b \neq 0$.

Definition 1.5.2. A finite sequence

$$M \xrightarrow{\rho} M' \xrightarrow{\rho'} M''$$

is said to be *exact (at M')* if im ρ = ker ρ'. In general a sequence

$$\cdots \xrightarrow{\rho_{i-1}} M_i \xrightarrow{\rho_i} M_{i+1} \xrightarrow{\rho_{i+1}} \cdots$$

is said to be *exact* if

$$M_{i-1} \xrightarrow{\rho_i} M_i \xrightarrow{\rho_i} M_{i+1}$$

is exact for each $i \in \mathbb{Z}$

Remark 1.5.3. Equivalently a sequence $M \xrightarrow{\rho} M' \xrightarrow{\rho'} M''$ is exact if $\rho' \circ \rho = 0$.

Lemma 1.5.4. *Consider a sequence*

$$0 \longrightarrow M \xrightarrow{\rho} M' \xrightarrow{\rho'} M'' \longrightarrow 0$$

The following are equivalent:

1. *The sequence is exact*
2. *ρ is injective and ρ' is surjective*

Proof. "1. $\implies$ 2.": by exactness ker ρ = im $0 = 0$ and im ρ' = ker $0 = M''$. "2. $\implies$ 1.": im $0 = 0$ = ker ρ and ker $0 = M''$ = im ρ' □

Corollary 1.5.5. *Given a left/right R-module map $\rho : M \to N$, the sequence*

$$0 \longrightarrow \text{ker } \rho \hookrightarrow M \twoheadrightarrow \text{im } \rho \longrightarrow 0$$

is exact.

1.5.2 Isomorphism Theorems

Proposition 1.5.6. *Let R be a subring of a ring S. Let $I \subset R$ be a left/right ideal. Then $\sigma : R/I \to S/IS, r + I \mapsto r + SI$ is a well defined ring map in the left case and so is $r + I \mapsto r + IS$ in the right case.*

Proof. Consider $\iota : R \hookrightarrow S, r \mapsto r$. Then since $\iota(I) \subset SI$, it follows that $\sigma : R/I \mapsto S/SI, r + I \mapsto \iota(r) + SI = r + SI$ is a well-defined ring map. □

Corollary 1.5.7. *Let R be a ring M a left R-module. Consider submodules $N, N' \subset M$ with $N' \subset N$. Then*

$$\frac{M/N'}{N/N'} \simeq M/N.$$

Proof. Follows from the 1st isomorphism theorem for modules: Consider the map

$$M/N' \to M/N$$
$$m + N' \mapsto m + N$$

This is a surjective left R-module map. Suppose $m + N = 0 + N$, then $m + N' \in N/N'$, hence the kernel of this map is N/N', so

$$\frac{M/N'}{N/N'} \simeq M/N.$$

□

Corollary 1.5.8. *Let R be a ring $I, J \subset R$ left/right ideals with $J \subset I$. Then*

$$\frac{R/J}{I/J} \simeq R/I.$$

Proof. Follows from the 1st isomorphism theorem for rings and the fact that

$$R/J \to R/I$$
$$r + J \mapsto r + I$$

is also a ring homomorphism. □

Lemma 1.5.9. *Consider ${}_RL \leq {}_RN \leq {}_RM$. Then*

$$0 \longrightarrow N/L \hookrightarrow M/L \twoheadrightarrow M/N \longrightarrow 0$$

is exact

Proof. This is an immediate consequence of Lemma 1.5.4 □

Theorem 1.5.10. *Let $N, L \leq M$. Then $N/(N \cap L) \simeq (N + L)/N$.*

Proof. We get the chain of modules $N \cap L \leq N \leq N + L$, hence we have an exact sequence

$$0 \longrightarrow N/(N \cap L) \xrightarrow{\rho} (N + L)/(N \cap L) \xrightarrow{\rho'} (N + L)/N \longrightarrow 0$$

Substituting $(N + L)/(N \cap L)$ for im ρ we preserve exactness, i.e.

$$0 \longrightarrow N/(N \cap L) \xrightarrow{\overline{\rho}} \text{im } \rho \xrightarrow{\rho'|_{\text{im } \rho}} (N + L)/N \longrightarrow 0$$

where $\overline{\rho}(n + N \cap L) = \rho(n + N \cap L)$. Note that for $n + N \cap L \in$ im ρ,

$$\rho' \,|_{\text{im } \rho}\, (n + N \cap L) = \rho'(n + N \cap L) = n + N = 0 \implies \rho' \,|_{\text{im } \rho} = 0.$$

It follows that im $\overline{\rho}$ = ker $\rho' \,|_{\text{im } \rho}$= ker $0 = (N+L)/N$, hence ρ is surjective. By exactness ρ is also injective hence $N/(N \cap L) \overset{\rho}{\simeq} (N + L)/N$. □

1.5.3 The Splitting Lemma

Lemma 1.5.11. *(The Splitting Lemma) Let R be a ring and A, B, C be left R-modules and*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

a short exact sequence. The following are equivalent,

1. *β is split epi.*
2. *α is split mono.*
3. *$\gamma : B \simeq A \oplus C$, $\beta\gamma = \pi_2$ and $\gamma^{-1}\alpha = \iota_1$.*

Proof. "3. $\implies$ 1. and 2.": There are maps

$$\iota : C \to B$$
$$x \mapsto \gamma((0, x))$$

and

$$\mu : B \to A$$
$$y \mapsto \pi_1(\gamma^{-1}(y))$$

Then

$$\beta\iota(x) = \beta\gamma(0, x) = \pi_1(0, x) = x = \mathrm{id}_C(x)$$

for each $x \in C$. We also have that

$$\mu\alpha(y) = \pi_1\gamma^{-1}\alpha(y) = \pi_1\iota_1(y) = y = \mathrm{id}_A(y)$$

for each $y \in A$.
"1. $\implies$ 3.": Suppose β is split epi. Then there is a ι which is a section of β. Note that if $b \in \ker\ \beta \cap \mathrm{im}\ \iota$, then $b = \iota(c)$ for some $c \in C$, hence $c = \beta(\iota(c)) = 0$, implying $b = 0$. Moreover

$$b = (b - \iota\beta(b)) + \iota\beta(b) \in \ker\ \beta + \mathrm{im}\ \iota$$

for each $b \in B$, hence $B = \ker\beta \oplus \mathrm{im}\ \iota$. Note that

$$A \simeq \mathrm{im}\ \alpha = \ker\ \beta$$

via α, since α is injective (since it is in particular a monomorphism) and that

$$C \simeq \mathrm{im}\ \iota$$

via ι since this is a section and therefore an injective map. We thus get that

$$B = \ker \beta \oplus \operatorname{im} \iota \simeq A \oplus C.$$

Here we use functiorality of the bifunctor $_ \oplus _ : \quad \text{mod}_R \times \quad \text{mod}_R \to \quad \text{mod}_R$.
By construction,

$$\beta\gamma(a, c) = \beta(\alpha(a) + \overline{\iota}(c)) = \beta\alpha(a) + \beta\iota(c) = c = \pi_2(a, c).$$

Secondly

$$\gamma^{-1}\alpha(a) = (\overline{\alpha}^{-1}\alpha(a), \beta\alpha(a)) = (a, 0) = \iota_1(a).$$

Here $\overline{\alpha}$ denotes α when seen as a function onto its image.
"2. $\implies$ 3.": this is dual to 1. $\implies$ 3. □

1.5.4 Free Modules

Lemma 1.5.12. *For a non-empty set X, the free additive group $R[X]$, where R is a ring, is an (R, R)-bimodule with generating set $\{\delta_{yx} : X \to R \mid x \in X\}$*

Proof. This is just a reformulation of Lemma 1.4.35. □

1.6 Ring theory

1.6.1 Matrix Rings

Definition 1.6.1. Let R be a ring and n, m be positive integers. We define *the set of $n \times m$ (n by m) matrices over R* to be the set

$$M_{n\times m}(R) = \prod_{i\in\{1,\dots,n\}, j\in\{1,\dots,m\}} R.$$

For an element $(a_{i,j}) \in M_{n\times m}(R)$ we define $(a_{ij}) := (a_{i,j})$, when no disambiguity arises from this notation. An element of $M_{n\times m}(R)$ is called an $(n \times m)$ *matrix*. We define $M_n(R) := M_{n\times n}(R)$.

Remark 1.6.2. Note that $M_{n\times m}(R)$ is an (R, R)-bimodule.

Example 1.6.3. Let $R = \mathbb{Z}$, $n = 2$ and $m = 3$ and consider $a_{11} = 1, a_{12} = 2, a_{13} = 3$ and $a_{21} = 2, a_{22} = 3, a_{23} = 2$. We opt to write the element (a_{ij}) as table with 2 rows and 3 columns, i.e.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix} := (a_{ij}).$$

For arbitrary rings positive integers n, m, we in general can write an element $(a_{ij}) \in M_{n\times m}$ as a table with n rows and m columns, i.e.

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} := (a_{ij})$$

Lemma 1.6.4. *Let R be a ring and n, m, l be positive integers. We define matrix multiplication to be the operation*

$$\cdot : M_{n\times m}(R) \times M_{m\times l}(R) \to M_{n\times l}(R)$$

defined by

$$(a_{ij})(b_{ij}) := \left(\sum_{k=1}^{m} a_{ik}b_{kj} \right).$$

Suppose we have an additional positive integer p and let $(a_{ij}) \in M_{n\times m}(R), (b_{ij}) \in M_{m\times l}(R), (c_{ij}) \in M_{l\times p}(R)$. Then

$$(a_{ij})\left((b_{ij})(c_{ij})\right) = \left((a_{ij})(b_{ij})\right)(c_{ij}).$$

Suppose $(a_{ij}) \in M_{n\times m}(R), (b_{ij}), (c_{ij}) \in M_{m\times l}(R)$. Then

$$(a_{ij})\left((b_{ij}) + (c_{ij})\right) = (a_{ij})(b_{ij}) + (a_{ij})(c_{ij})$$

Proof. Indeed for $(a_{ij}) \in M_{n\times m}(R), (b_{ij}) \in M_{m\times l}(R), (c_{ij}) \in M_{l\times p}(R)$,

$$\begin{aligned}(a_{ij})\left((b_{ij})(c_{ij})\right) &= (a_{ij})\left(\sum_{k=1}^{l} b_{yk}c_{kz}\right) = \left(\sum_{h=1}^{m} a_{xh}\sum_{k=1}^{l} b_{hk}c_{kz}\right) = \left(\sum_{h=1}^{m}\sum_{k=1}^{l} a_{xh}(b_{hk}c_{kz})\right) \\ &= \left(\sum_{k=1}^{l}\sum_{h=1}^{m}(a_{xh}b_{hk})c_{kz}\right) = \left(\sum_{k=1}^{l}\left(\sum_{h=1}^{m} a_{xh}b_{hk}\right)c_{kz}\right) = \\ &= \left(\sum_{h=1}^{m} a_{xh}b_{hk}\right)(c_{ij}) = \left((a_{ij})(b_{ij})\right)(c_{ij}) = \left((a_{ij})(b_{ij})\right)(c_{ij}).\end{aligned}$$

Furthermore, for $(a_{ij}) \in M_{n\times m}(R), (b_{ij}), (c_{ij}) \in M_{m\times l}(R)$,

$$\begin{aligned}(a_{ij})\left((b_{ij}) + (c_{ij})\right) = (a_{ij})(b_{ij} + c_{ij}) &= \left(\sum_{k=1}^{m} a_{ik}(b_{kj} + c_{kj})\right) = \left(\sum_{k=1}^{m} a_{ik}b_{kj} + a_{ik}c_{kj}\right) \\ &= \left(\sum_{k=1}^{m} a_{ik}b_{kj} + \sum_{k=1}^{m} a_{ik}c_{kj}\right) = \left(\sum_{k=1}^{m} a_{ik}b_{kj}\right) + \left(\sum_{k=1}^{m} a_{ik}c_{kj}\right) \\ &= (a_{ij})(b_{ij}) + (a_{ij})(c_{ij})\end{aligned}$$

□

Lemma 1.6.5. *Let R be a ring and n a positive integer. Then $(M_n(R), +, \cdot)$, where $\cdot$ is matrix multiplication of matrices in $M_n(R)$ and $M_n(R)$, is a ring called the $(n \times n)$ matrix ring (over R).*

Proof. $M_n(R)$ is trivially an additive group. Let $I := 1 := (\delta_{ij})$ (where $\delta_{ii} = 1$ for $i \in \{1, \ldots, n\}$ and $\delta_{ij} = 0$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$). Then for $(r_{ij}) \in M_n(R)$

$$1(r_{ij}) = \left(\sum_{k=1}^{n} \delta_{ik} r_{kj} \right) = \left(\delta_{ii} r_{ij} + \sum_{k \in \{1,\ldots,n\} \setminus \{i\}} \delta_{ik} r_{kj} \right) = (1 r_{ij} + 0) = (r_{ij}).$$

In a dual way one can prove that

$$(r_{ij})1 = (r_{ij}).$$

By Lemma 1.6.4 it follows that $(M_n(R), +, \cdot)$ is a ring. □

Example 1.6.6. A matrix ring is never a commutative ring: Take $(a_{ij}) \in M_n(R)$ where $a_{11} = 1$ and $a_{ij} = 0$ when $a_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Take $(b_{ij}) \in M_n(R)$ where $b_{1m} = 1$ and $b_{ij} = 0$ when $i \neq 1$ or $j \neq m$. Then it is easy to check that $(a_{ij})(b_{ij}) = (b_{ij})$ while $(b_{ij})(a_{ij}) = 0$.

Given a ring R and matrix $(a_{ij}) \in M_{n \times m}(R)$ induces a left R-linear map

$$(a_{ij}) : R^m \to R^n$$
$$v \mapsto (a_{ij})v$$

In general for left R-modules M and N with bases $\{m_1, \ldots, m_k\}$ respectively $\{n_1, \ldots, n_l\}$ we get a left R-linear map

$$(a_{ij}) : M \to N$$

defined by $m_i \mapsto \sum_{j=1}^{l} a_{ij} n_j$. This is the unique left R-module map making the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ R^k & \longrightarrow & R^l \end{array}$$

commute, where the vertical maps denote the canonical isomorphism induced by the respective bases. This leads to a canonical description of the morphisms of $\mathrm{Mod}_R^{\mathrm{fb}}$.

Lemma 1.6.7. *Let $\rho : M \to N$ be a left R-linear map of left R-modules with finite bases $\mathcal{M} = \{m_1, \ldots, m_k\}$ for M respectively $\mathcal{N} = \{n_1, \ldots n_l\}$. Then $\rho = (a_{ij})$ for some $(a_{ij}) \in M_{k \times l}(R)$.*

Proof. For each i there are unique $a_{i1}, \ldots, a_{il} \in R$ with

$$\rho(m_i) = \sum_{j=1}^{l} a_{ij} n_j$$

since an R-linear map is completely determined by its behavior on basis elements, it follows that $(a_{ij}) = \rho$ as left R-linear maps. □

1.6.2 Fields, Integral Domains & some Important Ideals

Definition 1.6.8. Let R be a ring. An element $r \in R$ is called a *unit* if there is an element $r' \in R$ such that $rr' = r'r = 1$. We denote the set of units of R by R^*.

Remark 1.6.9. I. Suppose there are two elements r_1, r_2 such that $rr_i = r_i r = 1$. Then

$$r(r_1 - r_2) = rr_1 - rr_2 = 1 - 1 = 0 \implies r_1 - r_2 = r_1 r(r_1 - r_2) = 0 \implies r_1 = r_2.$$

Hence an element satisfying $rr' = r'r = 1$ is unique. We denote it by r^{-1} and refer to it as the multiplicative inverse of r.

II. $(R^*, \cdot)$ is a group. We check that R^* is a submonoid of $(R, \cdot)$ and hence a monoid. Clearly $1 \in R^*$. Let $r, r' \in R^*$. Then

$$(rr')\left(r'^{-1}r^{-1}\right) = r\left(r'\left(r'^{-1}r^{-1}\right)\right) = r\left(\left(r'r'^{-1}\right)r^{-1}\right) = r\left(er^{-1}\right) = rr^{-1} = 1,$$

and similarly one can check that $\left(r'^{-1}r^{-1}\right)(rr') = 1$, hence $rr' \in R^*$ a. Let $r \in R^*$. Then $r^{-1}r = rr^{-1} = 1$, hence r^{-1} is a unit, hence R^* is a group.

III. If R is commutative it is sufficient to check that $rr' = 1$ to verify that r is a unit.

Definition 1.6.10. A commutative ring K where $K^* = K \setminus \{0\}$ is called a *field*. Removing the restriction of K being commutative, K is called a *division ring* or *skew field*.

Definition 1.6.11. An left ideal $M \subsetneq R$ is called a *left maximal* ideal, if for every left ideal $I \subset R$ with $M \subset I$, either $I = M$ or $I = R$. If M is an ideal with aforementioned property it is called a *maximal ideal*.

Definition 1.6.12. A ring R is called *simple* if the only two-sided ideals in R are the trivial ones, i.e. 0 and R.

Lemma 1.6.13. *Let S be a simple ring and R any ring. Consider a ring homomorphism $\sigma : S \to R$. Then σ is an embedding or R is the zero ring. In particular, the only field morphisms are embeddings.*

Proof. Note that ker σ is a two-sided ideal in S. Since S is simple, either ker $\sigma = 0$ or ker $\sigma = S$. In the first case σ is injective and in the second case $\sigma = 0$, and the only case where 0 can be a ring map is when $R = 0$. □

Lemma 1.6.14. *Any division ring is simple.*

Proof. Let D be a division ring and consider a non-zero ideal $I \subset R$. Then there is an $x \in I \setminus 0$. Since D is a division ring, there exists x^{-1} s.t. $1 = x^{-1}x \in I$, meaning $I = D$. □

Lemma 1.6.15. *Let R be a ring. Let $I \subset R$ be a two-sided ideal. Then I is a maximal ideal if and only if R/I is a simple.*

Proof. " $\implies$ ": Let $J/I \subset R/I$ be a non-zero two-sided ideal, i.e. assume $I \subsetneq J \subset R$ for some two-sided ideal J in R. Then $J = R$, hence $J/I = R/I$, implying R/I is simple. " $\impliedby$ ": Conversely, consider an ideal $J \subset R$ such that $I \subsetneq J$. Then $0 \neq J/I \subset R/I$, implying that $J/I = R/I$ and hence that $J = R$. □

Lemma 1.6.16. *Let M be a left R-module. Consider a chain C of submodules of M. Then*

$$\bigcup_{N \in C} N$$

is a submodule.

Proof. Let $m_1, m_2 \in \bigcup_{N \in C} N$. $m_1 \in N_1$ and $m_2 \in N_2$ for some $N_1, N_2 \in C$. WLOG $N_1 \subset N_2$, hence $m_1 \in N_2$, which means $m_1 + m_2 \in N_2 \subset \bigcup_{N \in C} N$. Clearly $0 \in \bigcup_{N \in C} N$. Let $r \in R$, clearly $rm_1 \in \bigcup_{N \in C} N$. □

Proposition 1.6.17. *Assuming choice, any proper left ideal in a non-zero ring is contained in left maximal ideal.*

Proof. Let R be a non-zero ring and $I \subset R$ a proper ideal. Consider the family

$$X := \{I \subset J \subset R : J \text{ is a proper ideal}\},$$

partially ordered by inclusion. Every chain in X has an upper bound, namely the union of the ideals in the chain (cf. the prior lemma). Indeed note that 1 is not an element of this union, making it a proper left ideal. By Zorn's lemma, X has a maximal element, and by definition such a maximal element is a maximal ideal in R containing I. □

Corollary 1.6.18. *Assuming choice, any non-zero ring has a maximal ideal.*

Proposition 1.6.19. *Let R be a commutative ring. Let $I \subset R$ be an ideal. Then I is a maximal ideal if and only if R/I is a field.*

Proof. " $\Longrightarrow$ ": We need to prove that every non-zero element of R/I is a unit. Consider $a + I \in R/I \setminus \{0 + I\}$, i.e. an element in R/I, where $a \notin I$. Since I is maximal we have that $I + Ra = R = R1$. This implies that we can find $b \in I$ and $r \in R$ such that $1 = b + ra$, hence

$$(r + I)(a + I) = (ra + I) + (b + I) = (ra + b) + I = 1 + I,$$

and since R/I is commutative, this implies $a + I$ is a unit.
" $\Longleftarrow$ ": Since R/I is a field, it is in particular a division ring. Then by Lemma 1.6.14 R/I is simple. Given any ideal $J \supsetneq I$, $J/I = R/I$, hence $J = R$, meaning I is maximal. □

Definition 1.6.20. Let R be a ring, $r \in R$. $d \in R$ is called a *left divisor of* r if $r \in Rd$ and a *right divisor of* r if $r \in dR$. If d is both a left and a right divisor of r, we write $d \mid r$. Hence if R is commutative, $d \mid r \iff r \in \langle d \rangle$.

Definition 1.6.21. Let R be a ring. An element $a \in R$ is called a *left/right zero divisor* if there is an element $r \in R \setminus 0$ such that $ar = 0$ respectively $ra = 0$. In a commutative ring an element is a left zero divisor if and only if it is a right zero divisor, hence we just call a left/right zero divisor in a commutative ring a *zero divisor*.

Definition 1.6.22. Let R be a ring. If the only left/right zero divisor of R is 0, then R is called a *left/right domain*. If R is a commutative ring and a domain it is called an *integral domain*.

Lemma 1.6.23. *Suppose $S \supset R$ is a ring extension where S is a left/right domain. Then so is R.*

Proof. Let $a \in R \setminus 0$, hence in particular in $S \setminus 0$ then for every $R \setminus 0$, $ar \neq 0$. □

Proposition 1.6.24. *A division ring D is a domain. Hence a field is an integral domain.*

Proof. Let $a \in D \setminus 0$. Then $\langle a \rangle = D$ since D is simple, hence $1 \in \langle a \rangle$, meaning there is some $b \in D$ such that $ba = ab = 1$. □

Lemma 1.6.25. *Let $x, y, a \in R$ where a is not a zero divisor. If $ax = ay$, then $x = y$. The same result can be proven if $xa = ya$. In particular, if R is a domain, then for $x, y \in R$, $a \in R \setminus 0$, $ax = ay$ xor $xa = ya$ implies $x = y$.*

Proof. We have that $a(x - y) = ax - ay = 0$ implies $x - y = 0$. The other result is dual. □

Definition 1.6.26. Let R be a commutative ring. An ideal $I \subsetneq R$ is said to be *prime* if for any $a, b \in R$ such that $ab \in I$, then $a \in I$ or $b \in I$

Lemma 1.6.27. *Let R be a commutative ring and $I \subset R$ and ideal. Then I is prime if and only if R/I is an integral domain*

Proof. " $\implies$ ": Let $a + I, b + I \in R/I$ be given such that $ab + I = 0 + I$. Then $ab \in I$, hence by assumption $a \in I$ or $b \in I$, meaning $a + I = 0$ or $b + I = 0$.
"$\impliedby$": Let $a, b \in R$ such that $ab \in I$. Then $ab + I = 0 + I$, hence by assumption $a + I = 0$ or $b + I = 0$, hence $a \in I$ or $b \in I$. □

Corollary 1.6.28. *A maximal ideal M in commutative ring R is prime.*

Proof. Indeed R/M is a field and a field is an integral domain, hence M is prime. □

Lemma 1.6.29. *Let $R \subset S$ be a commutative subring of S and $I \subset S$ be a two sided ideal. Then $R + I$ is a subring of R.*

Proof. Clearly $1 \in R + I$. Let $x, y \in R$ and $a, b \in I$, then

$$(x + a)(y + b) = \underbrace{xy}_{\in R} + \underbrace{xb + ay + ab}_{I} \in R + I.$$

□

Proposition 1.6.30. *Let $S \supset R$ be a commutative ring extension. Let $I \subset S$ be prime. Then $I \cap R \subset R$ is prime.*

Proof. By Proposition 1.6.27 S/I is an integral domain. Note that $S/I \supset (R + I)/I$. Hence from Lemma 1.6.23 $(R+I)/I$ is an integral domain. It follows from Theorem 1.5.10 that $(R + I)/I \simeq R/(I \cap R)$, hence $I \cap R$ is prime. □

Definition 1.6.31. Let R be a commutative ring. A non-unit and non-zero element $p \in R$ is called a *prime element* if for every $a, b \in R$, $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Lemma 1.6.32. *Let R be a commutative ring. Then for a non-zero $p \in R \setminus R^*$, p is prime if and only if $\langle p \rangle$ is prime.*

Proof. This is seen by the fact that $x \in \langle p \rangle$ is by definition equivalent to $p \mid x$. □

Definition 1.6.33. Consider a commutative ring R and an ideal $I \subset R$. We define the *radical* of I to be the set

$$\text{rad}(I) = \{r \in R : r^n \in I \text{ for some } n > 0\}.$$

An ideal with the property that $I = \text{rad}(I)$ is called a *radical ideal*.

Remark 1.6.34. Trivially we have that if $a \in I$, then $a = a^1 \in I$, hence $a \in \text{rad}(I)$. In other words, $I \subset \text{rad}(I)$.

Lemma 1.6.35. *The radical of an ideal $I \subset R$ is an ideal in R.*

Proof. Let $a, b \in \text{rad}(I)$ and $r \in R$. Clearly $0^1 = 0 \in I$, hence $0 \in \text{rad}(I)$. For some $n, m > 0$, $a^n, b^m \in I$. Thus, we also have that

$$(a+b)^{n+m} = \sum_{0}^{n+m} \binom{n+m}{k} a^{n+m-k} b^k, \tag{1}$$

For $k \in \{0, \ldots, m\}$, $n + m - k \geq n$, implying $a^{n+m-k} \in I$. For $k \in \{m+1, \ldots, n\}$, $b^k \in I$. Then using (1), it follows that $(a+b)^{n+m} \in I$ and hence that $a+b \in \text{rad}(I)$. Finally we also have that

$$(ra)^n = r^n a^n \in I \implies ra \in \text{rad}(I).$$

□

Lemma 1.6.36. *Let $I \subsetneq R$ be a prime ideal. Then I is a radical ideal.*

Proof. Let $a \in I$ and $n > 0$. We prove by induction in n that if $a^n \in I$ then $a \in I$. For $n = 1$, $a = a^1 \in I$. Suppose $a^{n+1} \in I$. Then $aa^n \in I$. Using that I is prime we get that $a \in I$ or $a^n \in I$. If we land in the first case, we are done. In the second case it follows by induction that $a \in I$. From the above it follows that if $a \in \text{rad}(I)$, then $a \in I$. Hence it follows from Remark 1.6.34 that $I = \text{rad}(I)$. □

Lemma 1.6.37. *Consider ideals $I, J \subset R$. Then*

$$\text{rad}(I \cap J) = \text{rad}(I) \cap \text{rad}(J) = \text{rad}(IJ)$$

Proof. For the first equality, note that

$$\begin{aligned} a \in \text{rad}(I \cap J) &\iff \exists n \geq 0 (a^n \in I \cap J) \iff \exists n \geq 0 (a^n \in I) \wedge \exists n \geq 0 (a^n \in J) \\ &\iff a \in \text{rad}(I) \cap \text{rad}(J). \end{aligned}$$

If $a \in \text{rad}(I) \cap \text{rad}(J)$, then $a^n \in I$ for some n and $a^m \in J$ for some m. Then $a^{n+m} \in IJ$, hence $a \in \text{rad}(IJ)$.
If $a \in \text{rad}(IJ)$ then $a^n = xy$ for some $x \in I$ and $y \in J$. In particular, a^n is in I and in J, hence $a \in \text{rad}(I) \cap \text{rad}(J)$. □

Lemma 1.6.38. *Let $I \subset R$ be an ideal. Then* $\text{rad}(I)$ *is the intersection of all prime ideals containing I.*

Proof. Note that

$$\bigcap_{\mathfrak{p} \text{ prime}, \mathfrak{p} \supset I} \mathfrak{p},$$

is a radical ideal, hence

$$\operatorname{rad}(I) \subset \operatorname{rad}\left(\bigcap_{\mathfrak{p} \text{ prime}, \mathfrak{p} \supset I} \mathfrak{p} \right) = \bigcap_{\mathfrak{p} \text{ prime}, \mathfrak{p} \supset I} \mathfrak{p}.$$

Suppose $a \notin \operatorname{rad}(I)$. Consider the family

$$X := \{J \subset R : J \supset I \text{ is an ideal}, a^n \notin J \text{ for all } n \geq 0\}.$$

This is a non-empty family of sets, since it contains I and it is partially ordered by inclusion. By Lemma 1.6.16 every chain in X has an upper bound, hence it contains a maximal element $\mathfrak{p} \supset I$ by Zorn's Lemma. Take a pair of elements $x, y \in R$ such that $x, y \notin \mathfrak{p}$. Then by maximality $\langle x \rangle + \mathfrak{p}, \langle y \rangle + \mathfrak{p} \notin X$, hence $a \in \operatorname{rad}(\langle x \rangle + \mathfrak{p}), \operatorname{rad}(\langle y \rangle + \mathfrak{p})$, hence a is in the intersection of the radicals of $\langle x \rangle + \mathfrak{p}$ and $\langle y \rangle + \mathfrak{p}$, which is just $\langle xy \rangle + \mathfrak{p}$. This ideal is then not in X, hence $xy \notin \mathfrak{p}$. This proves that $\mathfrak{p}$ is prime and therefor that a is not in the intersection of all prime ideals containing I. □

Proposition 1.6.39. *Given a prime ideal $\mathfrak{p} \subset S$ and a ring map $\sigma : R \to S$, then $\sigma^{-1}(\mathfrak{p}) \subset R$ is a prime ideal.*

Proof. Suppose $xy \in \sigma^{-1}(\mathfrak{p})$. Then $\sigma(x)\sigma(y) = \sigma(xy) \in \mathfrak{p}$, hence $\sigma(x) \in \mathfrak{p}$ or $\sigma(y) \in \mathfrak{p}$; stated equivalently $x \in \sigma^{-1}(\mathfrak{p})$ or $y \in \sigma^{-1}(\mathfrak{p})$. □

The following definition will be important way later on.

Definition 1.6.40. Let R be a commutative ring. We define the *spectrum of* R to be the set

$$\operatorname{Spec} R := \{I \subset R : I \text{ is a prime ideal}\}$$

Proposition 1.6.41. *Let R be a commutative ring and $I \subset R$ an ideal. Then there is a one-to-one correspondence between radical/prime/maximal ideals in R containing I and radical/prime/maximal ideals in R/I*

Proof. **Radical:** Let $J \subset R$ be a radical ideal containing I. Let $x + I \in \operatorname{rad}(J/I)$, then for some $n \geq 1$, $x^n + I \in J/I$, hence $x^n \in J$, implying $x \in \operatorname{rad}(J) = J$. This means $x + I \in J/I$.
Let $K \subset R/I$ be a radical ideal. Then $K = J/I$ for some ideal $J \subset R$ containing I. Let $x \in \operatorname{rad}(J)$. Then for some $n \geq 1$, $x^n + I \in J/I$, hence $x + I \in \operatorname{rad}(J/I) = J/I$,

implying $x \in J$.

Prime: J/I is prime if and only if $R/J \simeq \frac{R/I}{J/I}$ is an integral domain which is equivalent to J being prime.

Maximal: J/I is maximal if an only if $R/J \simeq \frac{R/I}{J/I}$ is maximal which is equivalent to J being maximal. □

Lemma 1.6.42. *Let I, J be ideals in a commutative ring R. Suppose $I = \langle a_1 \ldots, a_m \rangle$ for suitable $a_1, \ldots, a_m \in I$ and $I \subset \text{rad}(J)$. Then $I^n \subset J$ for some $n \geq 0$.*

Proof. Let $n_i \geq 0$ be given such that $a_i^{n_i} \in J$. Let $n = \sum_1^m n_i$. We prove the statement by induction in m. For $m = 1$ the statement is trivial.

$$\lambda_{i,j} \in R \quad (i \in \{1, \ldots, m\}, j \in \{1, \ldots, 2n\})$$

Then

$$\prod_1^n \left(\sum_1^m \lambda_{i,j} a_i \right) = \sum_{v \in \mathbb{N}^m} \mu_v a_1^{v_1} \cdots a_m^{v_m}.$$

A simple induction argument shows that if $v_i < n_i$ for some i then $v_j > n_j$ for some j, hence $a_i^{v_1} \cdots a_m^{v_m} \in J$ for each $v \in \mathbb{N}^m$ with $\sum_1^m v_i = n$. It follows that $\prod_1^n (\sum_1^m \lambda_{i,j} a_i) \in J$, hence $I^n \subset J$. □

1.6.3 The IBN Property

Definition 1.6.43. A ring R has the *invariant basis number (IBN) property* if for each $n, m \geq 0$, $R^n \simeq R^m$ (as left R-modules) implies $n = m$.

Definition 1.6.44. Let R be an arbitrary ring, we define $\mathbb{CFM}_{\mathbb{N}}(R)$, *the column finite matrices over* $\mathbb{N}$ to be the set

$$\left\{ (a_{ij}) \in \prod_{(i,j) \in \mathbb{N}^2} R : \text{ for each } j \text{ there exists } N \text{ such that } a_{ij} = 0 \text{ for all } i \geq N \right\}.$$

We define

$$(a_{ij})(b_{ij}) = \left(\sum_{k \in \mathbb{N}} a_{ik} b_{kj} \right)$$

for $(a_{ij})(b_{ij}) \in \mathbb{CFM}_{\mathbb{N}}(R)$.

Proposition 1.6.45. *The above multiplication is well-defined and with it $\mathbb{CFM}_{\mathbb{N}}(R)$ becomes a ring.*

Proof. Note that for $(a_{ij}), (b_{ij}) \in \mathbb{CFM}_{\mathbb{N}}(R)$, first of all that $\sum_{k\in\mathbb{N}} a_{ik}b_{kj} = \sum_1^N a_{ik}b_{kj}$ for a suitably large N, since $b_{kj} = 0$ for k greater than a suitably large N. secondly when i is large, $a_{ij} = 0$, hence $\sum_{k\in\mathbb{N}} a_{ik}b_{kj} = 0$, which shows $(a_{ij})(b_{ij}) \in \mathbb{CFM}_{\mathbb{N}}(R)$. Checking the ring axioms is a task completely analogous to doing the same matrix rings of R. □

The above is a simple example of a ring without the IBN property.

Example 1.6.46. We define a $\mathbb{CFM}_{\mathbb{N}}(R)$-linear map

$$\mathbb{CFM}_{\mathbb{N}}(R) \to \mathbb{CFM}_{\mathbb{N}}(R)^2$$
$$(a_{ij}) \mapsto ((a_{i,2j}), (a_{i,2j+1}))$$

which has inverse

$$\mathbb{CFM}_{\mathbb{N}}(R)^2 \to \mathbb{CFM}_{\mathbb{N}}(R)$$

taking a pair $((b_{ij}), (c_{ij}))$ "weaving" them row wise, i.e. mapping it to (a_{ij}) with $a_{i,2j} = b_{ij}$ and $a_{i,2j+1} = c_{ij}$. Then $\mathbb{CFM}_{\mathbb{N}}(R) \simeq \mathbb{CFM}_{\mathbb{N}}(R)^2$. In fact recursively using this argument, $\mathbb{CFM}_{\mathbb{N}}(R)^m \simeq \mathbb{CFM}_{\mathbb{N}}(R)^n$ for n, m.

Lemma 1.6.47. *A field K has the IBN property.*

Proof. This is an immediate consequence of dimension being an integer invariant of finite basis vector spaces, which we will see later. □

Proposition 1.6.48. *Assuming axiom of choice, a commutative ring has the IBN property.*

Proof. Suppose $f : R^n \simeq R^m$. Let I be a maximal ideal. Then f restrict to a R-module homomorphism

$$f|_{\prod_1^n I} : \prod_1^n I \to \prod_1^n I$$

We thus get an induced R/I-linear map

$$g : (R/I)^n \simeq R^n/I^n \to R^m/I^m(R/I)^m$$

following the same construction for the inverse of f we get an inverse of g, hence g defines an isomorphim of R/I vector space it follows that $n = m$, since R/I has the IBN property. □

1.6.4 Reduced Rings

Definition 1.6.49. An element x in a ring R is called *nilpotent* if $x^n = 0$ for some $n \geq 1$.

Definition 1.6.50. An left ideal $I \subset R$ is *nil*, if every element in I is nilpotent. I is *nilpotent* if $I^n = 0$ for some $n \geq 1$.

Remark 1.6.51. If I is nilpotent, then it is nil.

Lemma 1.6.52. *For a commutative ring R the set of nilpotent elements in R,* $\text{nil}(R)$ *is a two-sided ideal*

Proof. Clearly, 0 is nilpotent. Let $x, y \in \text{nil}(R)$. Then $x^n = 0$ and $y^m = 0$ for some $n, m \geq 1$. We claim that $(x+y)^{n+m} = 0$. This is a consequence of the binomial theorem. Lastly, for any $a \in R$, $(ax)^n = a^n x^n = 0$. □

Lemma 1.6.53. *In a commutative ring R,*

$$\text{nil}(R) = \bigcap_{\mathfrak{p} \leq R : \mathfrak{p} \text{ prime}} \mathfrak{p}$$

Proof. Indeed, using Lemma 1.6.38,

$$\text{nil}(R) = \text{rad}(0) = \bigcap_{0 \subset \mathfrak{p} \leq R : \mathfrak{p} \text{ prime}} \mathfrak{p}$$

□

Definition 1.6.54. A ring is called *reduced* if it has not nilpotent elements.

Definition 1.6.55. Given a commutative ring R, we form the *reduced ring of R* defined by

$$R_{\text{red}} := R/\text{nil}(R).$$

Lemma 1.6.56. *For a commutative ring R, R_{red} is reduced.*

Proof. Indeed, if $x^n + \text{nil}(R) = 0$ for some $n \geq 1$, then $x^n \in \text{nil}(R)$, hence x^n is contained in every prime ideal in R, which implies x is contained in every prime ideal in R, so $x \in \text{nil}(R)$. □

1.6.5 Comaximal ideals

In this subsection every ring will be assumed commutative.

Definition 1.6.57. Let R be a ring. A pair of ideals I, J in R are said to be *comaximal* if $I + J = R$.

Lemma 1.6.58. *Let I, J be comaximal ideals in a ring R. Then*

$$IJ = I \cap J$$

Proof. The first inclusion is implied by Lemma 1.4.49. Let $a \in I \cap J$. Since I and J are comaximal we can write $1 = i + j$ for suitable $i \in I$ and $j \in J$. Then

$$a = a(i + j) = ai + aj = ia + aj \in IJ,$$

since $ia, aj \in IJ$. □

Lemma 1.6.59. *Let I, J be comaximal ideals in a ring R. Then I^n and J^m are comaximal for every $n, m \geq 1$.*

Proof. **Claim 1:** We first show that I, J^m are comaximal for every $m \geq 1$ by way of induction in m. The base case is true by assumption. Let $m \geq 1$ and $x \in R$. By induction $R = I + J^m$, hence $x = a + b$ for suitable $a \in I$ and $b \in J^m$. Moreover, $1 = i + j$ for suitable $i \in I$ and $j \in J$. Then

$$x = a + b = a + b(i + j) = (a + bi) + bj \in I + J^{m+1}.$$

We now fix $m \geq 1$ it follows by a similar induction argument in n that I^n and J^m are comaximal. □

Lemma 1.6.60. *Let R be a ring. Consider ideals $I_1, \ldots, I_N$ in R and set $J_i := \bigcap_{j \neq i} I_j$. Suppose I_i and J_i are comaximal for each i. Then*

$$\bigcap_1^N I_i^n = \left(\prod_1^N I_i\right)^n = \left(\bigcap_1^N I_i\right)^n$$

Proof. Let $n \geq 1$. Note for each $N \geq 1$, $I_1 J_1 = I_1 \cap J_1$, by lemma 1.6.58, hence by induction we have that $\prod_1^N I_i = \bigcap_1^N I_i$, hence for each $n \geq 1$, $\left(\prod_1^N I_i\right)^n = \left(\bigcap_1^N I_i\right)^n$. By assumption and induction I_1^n and $\bigcap_2^{N+1} I_i^n = \left(\bigcap_2^{N+1} I_i\right)^n$ are comaximal, hence

$$\bigcap_1^{N+1} I_i = I_1^n \cap \bigcap_2^{N+1} I_i^n = \prod_1^{N+1} I_i^n = \left(\prod_1^{N+1} I_i\right)^n.$$

□

1.6.6 Principal Ideal Domains

Definition 1.6.61. An ideal $I \subset R$ is *principal*. A domain in which every ideal is principal is called a *principal ideal domain* or a *PID*.

Lemma 1.6.62. *Let R be a PID. The non-trivial maximal ideals of R are those generated by primes.*

Proof. A maximal ideal is generated by some p that is non-zero and a non-unit. Since a maximal ideal is prime we have that p is prime.
Let a prime p be given. Suppose $\langle p\rangle \subset \langle x\rangle$. Then $p = qx$. Then $q \in R^*$ or $x \in R^*$, since p is in particular irreducible. Then $\langle p\rangle = \langle q\rangle$ or $\langle p\rangle = R$, hence $\langle p\rangle$ is maximal. □

Lemma 1.6.63. *Let R be a PID. Irreducible elements in R are prime.*

Proof. p be irreducible. Suppose $\langle p\rangle \subset \langle p'\rangle$. Then $p = qp'$ for some q, then $q \in R^*$ or $p' \in R^*$. In the first case $\langle p\rangle = \langle p'\rangle$ and in the second case $\langle p'\rangle = R$. Then $\langle p\rangle$ is maximal, hence p is prime. □

The following is an immediate result of the prior two lemmas

Corollary 1.6.64. *The non-zero maximal ideals of a PID are those generated by irreducible elements*

1.6.7 Greatest Common Divisor and Least Common Multiples

Definition 1.6.65. A *greatest common divisor* of two elements a, b in a commutative ring is an element d where for every $c \in R$ such that $c \mid a$ and $c \mid b$ we have that $c \mid d$.

Remark 1.6.66. Note that $\gcd(a, 0) = a$ since if $a \mid a$ and any element divides 0.

Definition 1.6.67. A *Least common multiple* of two elements a, b in a commutative ring is an element $m \in R$ where for every $c \in R$ such that $a \mid c$ and $b \mid c$ we have that $m \mid c$.

Remark 1.6.68. $\operatorname{lcm}(a, 0) =$ since 0 is the only element that is a multiple of 0.

1.6.8 Unique Factorization Domains and Euclidean Domains

In our exploration of unique factorization domains and Euclidean Domains we will fix an integral domain R.

Definition 1.6.69. Let a non-unit and non-zero element $a \in R$ be given. a is said to be an *irreducible element* if for every $b, c \in R$

$$a = bc \implies b \in R^* \text{ or } c \in R^*.$$

Lemma 1.6.70. *A prime element is irreducible.*

Proof. Let $p \in R$ be prime. Consider $a, b \in R$ such that $p = ab$, then $p \mid ab$, hence either $p \mid a$ or $p \mid b$. Suppose $p \mid a$. Then $a = pr$ for some $r \in R$. This means $p = prb$, which by Lemma 1.6.25 means $rb = 1$, hence that b is a unit. In the case $p \mid b$, we can similarly show that a is a unit. It thus follows that p is irreducible. □

Definition 1.6.71. R is called a *Unique factorization domain (UFD)* if for every $r \in R \setminus \{R^* \cup \{0\}\}$ has unique factorization into irreducible elements, i.e. there are distinct irreducible elements $p_1, \ldots, p_n \in R$ unique and $v_1, \ldots, v_n \geq 1$ such that

$$r = \prod_1^n p_i^{v_i}.$$

Remark 1.6.72. By uniqueness we more precisely mean that given $q_1, \ldots, q_m \in R$ another sequence of distinct irreducible elements and $w_1, \ldots, w_m \geq 1$ such that

$$r = \prod_1^m q_i^{w_i},$$

then $m = n$ and there is some bijection $\omega : \{1, \ldots, n\} \to \{1, \ldots, n\}$ (i.e. a permutation $\omega \in \mathcal{S}_n$) and units $a_1, \ldots, a_n \in R$ such that

$$p_i = a_i q_{\tau(i)} \text{ and } v_i = w_{\tau(i)},$$

for each $i \in \{1, \ldots, n\}$.

Proposition 1.6.73. *Let R be a ring in which every element that is not zero or a unit can be written as a product of irreducible elements. R is a UFD if and only if every irreducible element is a prime.*

Proof. " $\implies$ ": Let $p \in R$ be irreducible and suppose there are $a, b \in R$ such that $p \mid ab$. We aim to prove that $p \mid a$ or $p \mid b$. Since $p \mid 0$, we are done if $a = 0$ or $b = 0$. So assume $a, b \neq 0$. In general for $x, y, z \in R$ if $x \mid y$, then $x \mid yz$. Hence if a is a unit, then $p \mid b = a^{-1}(ab)$, and similary if b is a unit then $p \mid a = b^{-1}(ab)$. So assume that a and b are not units. For some $q \in R$, $pq = ab$. q is not a unit, for otherwiser $p = abq^{-1}$ contradicting the irreduciblity of p. We can then find irreducible $q_1, \ldots, q_n, p_1, \ldots, p_m, p_{m+1}, \ldots, p_l \in R$ and $v_1, \ldots, v_m, w_1, \ldots, w_m, w_{m+1}, \ldots, v_l \geq 1$ such that

$$q = \prod_1^n q_i^{v_i} \text{ and } a = \prod_1^m p_i^{w_i} \text{ and } b = \prod_{m+1}^l p_i^{w_i}.$$

From this it follows that

$$p \prod_1^n q_i^{v_i} = \prod_1^l p_i^{w_i}.$$

Since the above is a factorization into irreducible it follows from the assumption that R is a UFD that there exists an $i \in \{1, \dots, l\}$ and a unit $s \in R$ such that $w_i = 1$ and $p = sp_i$. If $i \in \{1, \dots, m\}$ then

$$a = \prod_1^m p_j^{w_j} = s^{-1} p \prod_{j \in \{1,\dots,m\} \setminus \{i\}} p_j^{w_j} \implies p \mid a.$$

By a similar argument, if $i \in \{m+1, \dots, l\}$, then $p \mid b$. " $\Longleftarrow$ ": Suppose there are irreducible elements $p_1, \dots, p_n, q_1, \dots, q_m \in R$ and positive integers $v_1, \dots, v_n, w_1, \dots, w_m \geq 1$ such that

$$\prod_1^n p_k^{v_k} = \prod_1^m q_k^{w_k}.$$

Let $i \in \{1, \dots, n\}$. Then $p_i \mid \prod_1^m q_k^{w_k}$. By assumption p_i is prime, hence $p_i \mid q_{\tau(i)}$ for some $\tau(i) \in \{1, \dots, m\}$, hence for some $s_i \in R$, $q_{\tau(i)} = s_i p_i$, by irreducibility, we get that s_i is a unit. Similarly for $j \in \{1, \dots, m\}$, we can find $\omega(j) \in \{1, \dots, n\}$ and $t_j \in R^*$ such that $p_{\omega(j)} = t_j q_j$. Thus $p_i = s_i q_{\tau(i)} = s_i t_j p_{\omega(\tau(i))}$, hence $\omega(\tau(i)) = i$. Conversely one can show that $\tau(\omega(j)) = j$, hence $n = m$ and τ is a bijection. Now we show that $v_i = w_{\tau(i)}$ for each i. WLOG $v_i \geq w_{\tau(i)}$, Then

$$p_i^{v_i - w_{\tau(i)}} p^{w_{\tau(i)}} \prod_{k \in \{1,\dots,n\} \setminus \{i\}} p_k^{v_k} = p_i^{v_i} \prod_{k \in \{1,\dots,n\} \setminus \{i\}} p_k^{v_k} = p_i^{w_{\tau(i)}} \prod_{k \in \{1,\dots,n\} \setminus \{\tau(i)\}} p_k^{w_k},$$

which implies that

$$p_i^{v_i - w_{\tau(i)}} \prod_{k \in \{1,\dots,n\} \setminus \{i\}} p_k^{v_k} = \prod_{k \in \{1,\dots,n\} \setminus \{\tau(i)\}} p_k^{w_k} \implies p^{v_i - w_{\tau(i)}} \mid \prod_{k \in \{1,\dots,n\} \setminus \{\tau(i)\}} p_k^{w_k},$$

and if $v_i - w_{\tau(i)} \neq 0$ then since p_i is prime $p_i \mid p_k$ for some $k \in \{1, \dots, n\} \setminus \{\tau(i)\}$ which is not possible since p_i and p_k are distinct. So we conclude that $v_i = w_{\tau(i)}$. □

Definition 1.6.74. Let R be a UFD $\mathcal{P}$ be the set of prime/irreducible elements in R. We say two element $p, q \in \mathcal{P}$ are *associated* if there is a unit $a \in R$ such that $p = aq$.

Remark 1.6.75. Write $p \sim q$ if p and q are associated. Being associated is an equivalence relation on $\mathcal{P}$. Indeed for $p, q, r \in \mathcal{P}$. If $p = 1p$ implying $p \sim p$. $p \sim q$, then for some $a \in R^*$, $p = aq$, hence $q = a^{-1}p$, hence $q \sim p$. Suppose $p \sim q$, $q \sim p$, then for $a, b \in R^*$, $p = aq$ and $q = br$. Then $p = (ab)r$, hence $p \sim r$. We may then write any element as a product over $\mathcal{P}$

$$\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p},$$

where v_p is equal to 0 for all but finitely many p.

Lemma 1.6.76. *Let R be a UFD. Then $\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p} \mid \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{w_p}$ if and only if $v_p \leq w_p$ for every $p \in \mathcal{P}$.*

Proof. We that

$$\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{w_p} = \left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p} \right) \left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{u_p} \right) = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p + u_p}$$

for suitable $u_p \geq 0$. By uniqueness $v_p + u_p = w_p$ implying that $v_p \leq w_p$ for every $p \in \mathcal{P}$. Conversely if $v_p \leq w_p$ then there is some $u_p \geq 0$ such that $v_p + u_p = w_p$ and hence

$$\left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p} \right) \left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{u_p} \right) = \left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p + u_p} \right) = \left(\prod_{[p]_\sim \in \mathcal{P}/\sim} p^{w_p} \right)$$

□

Lemma 1.6.77. *Let $a, b \in R \setminus 0$. Then $a = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p(a)}$ and $b = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p(b)}$. One finds that*

$$\gcd(a,b) = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{\min(v_p(a), v_p(b))} \text{ and } \operatorname{lcm}(a,b) = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{\max(v_p(a), v_p(b))},$$

and these are unique up to multiplication by units.

Proof. Let $c \in R$ such that $c \mid a$ and $c \mid b$, then

$$c = \prod_{[p]_\sim \in \mathcal{P}/\sim} p^{v_p(c)}$$

with $v_p(c) \leq v_p(a)$ and $v_p(c) \leq v_p(b)$ by the above lemma, hence $v_p(c) \leq \min(v_p(a), v_p(b))$. Suppose $d \in R$ is a greatest common divisor of a and b, Then $d \mid \gcd(a,b)$ and $\gcd \mid d$, hence $v_p(d) = \max(v_p(a), v_p(b))$. Let $c \in R$ such that $a \mid c$ and $b \mid c$. Then $v_p(a) \leq v_p(c)$ and $v_p(b) \leq v_p(c)$ by the above lemma, hence $\max(v_p(a), v_p(b)) \leq v_p(c)$. Showing that $\operatorname{lcm}(a,b)$ is that unique up to multiplication by a unit is similar to the gcd-case. □

Lemma 1.6.78. *Let R be an integral $r = \prod_1^m p_i^{v_i} \in R$ where $p_1, \ldots, p_m \in R$ are distinct primes and $v_1, \ldots, v_m \geq 1$. Then $\operatorname{rad}(\langle r \rangle) = \langle \prod_1^m p_i \rangle$.*

Proof. Clearly $\prod_1^m p_i \in \operatorname{rad}(\langle r \rangle)$, since for $n = \max\ v_i$, $r = \prod_1^m p_i^{v_i} \mid \prod_1^m p_i^n$. Conversely, if $a \in \operatorname{rad}(\langle r \rangle)$, then for some $n \geq 0$, $r \mid a^n$, implying $\prod_1^m p_i \mid a^n$, hence $\prod_1^m p_i \mid a$. □

Lemma 1.6.79. *Let R be a UFD. The prime ideals of R are R, 0 and principal ideals generated by $\langle p \rangle$ for an irreducible element $p \in R$. This means that a proper non-zero prime ideal in a UFD contains no non-trivial prime ideal.*

Proof. This follows from Lemma 1.6.32 and Lemma 1.6.73. A non-trivial prime ideal contained in $\langle p \rangle$ is on the form $\langle q \rangle$. Then $q = ap$ for some $a \in K$. Since q is irreducible a is a unit hence $\langle p \rangle = \langle q \rangle$. □

Lemma 1.6.80. *A PID is a UFD.*

Proof. Let R be a PID. If we can prove that any non-unit non-zero element in R decomposes into a product of irreducible elements, we are done by Proposition 1.6.73, having the prior lemma in mind. Let $a \in R \setminus 0$ be a non-unit. Since R is Noetherian we can find a maximal ideal $\langle p_1 \rangle \supset \langle a \rangle$. Note that then $\langle p_1 \rangle$ is prime, hence p_1 is prime and that $a = a_1 p_1$ for some a_1. Define a_{n+1} to be an element such that $a_n = a_{n+1} p_n$ for some p_n. We get an ascending chain

$$\langle a \rangle \subset \langle a_1 \rangle \subset \dots$$

Since R is Noetherian, for some m, $\langle a_m \rangle = \langle a_n \rangle$ for $n \geq m$. Pick m to be the smallest such. If a_m was reducible, then $\langle a_m \rangle \subsetneq \langle a_{m+1} \rangle$. So it follows by induction that $a = a_m \prod_1^m p_i$; a product of irreducible elements.

□

Definition 1.6.81. A *Euclidean domain* is a domain R such there is a function $N : R \setminus 0 \to \mathbb{N}$ such that for every $x, y \in R$ there are $q, r \in R$ with $r = 0$ or $N(r) < N(y)$ and

$$x = qy + r$$

Theorem 1.6.82. *A Euclidean domain R is a PID.*

Proof. Let $I \subset R$ be a non-zero ideal. Let $d \in I$ be such that

$$N(d) = \min(\{N(a) : a \in I\}).$$

Let $x \in I$. Then there are $q, r \in R$ such that

$$x = qd + r$$

and $r = 0$ or $N(r) < N(x)$. Note that $x - r \in I$ and since $x \in I$, we have that $r \in I$. Then $N(x) \leq N(d)$, hence $r = 0$. Then $x = qd$, meaning $x \in \langle d \rangle$. We thus get that $I = \langle d \rangle$ and thus that R is a PID. □

Corollary 1.6.83. *A Euclidean domain is a UFD.*

Proof. A Euclidean domain is a PID, which is a UFD. □

1.6.9 Localizations of Commutative Rings

Definition 1.6.84. Given a ring R, a subset X of R is called *multiplicative* if it is a submonoid of $(R, \cdot)$

Definition 1.6.85. Let R be a commutative ring and X a multiplicative set in R. Given $(a, x), (b, y) \in R \times X$ we write $(a, x) \sim (b, y)$ if and only if there exists a $c \in X$ such that

$$c(ay - bx) = 0$$

Remark 1.6.86. This defines an equivalence relation on $R \times X$.

Definition 1.6.87. Let R be a commutative ring and X a multiplicative set. We define the *localization of R with respect to X* to be the set

$$X^{-1}R := (R \times X)/\sim .$$

an element $[(a, x)] \in X^{-1}R$ is denoted $\frac{a}{x}$.

Example 1.6.88.

1. An example of a multiplicative set is $X_x := \{x^n : n \geq 0\}$ for some $x \in R$. We define
$$R_x := X_x^{-1}R.$$
This construction is referred to as *the localization of R with respect x*

2. Another example of a multiplicative set is $X_{\mathfrak{p}} := R \setminus \mathfrak{p}$ where $\mathfrak{p} \subset R$ is a prime ideal. Indeed, if $a, b \in R \setminus \mathfrak{p}$, then $ab \notin \mathfrak{p}$, by the contrapositive of the definition of being a prime ideal. We define the *localization of R with respect to* $\mathfrak{p}$ to be
$$R_{\mathfrak{p}} := X_{\mathfrak{p}}^{-1}R$$
Note that $R \setminus \mathfrak{p}$ contains no zero divisors. We may therefore identify R with the subring of elements of the form $\frac{a}{1} \in R_{\mathfrak{p}}$.

3. Let X be the set of elements in R that are not zero divisors. Then $X^{-1}R$ is called the *total ring of fractions of R*. X is the largest multiplicative subset such that ι_X is injective. Indeed, if $X \subsetneq Y$ for another multiplicative set Y in R, then Y must contain a zero divisor, hence ι_Y is not injective.

Lemma 1.6.89. *Given a commutative ring R and a multiplicative set X. Then $X^{-1}R$ is the zero ring $\{\frac{0}{1}\}$ if and only if $0 \in X$.*

Proof. Suppose $0 \in X$. Then for each $\frac{a}{x} \in X^{-1}R$, $0(a \cdot 1 - x \cdot 0) = 0$, hence $\frac{a}{x} = \frac{0}{1}$.
Suppose $X^{-1}R = \{\frac{0}{1}\}$. Then $\frac{1}{1} = \frac{0}{1}$, hence for some $f \in X$, $f = f(1 \cdot 1 - 1 \cdot 0) = 0$. □

Lemma 1.6.90. *Let R be a commutative ring and X a multiplicative set. We define addition on $X^{-1}R$ by*

$$\frac{a}{x} + \frac{b}{y} := \frac{ay + bx}{xy} \quad \left(\frac{a}{x}, \frac{b}{y} \in X^{-1}R\right).$$

We define multiplication on $X^{-1}R$ by

$$\frac{a}{x}\frac{b}{y} := \frac{ab}{xy} \quad \left(\frac{a}{x}, \frac{b}{y} \in X^{-1}R\right).$$

We set $1 := 1_{X^{-1}R} := \frac{1}{1}$ and $0 := 0_{X^{-1}R} := \frac{0}{1}$. Then $(R, +, \cdot, 0, 1)$ is a commutative ring.

Proof. We first check that addition and multiplication are well-defined. Let $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}, \frac{d}{w} \in X^{-1}R$ be given with $\frac{a}{x} = \frac{c}{z}$ and $\frac{b}{y} = \frac{d}{w}$. Then there are $f, g \in R$ such that

$$f(az - cx) = 0 \text{ and } g(bw - dy) = 0.$$

$$gf((ay + bx)zw - (cw + dz)xy) = wyg(f(az - cx)) + zxf(g(bw - dy)) = 0,$$

which implies

$$\frac{a}{x} + \frac{b}{y} = \frac{ay + bx}{xy} = \frac{cw + dz}{zw} = \frac{c}{z} + \frac{d}{w}.$$

Moreover,

$$fg(abzw - cdxy) = fg(abzw - cdxy - bcxw + bcxw) = gbwf(az - cx) + fcxg(bw - dy) = 0,$$

hence

$$\frac{a}{x}\frac{b}{y} = \frac{ab}{xy} = \frac{cd}{zw} = \frac{c}{z}\frac{d}{w}.$$

Let $\frac{a_1}{x_1}, \frac{a_2}{x_2}, \frac{a_3}{x_3} \in X^{-1}R$ be given. Then

$$\begin{aligned}\frac{a_1}{x_1} + \left(\frac{a_2}{x_2} + \frac{a_3}{x_3}\right) &= \frac{a_1}{x_1} + \frac{a_2x_3 + a_3x_2}{x_2x_3} = \frac{a_1x_2x_3 + a_2x_1x_3 + a_3x_1x_2}{x_1x_2x_3} \\ &= \frac{(a_1x_2 + a_2x_1)x_3 + a_3x_1x_2}{x_1x_2x_3} = \frac{a_1x_2 + a_2x_1}{x_1x_2} + \frac{a_3}{x_3} \\ &= \left(\frac{a_1}{x_1} + \frac{a_2}{x_2}\right) + \frac{a_3}{x_3},\end{aligned}$$

and

$$\frac{a}{x} + 0 = \frac{a}{x} + \frac{0}{1} = \frac{a \cdot 1 + 0x}{1x} = \frac{a}{x}.$$

For an $\frac{a}{x} \in X^{-1}R$, we define $-\frac{a}{x} := \frac{-a}{x}$

$$\frac{a}{x} - \frac{a}{x} = \frac{a}{x} + \frac{-a}{x} = \frac{ax - ax}{x^2} = 0.$$

Commutativity of addition on $X^{-1}R$ follows readily from the commutativity of addition on R. Checking that multiplication on $X^{-1}R$ gives it the structure of a commutative monoid is also readily verifiable. multiplication being distributive over addition is also easy to verify. □

Lemma 1.6.91. *Let X be a multiplicative set of a commutative ring R. The ring map*

$$\begin{aligned} \iota_X : R &\to \left\{\frac{a}{1} : a \in R\right\} \\ a &\mapsto \frac{a}{1} \end{aligned}$$

has image

$$\left\{\frac{a}{1} : a \in R\right\}.$$

Furthermore, ι_X is injective if and only if X contains no zero-divisors.

Proof. The first statement is trivial. Suppose X contains no zero-divisors. If $\frac{a}{1} = 0$, then there is an $x \in X$ with

$$xa = 0,$$

hence $a = 0$. Conversely, if X contains a zero divisor x, we may find $a \in R \setminus 0$ with $xa = 0$, implying

$$\iota_X(a) = \frac{a}{1} = 0,$$

hence ι_X cannot be injective. □

Definition 1.6.92. Let R be a commutative ring. Let X be a multiplicative set. We define the *saturation* of X to be the set

$$\widehat{X} := \{r \in R : \exists r' \in R, r'r \in X\}.$$

X is *saturated*, if

$$\widehat{X} = X.$$

Remark 1.6.93. Let $x \in X$, then $1x \in X$, hence $x \in \widehat{X}$. We thus have $X \subset \widehat{X}$. Let $r, s \in \widehat{X}$, then for some $r', s' \in R$, $r'r \in X$ and $s's \in X$. Then $r's'rs \in X$, hence $rs \in \widehat{X}$. Clearly $1 \in \widehat{X}$. Thus $\widehat{X}$ is a multiplicative set. The saturation of X is clearly saturated. Let Y be a saturated multiplicative set containing X. Let $r \in \widehat{X}$. Then for some $r' \in R$, $r'r \in X \subset Y$. Thus $\widehat{X} \subset Y$, hence $\widehat{X}$ is the smallest saturated multiplicative set containing X.

Lemma 1.6.94. *Let R be a commutative ring and X a multiplicative set in R. Then X is saturated if and only if for every $x, y \in R$ when $xy \in X$ then $x, y \in X$*

Proof. " $\implies$ ": Suppose X is saturated. Let $x, y \in R$ s.t. $xy \in X$. Then $y \in \widehat{X} = X$ and since $yx = xy \in X$, $x \in \widehat{X} = X$.
" $\impliedby$ ": Let $r \in \widehat{X}$, then for some $r' \in R$, $r'r \in X$, which by assumption means $r \in X$. □

Lemma 1.6.95. *Let R be a commutative ring and X a multiplicative set. Consider the map*

$$\begin{aligned} \iota_X : R &\hookrightarrow X^{-1}R \\ r &\mapsto \frac{r}{1} \end{aligned}$$

Let $\frac{r}{x} \in X^{-1}R$. Then

$$\frac{r}{x} \in \left(X^{-1}R\right)^* \iff r \in Y := \iota_X^{-1}\left(\left(X^{-1}R\right)^*\right).$$

Furthermore, $\widehat{X} = Y$. Thus

$$\left(X^{-1}R\right)^* = \left\{\frac{r}{x} \in X^{-1}R : r \in \widehat{X}\right\}.$$

Proof. " $\implies$ ": Suppose $\frac{r}{x} \in (X^{-1}R)^*$. Then for some $\frac{s}{y} \in X^{-1}R$, $\frac{r}{x}\frac{s}{y} = \frac{s}{y}\frac{r}{x} = 1$. From this we get that

$$r\frac{s}{xy} = \frac{rx}{x}\frac{s}{xy} = \frac{r}{x}x\frac{1}{x}\frac{s}{y} = 1,$$

hence $r \in Y$.
" $\impliedby$ ": If $r \in Y$, then $r\frac{s}{y} = 1$ for some $\frac{s}{y} \in X^{-1}R$, hence $\frac{r}{x}\frac{sx}{y} = 1$, implying $\frac{r}{x} \in (X^{-1}R)^*$.
If $r \in \widehat{X}$. Then for some $r' \in R$, $r'r \in X$. Then

$$r\frac{r'}{r'r} = \frac{r'r}{r'r} = 1 \implies r \in Y.$$

Let $r \in Y$. Then for some $\frac{s}{y} \in X^{-1}S$, $r\frac{s}{y} = 1$, meaning

$$sr = sr\frac{1}{y}y = r\frac{s}{y}y = y \in X \implies r \in \widehat{X}.$$

□

For this subsection $\mathcal{C}$ denotes the category whose objects are pairs consisting of a commutative ring and a multiplicative set in that ring. A morphism of such objects, $(R, X) \to (S, Y)$ is ring map, mapping every element in X to an element in Y. Let $\mathcal{D}$ be the subcategory of $\mathcal{C}$ where an object in $\mathcal{D}$ is a pair $(R, X) \in \mathcal{C}$ with $X \subset R^*$.

Lemma 1.6.96. *Let $\sigma : (R, X) \to (S, S^*)$ be a morphism in $\mathcal{C}$. There is a unique ring homomorphism $\overline{\sigma} : X^{-1}R \to S$ such that*

$$\begin{array}{ccc} R & & \\ {\scriptstyle \iota_X}\downarrow & \searrow^{\sigma} & \\ X^{-1}R & \xrightarrow[\overline{\sigma}]{} & S \end{array}$$

commutes.

Proof. We define

$$\overline{\sigma} : X^{-1}R \to S$$
$$\frac{a}{x} \mapsto \sigma(a)\sigma(x)^{-1}$$

Suppose $\frac{a}{x} = \frac{b}{y}$. Then there is a $z \in X$ such that $z(ay - bx) = 0$. Then $\sigma(z), \sigma(y), \sigma(x) \in S^*$ and

$$\sigma(z)\sigma(y)\sigma(a) = \sigma(z)\sigma(x)\sigma(b) \implies \overline{\sigma}\left(\frac{a}{x}\right) = \sigma(a)\sigma(x)^{-1} = \sigma(b)\sigma(y)^{-1} = \overline{\sigma}\left(\frac{b}{y}\right).$$

One readily verifies that this is a ring map. Let $a \in R$. Then

$$\overline{\sigma}\iota_X(a) = \overline{\sigma}\left(\frac{a}{1}\right) = \sigma(a)\sigma(1)^{-1} = \sigma(a).$$

Note that since $\sigma(x) \in S^*$ for every $x \in X$ we cannot have $0 \in X$. Suppose $\tau : X^{-1}R \to S$ is another ring map satisfying $\sigma = \tau\iota_X$. For every $x \in X$ note that

$$1 = \tau\left(\frac{1}{1}\right) = \tau\left(\frac{x}{1}\frac{1}{x}\right) = \tau\left(\frac{x}{1}\right)\tau\left(\frac{1}{x}\right) = \tau\iota_X(x)\tau\left(\frac{1}{x}\right) = \sigma(x)\tau\left(\frac{1}{x}\right),$$

hence $\tau\left(\frac{1}{x}\right) = \sigma(x)^{-1}$. It thus follows for every $\frac{a}{x} \in X^{-1}R$ that

$$\tau\left(\frac{a}{x}\right) = \tau\left(\frac{a}{1}\right)\tau\left(\frac{x}{1}\right) = \tau\iota(a)\sigma(x)^{-1} = \sigma(a)\sigma(x)^{-1} = \overline{\sigma}\left(\frac{a}{x}\right).$$

Thus we conclude $\overline{\sigma} = \tau$. □

Remark 1.6.97. The contents of this lemma is that the functor

$$\mathrm{Loc} : \mathcal{C} \to \mathcal{D}$$
$$(R, X) \mapsto (X^{-1}R, \iota_X(X))$$
$$(\sigma : (R, X) \to (S, Y)) \mapsto (\overline{\sigma} : X^{-1}R \to Y^{-1}S, \frac{a}{x} \mapsto \sigma(a)\sigma(x)^{-1})$$

is left adjoint to the forgetful functor $\mathcal{F} : \mathcal{D} \to \mathcal{C}$. Since for each $(R, X) \in \mathcal{C}$ and $(S, Y) \in \mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}}((R, X), \mathcal{F}(S, Y)) \to \mathrm{Hom}_{\mathcal{D}}((X^{-1}R, \iota_X(X)), (S, Y))$$
$$\sigma \mapsto \overline{\sigma}$$

defines a bijection. Indeed, given a morphism $\tau \in \mathrm{Hom}_{\mathcal{D}}((X^{-1}R, \iota_X(X)), (S, Y))$, and setting $\sigma := \tau|_R \in \mathrm{Hom}_{\mathcal{C}}((R, X), \mathcal{F}((S, Y)))$, we have $\overline{\sigma} = \tau$. This shows (using Yoneda) that the functor

$$\mathrm{Hom}_{\mathcal{C}}((R, X), _)|_{\mathcal{D}} : \mathcal{D} \to \mathrm{Set}$$

is represented by $(X^{-1}R, \iota_X(X))$, hence X^{-1} is unique up to unique isomorphism among rings in $\mathcal{D}$ with this property.

As a gift of this functorial description of localization we get

Corollary 1.6.98. *If $\sigma : R \simeq S$ and X is a multiplicative set in R, then*

$$X^{-1}R \simeq \sigma(X)^{-1}S.$$

Lemma 1.6.99. *Consider a multiplicative set X in a commutatitive ring R and a saturated multiplicative set $Y \subset X^{-1}R$ with $Y \cap R \supset X$. Then*

$$(Y \cap R)^{-1}R \simeq Y^{-1}(X^{-1}R).$$

Proof. We have the canonical injection $R \hookrightarrow X^{-1}R$ from which we obtain the injective map

$$\begin{aligned} \sigma : (Y \cap R)^{-1}R &\to Y^{-1}(X^{-1}R) \\ \frac{a}{y} &\mapsto \frac{\frac{a}{1}}{y} \end{aligned}$$

It suffices to show that this map is surjective. Let $\frac{\frac{a}{x}}{\frac{b}{y}} \in Y^{-1}(X^{-1}R)$. Note that since Y is saturated $b \in Y$ and $\frac{1}{y} \in Y$. Since $y\frac{1}{y} = 1 \in Y$ it follows that $y \in Y$. Then

$$\frac{\frac{a}{x}}{\frac{b}{y}} = \frac{a}{x}\frac{1}{\frac{b}{y}} = \frac{a}{x}\frac{y}{b} = \frac{ay}{bx} = \sigma\left(\frac{ay}{bx}\right).$$

□

Definition 1.6.100. Let R be a commutatitive ring and X a multiplicative subset of R. Consider an ideal $I \subset R$. We define the *localization ideal of I in $X^{-1}R$* to be the set

$$X^{-1}I := \left\{\frac{a}{x} \in X^{-1}R : a \in I, x \in X\right\}$$

Lemma 1.6.101. *Let R be a commutatitive ring and X a multiplicative subset of R. Consider an ideal $I \subset R$. Then $X^{-1}I$ is a proper ideal in $X^{-1}R$ if and only if $X \cap I = \emptyset$.*

Proof. If $X^{-1}I = X^{-1}R$, then $\frac{a}{x} = 1$ for some $a \in I$ and $x \in I$. Then $\frac{a}{x}$ is a unit, hence $ab \in X$ for some $b \in R$. It follows that $\frac{ab}{xb} \in X^{-1}I$ with $ab \in X \cap I$. Conversely, if there is an $a \in X \cap I$, then $1 = \frac{a}{a} \in X^{-1}I$. □

Lemma 1.6.102. *Let R be a commutative ring and $X \subset R$ a multiplicative subset of R. Let $I \subset R$ be an ideal. Then $(X^{-1}R)I = X^{-1}I$. Consequently, $X^{-1}I$ is an ideal.*

Proof. Let $\sum_1^n \frac{r_i}{x_i} a_i \in (X^{-1}R)I$ where $\frac{r_i}{x_i} \in X^{-1}R$ and $a_i \in I$. Then

$$\sum_1^n \frac{r_i}{x_i} a_i = \frac{\sum_{i=1}^n \left(\prod_{j \in \{1,\dots,n\}, j \neq i} x_j \right) r_i a_i}{\prod_{j=1}^n x_i} \in X^{-1}I.$$

(In other words we just compute the sum of these fractions and see that the numerator is in I) The converse inclusion is trivial. □

Lemma 1.6.103. *Let R be a commutative ring and X a multiplicative subset of R. Let $I \subset X^{-1}R$ be an ideal. Then $X^{-1}(I \cap R) = I$.*

Proof. Let $\frac{a}{x} \in X^{-1}(I \cap R)$, where $a \in I$, $x \in X$.Then

$$\frac{a}{x} = \frac{1}{x} a \in I.$$

Conversely, let $\frac{a}{x} \in I$. Then $a = x\frac{a}{x} \in I$, hence $\frac{a}{x} \in X^{-1}(I \cap R)$. □

Corollary 1.6.104. *If $I \subset R$ is a prime ideal, then $X^{-1}I$ is prime.*

Proof. Let $\frac{a}{x}\frac{b}{y} \in X^{-1}I$ then $ab \in I$, hence $a \in I$ or $b \in I$, so $\frac{a}{x} \in X^{-1}I$ or $\frac{b}{y} \in X^{-1}I$ □

Definition 1.6.105. If R is a commutative ring and X is a multiplicative set in R, then given an ideal $I \subset R$, we define $X/I := \{x + I : x \in X\}$.

Remark 1.6.106. One readily verifies that X/I is a multiplicative set in R/I.

Lemma 1.6.107. *If R is a commutative ring and X is a multiplicative set in R, then for any ideal $I \subset R$,*

$$(X/I)^{-1}R/I \simeq (X^{-1}R)/(X^{-1}I)$$

Proof. We have a surjective map of rings

$$\sigma X^{-1}R \to (X/I)^{-1}R/I$$
$$\frac{a}{x} \mapsto \frac{a+I}{x+I}$$

induced by $R \twoheadrightarrow R/I$. We check that ker $\sigma = X^{-1}I$. Let $\frac{a+I}{x+I} = 0 + I$. Then $fa \in I$ for some. Then

$$\frac{a}{x} = \frac{af}{ax} \in X^{-1}I.$$

□

Lemma 1.6.108. *Let R be a commutative ring and X a multiplicative set.*

1. $\text{nil}(X^{-1}R) = \{\frac{a}{x} : a \in \text{nil}(R)\}$
2. *R is reduced if and only if $X^{-1}R$ is reduced.*
3. *Set $X_{\text{red}} = \{x + \text{nil}(R) : x \in X\}$. Then*

$$\left(X^{-1}R\right)_{\text{red}} \simeq X_{\text{red}}^{-1}R_{\text{red}}.$$

Proof. 1. If $\frac{a}{x} \in X^{-1}R$ with $\frac{a^n}{x^n} = 0$ for some $n \geq 1$. Then $a^n = 0$, hence $a \in \text{nil}(R)$. If $\frac{a}{x} \in X^{-1}R$ is given such that $a^n = 0$ for some $n \geq 1$, then $\frac{a^n}{x^n} = 0$, hence $\frac{a}{x} \in \text{nil}(X^{-1}R)$.
2. If R is not reduced, then trivially $X^{-1}R$ is not reduced, since R is a subring of $X^{-1}R$. Suppose $X^{-1}R$ is not reduced, then there is a non-zero element $\frac{a}{x} \in \text{nil}(X^{-1}R)$, hence a is a non-zero element in $\text{nil}(R)$, implying R is not reduced.
3. Follows directly from Lemma 1.6.107 □

1.6.10 Local Rings, Localizations of Integral Domains & Field of Fractions

Definition 1.6.109. A ring R is called *local* if it has unique maximal left ideal, $\mathfrak{m}_R$. A homomorphism of local rings is a homomorphism of rings $f : R \to S$ if $f^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.

Proposition 1.6.110. *Let R be ring. R is local if and only if $\mathfrak{m} := R \setminus R^*$ is a left ideal.*

Proof. " $\implies$ ": Let I be the unique maximal left ideal of R. Then since I is proper, $I \subset \mathfrak{m}$. Note that for every $x \in \mathfrak{m}$, $\langle x \rangle$ is a proper ideal in R, hence $x \in I$. Therefor $\mathfrak{m} = I$, hence $\mathfrak{m}$ is an ideal.
" $\impliedby$ ": Let $I \subsetneq R$ be an ideal. Then every element of I is a non-unit, hence $I \subset \mathfrak{m}$, thus $\mathfrak{m}$ is the unique maximal left ideal in R. □

Lemma 1.6.111. *Given a prime ideal $\mathfrak{p} \subset R$ of a commutative ring R, the multiplicative set $X = R \setminus \mathfrak{p}$ is saturated.*

Proof. This follows from Lemma 1.6.94, since if $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, then $ab \in \mathfrak{p}$. □

Lemma 1.6.112. *Given a prime ideal $\mathfrak{p} \subset R$ of a commutative ring R, the ring $R_{\mathfrak{p}}$ is a local ring.*

Proof. Let $\frac{a}{x}, \frac{b}{y} \in R_{\mathfrak{p}}$ be non-units, then $a, b \in \mathfrak{p}$, hence $ay + bx \in \mathfrak{p}$, which means $\frac{a}{x} + \frac{b}{x}$ is not a unit in $R_{\mathfrak{p}}$ (cf. the prior lemma and Lemma 1.6.95). If $\frac{c}{z} \in R_{\mathfrak{p}}$, then $ca \in \mathfrak{p}$, hence $\frac{c}{z}\frac{a}{x}$ is a non-unit in $R_{\mathfrak{p}}$. We conclude that the non-units of $R_{\mathfrak{p}}$ is an ideal, hence $R_{\mathfrak{p}}$ is local. Let $\frac{a}{x}, \frac{b}{y} \in R_{\mathfrak{p}} \setminus 0$. □

Lemma 1.6.113. *Consider a ring map $\sigma : R \to S$ and $\mathfrak{p} \subset R$ and $\mathfrak{q} \subset S$ prime ideals with $\sigma^{-1}(\mathfrak{q}) = \mathfrak{p}$. Then the localization $\sigma_{\mathfrak{p}} : R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is a local ring homomorphism.*

Proof. The maximal ideals $\mathfrak{m}_{\mathfrak{p}} \subset R$ and $\mathfrak{m}_{\mathfrak{q}} \subset S$ is the set of quotients with numerator in $\mathfrak{p}$ respectively $\mathfrak{q}$. If $\frac{a}{x} \in \sigma_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{q}})$, then

$$\frac{\sigma(a)}{\sigma(x)} = \sigma_{\mathfrak{p}}\left(\frac{a}{x}\right) \in \mathfrak{m}_{\mathfrak{q}},$$

hence $\sigma(a) \notin \mathfrak{q}$. This implies $a \notin \sigma^{-1}(\mathfrak{q}) = \mathfrak{p}$, hence $\frac{a}{x} \in \mathfrak{m}_{\mathfrak{p}}$. If $\frac{a}{x} \in \mathfrak{m}_{\mathfrak{p}}$, then $a \notin \mathfrak{p} = \sigma^{-1}(\mathfrak{q})$, hence $\frac{a}{x} \in \sigma_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{q}})$. We thus conclude that $\mathfrak{m}_{\mathfrak{p}} = \sigma_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{q}})$. □

Proposition 1.6.114. *When R is an integral domain and $X \subset R \setminus 0$ is a multiplicative set, then for every $\frac{a}{x}, \frac{b}{y} \in X^{-1}R$,*

$$\frac{a}{x} = \frac{b}{y} \iff ay = bx$$

Proof. " $\Longleftarrow$ ": trivial.
" $\Longrightarrow$ ": There is an $h \in X$, such that $h(ay - bx) = 0$. Then $ay - bx = 0$. □

Corollary 1.6.115. *When R is an integral domain, for a prime ideal $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is an integral domain.*

Theorem 1.6.116. *When R is an integral domain, and X is a multiplicative set of R not containing 0, we have that $Q(R) := (R \setminus 0)^{-1}R$ is a field and $X^{-1}R$ can be identified with a subring of $Q(R)$. Moreover $Q(R)$ is the smallest field containing R.*

Proof. By the assumption that R is an integral domain, 0 i a prime ideal, hence $Q(R)$ is a commutative ring and $R \setminus 0$ is saturated, so

$$Q(R)^* = \left\{\frac{r}{s} \in Q(R) : r \in R \setminus \{0\}\right\} = Q(R) \setminus \{0\}.$$

This means $Q(R)$ is a field. Let K be a field containing R as a subring. Let $\frac{r}{s} \in Q(R)$. Then $r \in K$ and $\frac{1}{s} = s^{-1} \in K$, hence $\frac{r}{s} = r\frac{1}{s} \in K$. This means $Q(R) \subset K$, hence $Q(R)$ is the smallest field containing R as a subring. Consider the subring

$$\left\{\frac{a}{x} \in Q(R) : x \in X\right\} \subset Q(R).$$

Set $Y := R \setminus 0$ and consider the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\text{id}} & R \\ \downarrow{\scriptstyle \iota_X} & \searrow^{\overline{\text{id}}} & \downarrow{\scriptstyle \iota_Y} \\ X^{-1}R & \dashrightarrow & Q(R) \end{array}$$

where the dashed arrow takes $\frac{a}{x} \mapsto \frac{a}{x}$. The diagram tells us that this defines an isomorphism onto its image, which is the subring discussed earlier. □

Remark 1.6.117. We call $Q(R)$ *the field of fractions of* R. From the above we conclude that if K is a field then we can canonically make the identification $K = Q(K)$, and in general $Q(R) = Q(Q(R)) = Q(Q(Q(R)))...$ and so on.

Corollary 1.6.118. *If* $R \simeq S$*, then*

$$Q(R) \simeq Q(S).$$

Definition 1.6.119. We define the *rational numbers* to be the field $\mathbb{Q} := Q(\mathbb{Z})$.

1.6.11 Discrete Valuation Rings

Definition 1.6.120. Let R be a non-field integral domain. R is a *discrete valuation ring (DVR)* if it is noetherian and local with principal maximal ideal.

Proposition 1.6.121. *Let* R *be a non-field integral domain. Then* R *is a DVR if and only if there is an irreducible element* $t \in R$ *such that for every* $z \in R \setminus 0$ *there are unique* $u \in R^*$ *and* $n \geq 1$ *satisfying* $z = ut^n$

Proof. " $\Longrightarrow$ ": Let $\mathfrak{m} = \langle t \rangle$ be the maximal ideal of R. Then t is prime hence irreducible by maximality. Let $z \in R \setminus 0$. Then either z is a unit in which case $z \notin \mathfrak{m}$ or z is not a unit hence $z \in \mathfrak{m}$. There is a $u \in R \setminus 0$ with $t \nmid u$ and a maximal n such that $z = ut^n$. Since $u \notin \langle t \rangle$, it is a unit by maximality. Suppose $u' \in R^*$ and $n' \geq 0$ are given such that $ut^n = u't^{n'}$. Then $n = n'$, since otherwise $t \mid u'$, hence $u = u'$.
" $\Longleftarrow$ ": Let $\mathfrak{m} = \langle t \rangle$. Every non-unit is of the form ut^n, where $n \geq 1$, hence $R \setminus R^* = \mathfrak{m}$, hence R is local by Proposition 1.6.110. Let $I \subsetneq R$ be an ideal, then $I \subset \mathfrak{m}$. Then $I = \langle t^r \rangle$, where $r = \min\{n \geq 0 : t^n \in I\}$. Indeed if $a \in I$, then $a = ut^n$ for some $n \geq r$, hence $a = ut^{n-r}t^r \in I$. Then R is a PID, and hence Noetherian. □

Remark 1.6.122. We refer to an element such as t as an *uniformizing parameter*.The uniformizing parameters of a DVR are of the form ut where u is a unit and t is a uniformizing parameter. Set $K = Q(R)$. Then every $z \in K \setminus 0$ can be written uniquely on the form $z = ut^n$ for a unit $u \in R$ and an integer n. The integer n is called *the order of* z denoted $\text{ord}(z)$. We set $\text{ord}(0) = \infty$. One sees that $R = \text{ord}^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$ and $\mathfrak{m} = \text{ord}^{-1}(\mathbb{Z}_{\geq 1} \cup \{\infty\})$. The order is independent of uniformizing parameter. Since if t is a uniformizing parameter and $z = ut^n$ for unique $u \in R^*$, $n \in \mathbb{Z}$, then for a unit $s \in R$, $z = \frac{a}{b} = \frac{vst^l}{yst^k} = \frac{v}{y}t^{l-k}$ for suitable units v, y, $l, k \geq 0$, hence by uniqueness $\frac{v}{y} = u$ and $l - k = n$.

Proposition 1.6.123. *The localization of* $\mathbb{Z}$ *with respect to a maximal ideal* $\langle p \rangle$ *(*$p \in \mathbb{Z}$ *is prime),* $\mathbb{Z}_{\langle p \rangle}$ *is a DVR whose quotient field is* $\mathbb{Q}$*.*

Proof. One notes that $\mathbb{Z}_{\langle p \rangle} = \{\frac{a}{n} \in \mathbb{Q} : p \nmid n\}$. Suppose $p = \frac{a}{n}\frac{b}{m}$. Then $mn \mid ab$, WLOG $mn = 1$, hence $m = 1$ and $n = 1$. Then $p = ab$ hence a or b is a unit, meaning p is irreducible in $\mathbb{Z}_{\langle p \rangle}$. For $\frac{a}{n} \in \mathbb{Z}_{\langle p \rangle}$ let $\nu_p\left(\frac{a}{n}\right) = \max(\{n \geq 0 : p^n \mid \frac{a}{n}\})$. One eaily checks that $p \mid \frac{a}{n}$ if and only if $p \mid a$, hence $\nu_p(\frac{a}{n}) = \nu_p(a)$. We thus get that

$$\frac{a}{n} = \frac{q}{n}p^{\nu_p(a)},$$

where $p \nmid q$. Note that $\frac{n}{q}$ is the inverse of $\frac{q}{n}$. The uniqueness of this decomposition follows from p not being being a unit. Proposition 1.6.121 shows that $\mathbb{Z}_{\langle p \rangle}$ is a DVR. Every element in $\mathbb{Q}$ can be written as $\frac{a}{b} = \frac{s}{t}p^{\nu_p(a)-\nu_p(b)} = \frac{s}{t}p^{\text{ord}(\frac{a}{b})}$, where $p \nmid s, t$, hence $\mathbb{Q} \subset Q(\mathbb{Z}_{\langle p \rangle}) \subset \mathbb{Q}$. □

Lemma 1.6.124. *Let R be a DVR. Set $K = Q(R)$. Let $\mathfrak{m} = \langle t \rangle$ be the maximal ideal in R. If $z = \frac{a}{b}t^{\text{ord}(z)} \in K \setminus R$, then $z^{-1} \in \mathfrak{m}$.*

Proof. Note that

$$K \setminus R = \left\{ u\frac{1}{t^n} : u \in R^*, n \geq 0 \right\},$$

hence $z = ut^{-n}$ for suitable $u \in R^*$, $n \geq 0$. Then $z^{-1} = u^{-1}t^n \in \mathfrak{m}$. □

Proposition 1.6.125. *Let S be a DVR containing a subring R which is also a DVR. Set $K = Q(R)$ and suppose $S \subset K$. Let $\mathfrak{m} = \langle t \rangle$ be the maximal ideal in R. If the maximal ideal of S, $\mathfrak{n} = \langle s \rangle$, contains $\mathfrak{m}$, then $S = R$.*

Proof. Since $t \in \mathfrak{n}$, $t = us^n$ for some $n \geq 1$. By irreducibility of t, $t = us$. Let $v \in S^*$. Then $v^{-1} \notin \mathfrak{n} \supset \mathfrak{m}$, hence $v \in R$ by the prior lemma. This means $R^* = S^*$. Thus if $x \in S$, $x = vs^n = ut^n$ for some $n \geq 0$, $v \in S^* = R^*$, hence $x \in R$. □

Definition 1.6.126. An *order function* on a field K is a function $\nu : K \to \mathbb{Z} \cup \{\infty\}$, satisfying:

1. for every $a \in K$, $\nu(a) = \infty \iff a = 0$,
2. for every $a, b \in K$, $\nu(ab) = \nu(a) + \nu(b)$,
3. for every $a, b \in K$, $\nu(a + b) \geq \min(\nu(a), \nu(b))$.

Definition 1.6.127. Let K be a field with an order function ν. We define the ring induced by ν to be the set

$$R_\nu(K) := R_\nu := \nu^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\}) = \{r \in K : \nu(r) \geq 0\}.$$

Define the ideal induced by ν to be the set

$$\mathfrak{m}_\nu := \nu^{-1}(\mathbb{Z}_{\geq 1} \cup \{\infty\}) = \{r \in R_\nu : \nu(r) > 0\} \subset R_\nu$$

Lemma 1.6.128. *Let K be a field with an order function ν. We collect the following facts about R_ν and $\mathfrak{m}_\nu$:*

(i) R_ν is subring of K and hence is an integral domain.

(ii) For every $u \in R_\nu$,

$$u \in R_\nu^* \iff \nu(u) = 0.$$

(iii) $\mathfrak{m}_\nu$ is the unique maximal ideal of R_ν, hence R_ν is local.

(iv) $R_\nu = K$ if and only if ν is trivial. If ν is non-trivial, R_ν is not a field.

Proof. (i) Property 2. ensures that R_ν is closed under multiplication while property 3. ensures that R_ν is closed under addition. $0 \in R_\nu$ by property 1. Note that $\nu(1) = \nu(1 \cdot 1) = \nu(1) + \nu(1)$, hence $\nu(1) = 0$, hence $1 \in R_\nu$. Let $u \in R_\nu^*$. Then $0 = \nu(1) = \nu(uu^{-1}) = \nu(u) + \nu(u^{-1})$, hence $\nu(u^{-1}) = -\nu(u)$ and since $\nu(u) \geq 0$, it follows that $\nu(u^{-1}) = 0$. We thus in particular find that $\nu(-1) = 0$, hence for any $r \in R_\nu$ we have that $\nu(-r) = \nu(-1 \cdot r) = \nu(-1) + \nu(r) = \nu(r)$. It thus follows that $-r \in R_\nu$. We thus get that R_ν is a subring of K.
(ii) " $\Longrightarrow$ ": This was already proven in the proof of (i).
" $\Longleftarrow$ ": Let $u \in R_\nu$ such that $\nu(u) = 0$. For some $v \in K \setminus 0$, $uv = vu = 1$. Then $0 = \nu(1) = \nu(uv) = \nu(u) + \nu(v) = \nu(v)$, hence $v \in R_\nu$, which means u is a unit in R_ν.
(iii) $\mathfrak{m}_\nu = R_\nu \setminus R_\nu^*$, hence it is sufficient to prove that $\mathfrak{m}_\nu$ is an ideal by Proposition 1.6.110. $\mathfrak{m}_\nu$ is closed under addition by property 3. Let $r \in R_\nu$, $x \in \mathfrak{m}_\nu$. Then $\nu(rx) = \nu(r) + \nu(x) \geq 0 + 1 = 1$, hence $rx \in \mathfrak{m}_\nu$. (iv) " $\Longrightarrow$ ": Suppose ν is not trivial. Then there is some $x \in K \setminus 0$ such that $\nu(x) \geq 1$, then $\nu(x^{-1}) \leq 1$, hence $x^{-1} \notin R_\nu$.
" $\Longleftarrow$ ": Suppose ν is trivial. Then for $x \in K$

$$\nu(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{otherwise} \end{cases},$$

in any case $\nu(x) \geq 0$, hence $x \in R_\nu$.

□

Theorem 1.6.129. *Let K be a field and ν a non-trivial order function on K. Then $R = \{z \in K : \nu(z) \geq 0\}$ is a DVR with maximal ideal $\mathfrak{m} := \{z \in R : \nu(z) > 0\}$ such that $Q(R_\nu) = K$. Conversely, if R is a DVR with quotient field K, then* ord $: K \to \mathbb{Z} \cup \{\infty\}$ *is an order function. We thus obtain a one-to-one correspondence between DVR's and order functions.*

Proof. By the above lemma it is sufficient to check that R_ν is a PID. Let $I \subset R_\nu$ be a non-zero ideal. Let $s \in I$ be given such that $\nu(s) = \min(\nu(I))$. Let $x \in I$. For some $q \in K$, $x = qs$. Since

$$\nu(x) = \nu(q) + \nu(s) \implies \nu(q) = \nu(x) - \nu(s) \geq 0,$$

it follows that $s \mid x$ in R_ν, hence $I = \langle s \rangle$. Let $k \in K$. Then either $k \in R_\nu$ or $k^{-1} \in R_\nu$. In the first trivially $k \in Q(R_\nu)$. In the second case $k = (k^{-1})^{-1} = \frac{1}{k^{-1}} \in Q(R_\nu)$.
For the second statement, we have for $a \in R$ that $\text{ord}(a) = \infty$ if and only if $a = 0$. Let $a, b \in K$. Then $a = ut^m$ and $b = vt^l$ for unique $u, v \in R^*$ and $l := \text{ord}(b)$, $m := \text{ord}(a)$.First, we get that

$$ab = uvt^{m+l} \implies \text{ord}(ab) = m + l = \text{ord}(a) + \text{ord}(b),$$

hence ord satisfies property 2. Secondly,

$$a + b = \underbrace{\left(ut^{m-\min(m,l)} + vt^{l-\min(m,l)}\right)}_{q} t^{\min(m,l)}.$$

Note that $q \in R$ since $m - \min(m,l), l - \min(m,l) \geq 0$. We then have that $q = \alpha t^d$ for a unique unit $\alpha \in R^*$ and $d := \text{ord}(q) \geq 0$. This implies that $a + b = \alpha t^{d+\min(l,m)}$, hence

$$\text{ord}(a + b) = d + \min(m,l) \geq \min(m,l) = \min(\text{ord}(a), \text{ord}(b)),$$

proving property 3. □

Remark 1.6.130. One notes from the above proof that the uniformizing parameter for R_ν is $t \in R_\nu$ where $\nu(t) = \min(\nu(\mathfrak{m}_\nu))$.

Lemma 1.6.131. *Let R be a DVR with $K := Q(R)$. If $a_1, \ldots, a_n \in K$ where for some i $\text{ord}(a_i) < \text{ord}(a_j)$ for every $j \neq i$, then $\text{ord}(\sum_1^n a_j) = a_i$ and $\sum_1^n a_j \neq 0$*

Proof. We prove the first statement using induction in $n \geq 2$. Consider first the case $n = 2$. WLOG $m_1 := \text{ord}(a_1) < \text{ord}(a_2) := m_2$. Write $a_1 = u_1 t^{m_1}$ and $a_2 = u_2 t^{m_2}$ for suitable $u_1, u_2 \in R^*$. Then $a_1 + a_2 = (u_1 + u_2 t^{m_2 - m_1}) t^{m_1}$ and since $t \nmid u_1$ and $t \mid u_2 t^{m_2-m_1}$, it follows that $t \nmid u := u_1 + u_2 t^{m_2-m_1}$, hence u is a unit in R, meaning $\text{ord}(a_1) = \text{ord}(a_1 + a_2)$. WLOG $i = n + 1$. Note that $\text{ord}(a_n + a_{n+1}) = \text{ord}(a_{n+1})$, hence setting $a_n' = a_n + a_{n+1}$, we get $\text{ord}(a_n') = \text{ord}(a_{n+1}) < a_j$ for every $j \in \{1, \ldots, n-1\}$. By induction it follows that

$$\text{ord}(\sum_1^{n+1} a_j) = \text{ord}(\sum_1^{n-1} a_j + (a_n + a_{n+1})) = \text{ord}(\sum_1^{n-1} a_i + a_n') = \text{ord}(a_n') = \text{ord}(a_{n+1}).$$

Since $\text{ord}(\sum_1^n a_j) < \text{ord}(a_j) \leq \infty$, it follows that $\sum_1^n a_j \neq 0$ by property 1. of order functions. □

Lemma 1.6.132. *Let R be a DVR with maximal ideal $\mathfrak{m}$, $L := Q(R)$ and a subring K that is a field. Suppose that the composition $\sigma : K \hookrightarrow R \twoheadrightarrow L$ is an isomorphism. Then for any $z \in R$ there is a unique $\lambda \in K$ such that $z - \lambda \in \mathfrak{m}$.*

Proof. **The case $z \in \mathfrak{m}$:** Pick $\lambda = 0$. By the prior lemma for any $\mu \in K \setminus 0$

$$\text{ord}(z - \mu) = \text{ord}(\mu) = 0,$$

hence $z - \mu \notin \mathfrak{m}$.
The case $z \notin \mathfrak{m}$: Then z is a unit in R. Clear σ fixes K, hence it is a K-algebra isomorphism. For some $\lambda \setminus 0$, $\lambda = \sigma(\lambda) = z$, hence $z - \lambda = 0 \in \mathfrak{m}$. Since σ is injective it follows that λ is unique. □

Proposition 1.6.133. *We assume the same setup as the prior lemma. Let t be a uniformizing parameter of R and $z \in R$. For any $n \geq 0$ there are unique $\lambda_0, \ldots, \lambda_n \in K$, $z_n \in R$ such that*

$$z = \sum_0^n \lambda_i t^i + z_n t^{n+1}.$$

Proof. **Existence:** For the case $n = 0$ the statement follows from Lemma 1.6.132. Assuming the statement is true for some $n \geq 0$, we can write

$$z = \sum_0^n \lambda_i t^i + z_n t^{n+1}.$$

If $z_n \in R^* = K \setminus 0$, pick $\lambda_{n+1} := z_n$ and $z_{n+1} := 0$. Otherwise write $z_n = ut^l$, $l \geq 1$ and pick $\lambda_{n+1} := 0$, $z_{n+1} := ut^{l-1}$.
Uniqueness: Suppose there are $\lambda_0, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in K$ and $z_n, w_n \in R$ such that

$$\sum_0^n \lambda_i t^i + z_n t^{n+1} = \sum_0^n \mu_i t^i + w_n t^{n+1},$$

Then $\sum_0^n (\lambda_i - \mu_i)t^i + (z_n - w_n)t^{n+1} = 0$, hence $\text{ord}((\lambda_i - \mu_i)t^i) = \text{ord}((\lambda_j - \mu_j)t^j)$ for every i, j hence $(\lambda_i - \mu_i)t^i = 0$ for every i, implying $\lambda_i = \mu_i$. Similarly one can conclude that $z_n = w_n$. □

For what follows we assume some easy linear algebra that will be covered later

Lemma 1.6.134. *We keep the same setup as the above two results. Then $\dim_K \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ for every $n \geq 0$.*

Proof. We use induction in $n \geq 0$. We first prove that

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} = \{\lambda t^n + \mathfrak{m}^{n+1} : \lambda \in K\}.$$

This is clear since any element in $\mathfrak{m}^n$, is equal to $\lambda t^n + zt^{n+1}$ for some $\lambda \in K$, $z \in R$ as by Lemma 1.6.131 $\text{ord}(\sum_0^n \lambda_i t^i) = l$ where l is the smallest index such that $\lambda_l \neq 0$. Hence in the image of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ such an element is given by $\lambda t^n + zt^{n+1} + \mathfrak{m}^{n+1} = \lambda t^n + \mathfrak{m}^{n+1}$. It follows that

$$\sigma : K \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$
$$\lambda \mapsto \lambda t^n + \mathfrak{m}^{n+1}$$

is a surjective K-algebra map. Suppose $\lambda t^n \in \mathfrak{m}^{n+1}$. Then $\text{ord}(\lambda) > 0$, hence $\lambda = 0$. We thus conclude that $\mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq K$, meaning $\dim\ \mathfrak{m}^n/\mathfrak{m}^{n+1} = \dim\ K = 1$. □

Lemma 1.6.135. *We keep the setup from the prior results. For each $n \geq 0$,* $\dim\ R/\mathfrak{m}^n = n$. *It follows that* $\text{ord}(z) = \dim\ R/\langle z \rangle = \dim\ R/\mathfrak{m}^{\text{ord}(z)}$ *for each $z \in R$.*

Proof. We prove the result by induction in n. The base case is trivial. By Lemma 1.5.9 and Lemma 1.7.12 and the induction hypothesis it follows that

$$\dim\ R/\mathfrak{m}^{n+1} = \dim\ \mathfrak{m}^n/\mathfrak{m}^{n+1} + \dim\ R/\mathfrak{m}^n = n + 1,$$

where we also use the prior lemma. We now that $z = \lambda t^{\text{ord}(z)}$, hence $\langle z \rangle = \langle t \rangle^{\text{ord}(z)} = \mathfrak{m}^{\text{ord}(z)}$, hence $\text{ord}(z) = \dim\ R/\mathfrak{m}^{\text{ord}(z)}$. □

1.6.12 Torsion & Annihilators

Definition 1.6.136. An element of a ring R is *regular* if it is both not a left and right zero divisor.

Definition 1.6.137. Consider a left R-module M. $m \in M$ is a *torsion element* if there is a regular element $r \in R$ such that

$$rm = 0.$$

Definition 1.6.138. We define the *annihilator* of a subset X of a left R-module M to be the set

$$\text{Ann}(X) := \text{Ann}_R(X) := \{r \in R : rx = 0 \text{ for all } x \in X\}$$

Lemma 1.6.139. *The annihilator of subset $X \subset {}_RM$ is a left ideal in R.*

Proof. Indeed, if $r_1, r_2 \in \text{Ann}(X)$ and $a \in R$, then given $x \in X$,

$$(r_1 + ar_2)x = r_1x + ar_2x = 0.$$

□

1.6.13 Graded Rings

Definition 1.6.140. Let R be a ring. A *grading* on R is pair consisting of a monoid M an a family of additive subgroups $\{R_i : i \in M\}$ of R such that

$$R_i R_j \subset R_{ij}$$

such that

$$R = \sum_{i \in M} R_i$$

with $\sum_{i \in M} R_i$ being direct. A triplet consisting of a ring R with a grading $(M, \{R_i\})$ is called a *graded ring with respect to* M. A *graded ring* is a graded ring with respect to $(\mathbb{N}, +)$.

Definition 1.6.141. An element $f \in R$ in a graded ring $(R, M, \{R_i\})$ is said to be i-*homogeneous* if $f \in R_i$. When $M = (\mathbb{N}, +)$ we say that an element is *homogeneous of degree* n if it is n-homogeneous.

Remark 1.6.142. Note that 0 is i-homogeneous for each $i \in M$.

Lemma 1.6.143. *Consider a graded ring $(R, M, \{R_i\})$. Then for each $f \in R$ there is a unique decomposition,*

$$f = \sum_{i \in M} f_i$$

with $f_i \in R_i$. In other words, there is a unique subset $\{i_1, \ldots, i_n\} \subset M$ and unique non-zero $f_{i_k} \in R_{i_k}$ with

$$f = \sum_{1}^{n} f_{i_k}$$

Remark 1.6.144. We call $f_{i_1}, \ldots, f_{i_n}$ the homogeneous components of f.

Proof. This is by definition. □

Lemma 1.6.145. *Consider a graded ring R. R_0 is a subring of R.*

Proof. Note first that R_0 is an additive subgroup of R, hence it suffices to show that R_0 is a multiplicative submonoid of R. Clearly if $f, g \in R_0$, then $fg \in R_{0+0} = R_0$. Moreover, $1 \in R_n$ for some n, and $1 = 1 \cdot 1 \in R_n R_n = R_{2n}$, hence $1 \in R_n \cap R_{2n}$, implying $n = 2n$, hence $n = 0$. □

Proposition 1.6.146. *For a graded ring R, R_n is an (R_0, R_0)-bimodule for any n.*

Proof. It is clear that $fg \in R_n$ for $f \in R_0$ and $g \in R_n$, hence R_n is a left (and dually a right) R_0-submodule of ${}_{R_0}R_{R_0}$. □

Definition 1.6.147. An ideal I is called *homogeneous* of a graded ring $(R, M, \{R_i\})$ if the homogeneous if for each $f \in I$ the homogeneous components of f are in I.

Remark 1.6.148. Note that I is a left (and right) R_0-submodule of $(R, \mathbb{N}, \{R_n\})$, hence $I \cap R_n$ is a left (and right) R_0-submodule of R_n. This is called the *n-homogeneous part of I*. We can write

$$I = \bigoplus_{n\in\mathbb{N}} (R_n \cap I).$$

Lemma 1.6.149. *Consider a graded ring R and a two-sided ideal $I \subset R$. Then R/I is a graded ring with grading*

$$R/I = \bigoplus_{n\in\mathbb{N}} (R_n + I)/I$$

where I_n is the homogeneous part of I.

Proof. Indeed, for $r_n + x + I \in (R_n + I)/I$ and $r_m + y + I \in (R_m + I)/I$, then

$$(r_n + x)(r_m + y) + I = r_n r_m + (xr_m + yr_n + xy) + I \in (R_{nm} + I)/I.$$

□

Remark 1.6.150. Note that $R/I \simeq \bigoplus_{n\in\mathbb{N}} R_n/I_n$ where I_n is the n-homogeneous part of I.

Definition 1.6.151. Consider a graded ring R. We define $R_+ := \sum_{d>0} R_n$.

Lemma 1.6.152. *Given a graded ring R, R_+ defines a two-sided ideal in R.*

Proof. Indeed, if $f \in R$ and $\sum_r^n f_i \in R_+$, then $f \in R_l$ for some l, hence

$$f f_i \in R_{i+l},$$

meaning

$$f \sum_r^n f_i = \sum_r f f_i \in \sum_r^n R_{i+l} \subset R_+.$$

□

Definition 1.6.153. In a graded ring R, given an element $f \in R \setminus 0$ with homogeneous components f_i, we define *the degree of f* to be

$$\deg\ f := \max(\{i \in \mathbb{N} : f_i \neq 0\}).$$

We thus have a function

$$\deg : R \setminus 0 \to \mathbb{N}$$
$$f \mapsto \deg\ f$$

Remark 1.6.154. We can extend the degree function to all of R. Our conventions to what deg 0 is will differ in various contexts: Sometimes we extend deg to a function $R \to \mathbb{N} \cup \{\infty\}$, hence deg $0 = \infty$, sometimes we take deg 0 to be 0, some deg is extended to a multivalued function with deg $0 = \mathbb{N}$.

Lemma 1.6.155. *Let R be a graded ring, and $f, g \in R$, then*

$$\text{deg } (f + g) \leq \max(\text{deg } f, \text{deg } g)$$

holding with equality when deg $f \neq$ deg g.

Proof. Set $d := \max(\text{deg } f, \text{deg } g)$. Denote the homogeneous components of f and g by f_i respectively g_i. The homogeneous components of $f + g$ are $f_i + g_i$. Then $f_i = g_i = 0$ for every $i > d$ implies $f_i + g_i = 0$ for every $i > d$. It thus follows that deg $f + g \leq d$.
If deg $f <$ deg g, then $f_d + g_d = g_d$. It follows that deg $(f + g) = d$. □

Lemma 1.6.156. *Let R be a graded ring and a domain, and $f, g \in R \setminus 0$. Then*

$$\text{deg } fg = \text{deg } f + \text{deg } g.$$

Proof. Let f_i and g_i denote the homogeneous components of f resp. g and let d denote deg f+deg g. The homogeneous components of fg are $\sum_{i,j \in \mathbb{N}: i+j=k} f_i g_i$. When $i <$ deg f and $i + j = d$, then $j >$ deg g, hence $f_i g_j = 0$. Similarly, when $j <$ deg g and $i + j = d$, then $f_i g_j = 0$. It follows that

$$(fg)_d = f_{\text{deg } f} g_{\text{deg } g} \neq 0.$$

For $k > d$, when $i + j = k$, then $i > d$ or $j > d$. It follows that $f_i g_j = 0$, hence $(fg)_k = 0$.
We conclude that deg $fg = d$. □

Remark 1.6.157. When R is not a domain, deg $fg \leq d$ (when $fg \in R \setminus 0$ or deg 0 is defined to be 0), which is easily seen from the last argument of the above proof.

Lemma 1.6.158. *Let R be a graded ring. Let $f, g \in R \setminus 0$ be given. The following are equivalent*

1. *fg is not homogeneous.*
2. *f has distinct homogeneous components f_i, f_j for which $f_i g_n \neq 0$ and $f_j g_n \neq 0$ for some homogeneous component g_n of g, or g has distinct homogeneous components g_k, g_l for which $g_k f_m \neq 0$ and $g_k f_m \neq 0$ for some homogeneous component f_m of f.*

Proof. "1. $\implies$ 2.": Suppose first fg is not homogeneous. Then $fg \neq 0$. Hence $fg_k \neq 0$ for some k. Then $f_i g_k \neq 0$ for some i. If there is an f_j distinct from f_i that also satisfies $f_j g_k \neq 0$, then we are done. Suppose this is note the case. Then there is an g_h with $f_i g_h \neq 0$, for otherwise $fg = f_i g_k$.
"2. $\implies$ 1.": Suppose then that the second condition is met. In the first case, let k largest index for which there is a pair of homogeneous components f_i and f_j of f satisfying $f_i g_k, f_j g_k \neq 0$. Assume that i and j is the second largest respectively largest index satisfying the condition. Then one observes that

$$(fg)_{i+k} = f_i g_k \text{ and } (fg)_{j+k} = f_j g_k$$

hence fg has a pair of distinct homogeneous components and is thus not homogeneous. □

Corollary 1.6.159. *Let R be a graded ring and $x, f \in R$ with $x \neq 0$ homogenous with x not being a right zero divisor. If xf is homogeneous then so is f.*

Corollary 1.6.160. *Let R be an integral domain and a graded ring. For $f, g \in R \setminus 0$, if fg is homogeneous so is f and g.*

Lemma 1.6.161. *In a commutative graded ring R a homogeneous ideal $\mathfrak{p} \subset R$ is prime if and only if for every pair of homogeneous elements $a, b \in R$ if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*

Proof. The first statement is trivial. Suppose for every pair of homogeneous elements $a, b \in R$ if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Let $f, g \in R$ with $fg \in \mathfrak{p}$. Write $f = \sum_m^n f_i$ and $g = \sum_k^l g_i$ be the homogeneous decompositions. Then fg has homogeneous components $\sum_{i,j \in N : i+j=k} f_i g_j$. Note that for some $k \leq \deg\ f$ and $l \leq \deg\ g$, $f_i g_j = 0$ for $i + j > k + l$ and

$$\sum_{i,j \in \mathbb{N} : i+j=k+l} f_i g_j = f_k g_l \in \mathfrak{p},$$

Then $f_k \in \mathfrak{p}$ or $g_l \in \mathfrak{p}$. Assume $f_k \in \mathfrak{p}$. Then $f_k g \in \mathfrak{p}$, hence $(f - f_k)g \in \mathfrak{p}$. By induction in the number of terms in the product, it follows that $f - f_k \in \mathfrak{p}$ or $g \in \mathfrak{p}$. In the second case we are done. In the first case $f = (f - f_k) + f_k \in \mathfrak{p}$. The case $g_l \in \mathfrak{p}$ follows from a symmetric argument. □

Lemma 1.6.162. *Consider a left ideal I in a graded ring R. Then I is homogeneous if and only if it is generated from the left by a set of homogeneous elements.*

Proof. Indeed, if I is a left ideal generated by homogeneous elements from the left, then the homogeneous decomposition of any $f \in I$ has homogeneous components that must correspond some finite subset of the generators by uniqueness of homogeneous decomposition.
Conversely if I is homogeneous, I is generated from the left by the set

$$\{a \in R : a \text{ is a homogenous component of some } f \in I\}.$$

□

Lemma 1.6.163. *Consider a left ideal $I = \sum_1^n Ra_i$ finitely generated from the left in a graded ring R. I is homogeneous if and only if it is generated by some finite set of homogeneous elements.*

Proof. If I is generated by a finite set of homogeneous elements, then it is homogeneous by the prior lemma. If I is homogeneous it is generated by the set of homogeneous components of all of its generators which by assumption is a finite set. □

Proposition 1.6.164. *If I is a homogeneous ideal of a graded, commutative ring R, then* $\text{rad}(I)$ *is homogeneous.*

Proof. Let $f = \sum_0^d f_i \in \text{rad}(I)$. We must prove that $f_i \in \text{rad}(I)$. Note that $f^n = f_d^n + r \in I$ where deg $r < dn$. Then $f_d^n \in I$, hence $f_d \in \text{rad}(I)$. We thus have that $f - f_d \in \text{rad}(I)$ by induction in degree it follows that $f \in \text{rad}(I)$. □

Lemma 1.6.165. *If $\{I_\alpha\}_{\alpha \in A}$ is a family of homogeneous left ideals in a graded ring R, then $\sum_{\alpha \in A} I_\alpha$ and $\bigcap_\alpha I_\alpha$ are homogeneous left ideals in R.*

Proof. Indeed, for $(f_\alpha) \in \bigoplus_{\alpha \in A} I_\alpha$, we may for some $d \geq 0$ write $(f_\alpha) = (\sum_0^d f_{\alpha,i})$, where $f_{\alpha,i}$ is homogeneous of degree i for each i and α. Then

$$\sum_{\alpha \in A} f_\alpha = \sum_0^d \sum_{\alpha \in A} f_{\alpha,i}.$$

Note that since each I_α is homogeneous $f_{\alpha,i} \in I_\alpha$. Then $\sum_{\alpha \in A} f_{\alpha,i} \in \sum_{\alpha \in A} I_\alpha$, which means $\sum_{\alpha \in A} I_\alpha$ is homogeneous.
Consider $f = \sum_0^d f_i \in \bigcap_{\alpha \in A} I_\alpha$. Then for each $\alpha \in A$, $f \in I_\alpha$, hence $f_i \in I_\alpha$. We thus have that $f_i \in \bigcap_{\alpha \in A} I_\alpha$, which means $\bigcap_{\alpha \in A} I_\alpha$ is homogeneous. □

Lemma 1.6.166. *Let $I, J \subset R$ be homogeneous ideals in a graded ring R. Then IJ is homogeneous.*

Proof. Let $f = \sum_0^d f_i \in I$ and $g = \sum_0^e g_j \in J$. Then each $f_i \in I$ and each $g_j \in J$, meaning $f_i g_j \in IJ$ for each $0 \leq i \leq d$ and $0 \leq j \leq e$. Then since $fg = \sum_{k=0}^{d+e} \sum_{i,j:i+j=k} f_i g_j$ where $\sum_{i,j:i+j=k} f_i g_j$ is homogeneous of degree k for each $0 \leq k \leq d + e$ we get that IJ is homogeneous. □

We consider a way of extending a grading to certain localizations of a graded ring

Lemma 1.6.167. *Let R be a graded ring. Then*

$$H_R := \{x \in R : x \text{ homogeneous}\}$$

is a multiplicative set.

Proof. This is trivial. □

Lemma 1.6.168. *Let R be a graded ring and X a multiplicative subset contained in H_R. We define a triple on $(X^{-1}R, \mathbb{Z}, \{R_n\})$ with*

$$R_n := \left\{\frac{a}{x} : a \in R_i, x \in R_j, i - j = n\right\}$$

for $n < 0$. This defines a grading.

Proof. We check that R_n is an additive subgroup for $n < 0$. Indeed, for $\frac{a}{x}, \frac{b}{y} \in R_n$, we have $a \in R_{l_1}$ and $x \in R_{k_1}$ with $l_1 - k_1 = n$ and $b \in R_{l_2}$ and $y \in R_{k_2}$ with $l_2 - k_2 = n$, then $bx \in R_{l_2+k_1} = R_{k_2+k_1+n}$ and $ay \in R_{l_1+k_2} = R_{k_1+k_2+n}$, hence $bx + ay \in R_{k_1+k_2+n}$ and $xy \in R_{k_1+k_2}$. Since $k_1 + k_2 + n - (k_1 + k_2) = n$, it follows that $\frac{a}{x} + \frac{b}{y} \in R_n$.
Let $\frac{f}{x} \in X^{-1}R$. Then

$$\frac{f}{x} = \sum_{n\in\mathbb{N}} \frac{f_n}{x} \in \sum_{n-\deg\ x\in\mathbb{N}} R_k.$$

Suppose $\frac{a}{x} \in R_{n-\deg\ x} \cap R_{m-\deg x}$. Then $a \in R_n \cap R_m$, hence $n = m$ or $a = 0$, hence $n - \deg\ x = m - \deg\ x$ or $\frac{a}{x} = 0$. □

Lemma 1.6.169. *Given a homogeneous prime ideal $\mathfrak{p}$ in a commutative graded ring R, define $R_{(\mathfrak{p})}$ to be the localization of R with respect to $X := R_0 \cap R \setminus \mathfrak{p}$. Then $R_{(\mathfrak{p})}$ is a graded, local ring whose maximal ideal is homogeneous.*

Proof. Suppose $\frac{a}{f} \in R_{(\mathfrak{p})}$ is not a unit in $R_{\mathfrak{p}}$, then $a \in \mathfrak{p}$, hence $a \notin X$ (note that X is clearly saturated), hence $\frac{a}{f}$ is not a unit in $R_{(\mathfrak{p})}$. Conversely, if $\frac{a}{f} \in R_{(\mathfrak{p})}$ is a unit in $R_{\mathfrak{p}}$, then there is a $\frac{b}{g} \in R_{\mathfrak{p}}$ with

$$\sum_{n\in\mathbb{N}} fg_n = \sum_{n\in\mathbb{N}} ab_n.$$

In particular, since $a, f \in R_0$,

$$fg_0 = ab_0 \implies \frac{a}{f}\frac{b_0}{g_0} = 1 \implies f \in \left(R_{(\mathfrak{p})}\right)^*.$$

It thus follows that the set of non-units in $R_{(\mathfrak{p})}$ is the ideal $R_{(\mathfrak{p})} \cap \mathfrak{m}_{\mathfrak{p}}$, hence $R_{(\mathfrak{p})}$ is local with maximal ideal $\mathfrak{m}_{(\mathfrak{p})}$ say. Note that $\frac{a}{f} \in \mathfrak{m}_{(\mathfrak{p})}$ implies $a \in \mathfrak{p}$, hence $a_i \in \mathfrak{p}$ for each i. It thus follows that $\frac{a_i}{f} \in \mathfrak{m}_{(\mathfrak{p})}$, hence $\mathfrak{m}_{(\mathfrak{p})}$ is homogeneous. □

1.7 Vector Spaces

Definition 1.7.1. A *vector space* is a module over a field K.

For a field K, we define $\mathrm{Vect}_K := \mathrm{Mod}_K$.

1.7.1 Finite Dimensional Vector Spaces

Lemma 1.7.2. *Assuming the Axiom of Choice, any vector space V over a field K has a basis.*

Proof. Let X be the set of all linearly independent subsets of V. Let C be a chain in X. Suppose

$$\sum_1^n a_i v_i = 0$$

where

$$v_1, \ldots, v_n \in \bigcup_{\mathcal{V} \in C} \mathcal{V}$$

Then for each i, $v_i \in \mathcal{V}_i$ for some $\mathcal{V}_\rangle \in C$. Then for each i, $\mathcal{V}_\rangle \subset \mathcal{V}_m$ for some $m \in \{1, \ldots, n\}$. It thus follows that $a_i = 0$ for every i. Then $\bigcup_{\mathcal{V} \in C} \mathcal{V} \in X$ and is an upper bound of C. Then X has a maximal element $\mathcal{V}$. A maximally linear independent set is a basis, hence V has a basis. Indeed, given any $v \in V$ either $v \in \mathcal{V}$ or there are $a_1, \ldots, a_{n+1} \in K$ with $a_{n+1} \in K \setminus 0$ and $v_1, \ldots, v_n$ such that

$$a_{n+1} v + \sum_1^n a_i v_i = 0$$

since otherwise $\mathcal{V} \cup \{v\} \supsetneq \mathcal{V}$ would be linearly independent over K, contradicting maximality. We thus have

$$v = \sum_1^n -a_{n+1}^{-1} a_i v_i.$$

This means $\mathcal{V}$ generates V, so $\mathcal{V}$ is a basis. □

Lemma 1.7.3. *Consider a vector space V over a field K, if $\{v_1, \ldots, v_n\} \subset V$ are linearly independent over K and $\{w_1, \ldots, w_m\}$ generates V over K, then*

$$n \leq m.$$

Proof. If $v \neq 0$

$$v = \sum_1^m a_i w_i$$

for $a_i \in K$ where wlog $a_1 \neq 0$. Then

$$w_1 = a_1^{-1} v + \sum_2^m a_1^{-1} a_i w_i$$

hence $\{v, w_2, \ldots, w_m\}$.
Assume that if $\{v_1, \ldots, v_n\}$ are linearly independent that $n \leq m$. Suppose $\{v_1, \ldots, v_{n+1}\}$ are linearly independent. Using linear independence of $\{v_1, \ldots, v_n\}$ over K, we find that

$$\{v_1, \ldots, v_n, w_{n+1}, \ldots, w_m\}$$

which is a set of cardinality m. Then

$$v_{n+1} = \sum_1^n b_i v_i + \sum_1^{m-n} c_i w_{n+i}$$

for suitable $b_i, c_j \in K$. Then $n < m$ and $c_k \neq 0$ for some k for otherwise $\{v_1, \ldots, v_{n+1}\}$ are linearly dependent. Then $\{v_1, \ldots, v_{n+1}, w_{n+2}, \ldots, w_m\}$, a set of cardinality m, generates V over K. It thus follows that since

$$\{v_1, \ldots, v_{n+1}\} \subset \{v_1, \ldots, v_{n+1}, w_{n+2}, \ldots, w_m\}.$$

which means $n + 1 \leq m$. □

Corollary 1.7.4. *If $\mathcal{V}, \mathcal{V}'$ are finite bases of the same vector space, then*

$$\#\mathcal{V} = \#\mathcal{V}'.$$

Definition 1.7.5. Given a vector space V over a field K with finite basis $\mathcal{V}$, we define the *dimension of V over K* to be

$$\dim\ V := \dim_K\ V := \#\mathcal{V}.$$

Such a vector space is called a *finite dimensional vector space*. The category of such vector spaces is denoted $\mathrm{Vect}_K^{\mathrm{fd}}$. When V does not have a finite basis we define

$$\dim\ V := \infty$$

Remark 1.7.6. We see that dim defines an invariant of vector spaces over K valued in $(\mathbb{Z} \cup \{\infty\})$. Indeed, if $l : V \hookrightarrow W$ and B is a basis for B, then if

$$\sum_{v \in B} a_v l(v) = 0$$

we get that

$$0 = l\left(\sum_{v \in B} a_v v\right) \implies \sum_{v \in B} a_v v = 0 \implies a_v = 0 \text{ for all } v,$$

hence $l(B)$ is linearly independent over K. Then if C is a basis of W over K, it follows that $\#B = \#l(B) \leq \#C$, hence

$$\dim\ V \leq \dim\ W.$$

Definition 1.7.7. Let V be an n-dimensional vector space over a field K with basis $\mathcal{V} = \{v_1, \ldots, v_n\}$. Let $\mathcal{W} = \{w_1, \ldots, w_n\} \subset V$. Write $w_i = \sum_1^n a_{ij} v_j$ for suitable (unique!) $a_{ij} \in K$ and consider the matrix

$${}_{\mathcal{V}}T_{\mathcal{W}} := (a_{ij})^T \in M_n(K).$$

When ${}_{\mathcal{V}}T_{\mathcal{W}}$ is invertible we call it *the basis transformation of $\mathcal{V}$ to $\mathcal{W}$* or *a change-of-basis*

Remark 1.7.8. We canonically identify ${}_{\mathcal{V}}T_{\mathcal{W}}$ with an endomorphism on V:

$${}_{\mathcal{V}}T_{\mathcal{W}} : V \to V, v = \sum_1^n \alpha_i v_i \mapsto \sum_1^n \left(\sum_1^n a_{ji} a_j\right) v_i$$

Note that

$${}_{\mathcal{V}}T_{\mathcal{W}} v_i = \sum_1^n \left(\sum_1^n a_{kj} \delta_{ki}\right) v_j = \sum_1^n a_{ij} v_j = w_i$$

Lemma 1.7.9. *Let V be a vector space over a field K. Let $\mathcal{V}$ be a basis of V and $l \in \mathrm{Aut}_K(V)$. Then $l(\mathcal{V})$ is basis of V*

Proof. Let $v \in V$. Then

$$l^{-1}(v) = \sum_1^n a_i v_i$$

for some $v_1, \ldots, v_n \in \mathcal{V}$ and $a_1, \ldots, a_n \in K$. It follows that

$$v = \sum_1^n a_i l(v_i).$$

Suppose $l(v_1), \ldots, l(v_n) \in l(\mathcal{V})$ are given with $a_1, \ldots, a_n \in K$ such that

$$\sum_1^n a_i l(v_i) = 0.$$

Then

$$\sum_1^n a_i v_i = l^{-1}\left(\sum_1^n a_i l(v_i)\right) = 0,$$

hence $a_i = 0$ for each i. □

Theorem 1.7.10. *Let V be an n-dimensional vector space over a field K with basis $\mathcal{V} = \{v_1, \ldots, v_n\}$. Consider $\mathcal{W} = \{w_1, \ldots, w_n\} \subset V$. Then $\mathcal{W}$ is a basis of V over K if and only if ${}_{\mathcal{V}}T_{\mathcal{W}}$ is invertible.*

Proof. " $\Longrightarrow$ ": write $v_i = \sum_1^n b_{ij} w_j$. Note that $w_i =_{\mathcal{V}} T_{\mathcal{W}} v_i =_{\mathcal{V}} T_{\mathcal{W}\mathcal{W}} T_{\mathcal{V}} w_i$, and that $v_i =_{\mathcal{W}} T_{\mathcal{V}} w_i =_{\mathcal{W}} T_{\mathcal{V}\mathcal{V}} T_{\mathcal{W}} v_i$. Since a module homomorphism is uniquely characterized by its behavior on the basis elements, it follows that

$${}_{\mathcal{V}}T_{\mathcal{W}\mathcal{W}}T_{\mathcal{V}} =_{\mathcal{W}} T_{\mathcal{V}\mathcal{V}}T_{\mathcal{W}} = I_n,$$

hence ${}_{\mathcal{V}}T_{\mathcal{W}}$ is invertible with inverse ${}_{\mathcal{W}}T_{\mathcal{V}}$.
" $\Longleftarrow$ ": by the prior lemma, $\mathcal{W} =_{\mathcal{V}} T_{\mathcal{W}}(\mathcal{V})$ is a basis of V. □

Lemma 1.7.11. *(Rank-Nullity) Let V be a finite dimensional vector space and W any vector space. Then for a linear map $l : V \to W$,*

$$V \simeq \ker\, l \oplus \operatorname{im} l.$$

In particular,

$$\dim\, V = \dim\, \ker\, l + \dim \operatorname{im} l.$$

Proof. We have a short exact sequence

$$0 \longrightarrow \ker\, l \hookrightarrow V \twoheadrightarrow \operatorname{im} V \longrightarrow 0$$

with id_V being a retraction of $\ker\, l \hookrightarrow V$. By the splitting lemma, We are done □

Lemma 1.7.12. *(Generalized Rank-Nullity) Given a short exact sequence of vector spaces*

$$0 \longrightarrow V \xrightarrow{\rho} V' \xrightarrow{\rho'} V'' \longrightarrow 0$$

we have that

$$\dim\, V' = \dim\, V + \dim\, V''$$

Proof. This follows from Lemma 1.5.4 and rank-nullity. Indeed,

$$V' = \ker\, \rho' \oplus \operatorname{im} \rho' = \operatorname{im} \rho \oplus V'' \simeq V \oplus V''$$

where the isomorphism is induced by ρ considered as a map onto its image. □

Proposition 1.7.13. *For sequence of vector spaces*

$$0 \longrightarrow V_1 \xrightarrow{\rho_1} V_2 \xrightarrow{\rho_2} V_3 \xrightarrow{\rho_3} V_4 \longrightarrow 0$$

we have that dim V_4 = dim V_3 − dim V_2 + dim V_1.

Proof. Set W := im ρ_2 = ker ρ_3. Then

$$0 \longrightarrow V_1 \xrightarrow{\rho_1} V_2 \xrightarrow{\rho_2} W \longrightarrow 0$$

and

$$0 \longrightarrow W \hookrightarrow V_3 \xrightarrow{\rho_3} V_4 \longrightarrow 0$$

are exact, hence by the above lemma dim V_2 = dim V_1 + dim W implying dim W = dim V_2 − dim V_1. Moreover,

$$\text{dim } V_3 = \text{dim } V_4 + \text{dim } W = \text{dim } V_4 + \text{dim } V_2 - \text{dim } V_1,$$

hence dim V_4 = dim V_3 − dim V_2 + dim V_1. □

Lemma 1.7.14. *Let $U \subset W \subset V$ be vector spaces where V/U finite dimensional. Then*

$$\text{dim } V/U = \text{dim } V/W + \text{dim } W/U,$$

In particular we get that V/W and W/U are finite dimensional.

Proof. This follows directly from the above proposition and Lemma 1.5.9. □

Lemma 1.7.15. *Let*

$$0 \xrightarrow{\rho_0} V_1 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_{n-1}} V_n \xrightarrow{\rho_n} 0$$

be an exact sequence of finite dimensional vector spaces. Then $\sum_1^n (-1)^i$ dim $V_i = 0$

Proof. In the case $n = 1$, it's easy to see that $V_1 = 0$. Denote the first 0-map In general for a finite sequence of elements in an additive group, $a_0, \ldots, a_n$, say $\sum_1^1 (-1)^i (a_{i-1} + a_i) = -a_0 + (-1)^{n-1} a_n$. By the rank nullity theorem and exactness we have for each $i \in \{1, \ldots, n\}$ that dim V_i = dim ker ρ_i + dim im ρ_i = dim im ρ_{i-1} + dim im ρ_i. Hence picking a_i = dim im ρ_i, it follows that

$$\sum_1^n (-1)^i \text{dim } V_i = \sum_1^n (-1)^i (a_{i-1} + a_i) = -a_0 + (-1)^n a_n = -\text{dim im } \rho_0 + (-1)^n \text{dim im } \rho_n$$
$$= -\text{dim im } 0 + (-1)^n \text{dim im } 0 = 0.$$

□

Definition 1.7.16. Consider a vector space V and subspace $W \subset V$. We define the *codimension of W in V* to be

$$\text{codim}^V\ W := \text{codim}\ W = \dim\ V - \dim\ W.$$

Remark 1.7.17. Given a basis $\mathcal{V}$ of V, there is a basis $\mathcal{W} \subset \mathcal{V}$ of W. It thus follows that we can express codim W as the cardinality of the set $\mathcal{V} \setminus \mathcal{W}$.

Lemma 1.7.18. *Given a vector space V with subspaces $W' \subset W \subset V$. Then*

$$\text{codim}\ W \leq \text{codim}\ W'.$$

Proof. Indeed since $\dim\ W \geq \dim\ W'$, one sees that

$$\text{codim}\ W = \dim\ V - \dim\ W \leq \dim\ V - \dim\ W' = \text{codim}\ W'.$$

□

Lemma 1.7.19. *Let V be a finite dimensional vector space and $V_1, \ldots, V_n$ be subspaces. Then*

$$\text{codim} \bigcap_1^n V_i \leq \sum_1^n \text{codim}\ V_i.$$

Proof. Pick a basis of V, B and bases of $V_1, \ldots, V_n, \bigcap_1^n V_i$, denoted respectively $\mathcal{V}_1, \ldots, \mathcal{V}_n, \mathcal{V} \subset B$. Then

$$\text{codim} \bigcap_1^n V_i = \#\left(B \setminus \mathcal{V}\right) = \#\left(\bigcup_1^n B \setminus \mathcal{V}_i\right) \leq \sum_1^n \#\left(B \setminus \mathcal{V}_i\right) = \sum_1^n \text{codim}\ V_i.$$

□

1.7.2 Dual Vector Spaces

Definition 1.7.20. Let V be a vector space over a field K. The *dual vector space of V (over K)* is

$$\text{Dual}(V) := V' := \text{Vect}_K(V, K)$$

Remark 1.7.21. Note that this assignment is functorial. In particular, it is a contravariant endofunctor on Vect_K.

Lemma 1.7.22. *Suppose V is a finite dimensional vector space over K. Then*

$$V \simeq V'$$

Proof. WLOG $V \simeq K^n$ for some n, so it suffices to prove that $K^n \simeq \text{Vect}_K(K^n, K)$. K^n is equipped with a bilinear map

$$\langle \cdot, \cdot \rangle : K^n \times K^n \to K$$
$$(v, w) \mapsto \sum_1^n v_i w_i$$

For each $v \in K^n$ we thus have a K-linear map

$$l_v : K^n \to K$$
$$w \mapsto \langle v, w \rangle$$

We thus define a K-linear map

$$L : V \to V'$$
$$v \mapsto l_v$$

with inverse

$$L^{-1} : V' \to V$$
$$l \mapsto (l(e_1), \ldots, l(e_n))$$

where e_i is the i'th standard basis vector of K^n. □

Proof. Pick a basis $\{v_1, \ldots, v_n\}$ of V. Define for each $i \in \{1, \ldots, n\}$,

$$l_{v_i} : V \to K$$

by

$$l_{v_i}(v_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

we verify that $\{l_{v_1}, \ldots, l_{v_n}\}$ defines a basis for V' which would show that dim $V' = n$, implying $V \simeq V'$. Suppose

$$\sum_1^n b_i l_{v_i} = 0,$$

then

$$\sum_1^n b_i v_i = \sum_1^n \left(\sum_1^n b_j l_{v_j}(v_i) \right) v_i = 0$$

hence $b_i = 0$ for each i. Consider an arbitrary linear map $l \in V'$. Then for each $w = \sum_1^n a_i v_i \in V$

$$l(v) = \sum_1^n a_i l(v_i) = \sum_1^n l(v_i) l_{v_i}(w) = \left(\sum_1^n l(v_i) l_{v_i} \right)(w)$$

□

Remark 1.7.23. The basis constructed on V' in the second proof, is called the *dual basis on V (with respect to the basis $\{v_1, \ldots, v_n\}$).*

Proposition 1.7.24. *Given a field K and denoting the identity functor on* $\mathrm{Vect}_K^{\mathrm{fd}}$ *by* Id*,*

$$\mathrm{Dual} \cong \mathrm{Id}$$

Proof. Initially, we do not need the assumption of working with only finite dimensional vector spaces to define a natural transformation with injective components. We define a natural transformation ev : Id $\Rightarrow$ Dual to have components

$$\{ev^V : V \to V'', v \mapsto ev_v\}$$

consider a linear map $L : V \to W$. Let $v \in V$ and $l \in W''$. Then

$$\begin{aligned}\left(((L^*))^* \,\mathrm{ev}^V(v)\right)(l) = ((L^*)^*(\mathrm{ev}_v))(l) = (\mathrm{ev}_v L^*)(l) = \mathrm{ev}_v(lL) = l(L(v)) \\ = \mathrm{ev}_{L(v)}(l) = \left(\mathrm{ev}^W L(v)\right)(l).\end{aligned}$$

In other words the square

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{ev}^V} & V'' \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle (L^*)^*} \\ W & \xrightarrow{\mathrm{ev}^W} & W'' \end{array}$$

commutes, hence ev is a natural transformation. Suppose $V \neq 0$. Then V is not the zero object in Vect_K, hence $V' \neq 0$. Then if $v \neq 0$, then $l(v) \neq 0$ for some $l \in V'$, hence ev^V is injective for an arbitrary vector space V.
Suppose V is a finite dimensional vector space and consider a K-linear map $\epsilon \in V''$. When $\epsilon(l) = 0$ for every $l \in V'$, then $\epsilon = \mathrm{ev}_0$. Suppose this is not the case. Then there is some $l \in V' \setminus 0$ for which $\epsilon(l) \neq 0$. Moreover, there is a $v \in V$ with $l(v) = \epsilon(l) \neq 0$. $\dim \ker \epsilon = \dim V' - 1$. So if $\lambda \in V'$ is another functional on V with $\epsilon(\lambda) \neq 0$, then $\lambda = \alpha l$ for some $\alpha \in K$. It thus follows that

$$\epsilon(\lambda) = \epsilon(\alpha l) = \alpha\epsilon(l) = \alpha l(v) = (\alpha l)(v) = \lambda(v).$$

It thus follows that $\epsilon = \mathrm{ev}_v$. Then each component of ev is an isomorphism, hence ev is natural isomorphism. □

1.7.3 Projective Space

Definition 1.7.25. Let V be a vector space over some field K. For $v, w \in V \setminus 0$ we write $v \sim w$ if there exists a $\lambda \in K \setminus 0$ such that $w = \lambda v$

Remark 1.7.26. This is an equivalence relation. Indeed $v = 1 \cdot v$, hence $v \sim v$. If $v \sim w$, then $w = \lambda v$, hence $v = \lambda^{-1} w$, meaning $w \sim v$. Suppose $v \sim w$ and $w \sim u$. Then $w = \lambda v$ and $u = \mu w$, hence $u = \mu\lambda v$, implying $v \sim u$.

Definition 1.7.27. We define *the projective space of V over K* to be the set

$$\mathbb{P}(V) := (V \setminus 0)/\sim .$$

We furthermore define $\mathbb{P}^n := \mathbb{P}^n(K) := \mathbb{P}(K^{n+1})$ which is called *the projective n-space over K*. We denote an element $[v] = [(v_1, \ldots, v_{n+1})] \in \mathbb{P}^n$ by $[v_1, \ldots, v_n]$. We call $\mathbb{P}^1$ *the projective line over K* and $\mathbb{P}^2$ *the projective plane over K*.

Remark 1.7.28. Note that $[\lambda v] = [v]$ for every $\lambda \in K \setminus 0$ and $[v] \in \mathbb{P}(V)$, hence $[\lambda v_1, \ldots, \lambda v_{n+1}] = [v_1, \ldots, v_{n+1}]$ for every $[v_1, \ldots, v_{n+1}] \in \mathbb{P}^n$.
Consider the category of Vector Spaces with morphism being maps that are homogeneous of degree 1. Consider also category of sets P with a $(K^*, \cdot)$-action, satisfying $p = \lambda p$ for every $p \in P$, with morphisms being maps that are homogeneous of degree 1 with respect to this K^*-action. Then $V \mapsto \mathbb{P}(V)$ and $f : V \to W \mapsto \widehat{f} : \mathbb{P}(V) \to \mathbb{P}(W), [v] \mapsto [f(v)]$, defines a functor. Restricting all sets to be topological spaces (with vector spaces being topological vector spaces) and all maps to be continuous, we get two subcategories of category of topological spaces such that the functor $(V, f) \mapsto (\mathbb{P}(f), \widehat{f})$ restricts to a functor between these categories. Indeed if $f : X \to Y$ is continuous and $\pi : X \to X/\sim_X$, $\tau : Y \to Y/\sim_Y$, denotes quotient maps to some quotient spaces and $\widehat{f} : X/\sim_X \to Y/\sim_Y, [x] \mapsto [f(x)]$ is well-defined, then $\widehat{f}\pi = \tau f$, hence $\widehat{f}$ is continuous, by the universal property of quotient space.

Definition 1.7.29. For each $i \in \{1, \ldots, n+1\}$ we define *the i'th copy of K^n in $\mathbb{P}^n$* to be the set

$$U_i := \{[v_1, \ldots, v_{n+1}] \in \mathbb{P}^n : v_i \neq 0\} .$$

We furthermore define *the i'th hyperplane at infinity in $\mathbb{P}^n$* to be the set

$$H_{\infty,i} := \{[v_1, \ldots, v_{n+1}] \in \mathbb{P}^n : v_i = 0\} .$$

We define the hyperplane at infinity in $\mathbb{P}^n$ to be $H_\infty := H_{\infty,n+1}$

Remark 1.7.30.
1. Suppose V is a topological vector space. Consider the map $m_\lambda : \mathbb{P}(V) \to \mathbb{P}(V), [v] \mapsto [\lambda v]$ for $\lambda \in K \setminus 0$. One clearly has that $m_\lambda = \mathrm{id}$, hence it is continuous.

2. One notes that $\mathbb{P}^n = \bigcup_1^{n+1} U_i$. Note that $\varphi : K^n \to U_i, v \mapsto [v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n]$ is a bijection. Indeed the map

$$\varphi^{-1} : U_i \to K^n, [v_1, \ldots, v_i, \ldots, v_{n+1}] \mapsto (v_1/v_i, \ldots, v_{i-1}/v_i, v_{i+1}/v_i, \ldots, v_{n+1}/v_i)$$

is well-defined, since

$$((\lambda v_1)/(\lambda v_i), \ldots, (\lambda v_{i-1})/(\lambda v_i), (\lambda v_{i+1})/(\lambda v_i), \ldots, (\lambda v_{n+1})/(\lambda v_i)) = (v_1/v_i, \ldots, v_{i-1}/v_i, v_{i+1}/v_i, \ldots, v_{n+1}/v_i).$$

Clearly φ and φ^{-1} are mutual inverses. Suppose K is a topological field. Then K^m becomes a topological vector space and we can endow $\mathbb{P}^m$ with the quotient topology. Note that φ is continuous, since it is given by pre-composition of $\iota : K^n \to \pi^{-1}(U_i) \setminus 0, v \mapsto (v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_{n+1})$ with $\pi \mid_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$, which are continuous maps. Let $U \subset K^n$ be open. Let $Q := \{v \in K^{n+1} : v_i \in K, (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}) \in KU\} \simeq_{\mathcal{S}_{n+1}} K \times KU$. Define $O := Q \setminus 0$. One easily verifies that $O = Q \cap K^{n+1} \setminus 0$ and that $\pi^{-1}(\pi(O)) = O$, implying that $\pi(O)$ is open. One checks that $\varphi(U) = U_i \cap \pi(O)$, hence $\varphi(U)$ is open in U_i, hence φ^{-1} is continuous. We thus conclude that K^n is homeomorphic to U_i for each i. Hence $\mathbb{P}^n$ is locally homeomorphic to K^n.

3. Consider the map $\pi \mid_{S^n(\mathbb{R})} : S^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$, $S^n(K) = \{v \in \mathbb{R}^{n+1} : \|v\| = 1\}$. Then for $[v] \in \mathbb{P}^n(\mathbb{R})$, $[v] = [1/\|v\|v] = \pi_{S^n(\mathbb{R})}(1/\|v\|v)$, hence $\mathbb{P}^n(\mathbb{R})$ is compact. Since $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, it follows from functoriality that $\mathbb{P}^n(\mathbb{C}) \simeq \mathbb{P}^{2n+1}(\mathbb{R})$. Moreover for every $[v] = [w]$ for $v, w \in S^n(\mathbb{R})$, then $w = \lambda v$, hence $1 = \|\lambda v\| = |\lambda|\|v\| = |\lambda|$, hence $w = \pm v$. So $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to the northern hemisphere, i.e. $S^n(\mathbb{R})/(x \sim -x)$.

4. Another thing to note is that $\mathbb{P}^n = U_i \sqcup H_{\infty,i}$

1.7.4 The Projective Span

Definition 1.7.31. For $[v_1], \ldots, [v_m] \in \mathbb{P}^n$ we define

$$\text{Span}([v_1], \ldots, [v_m]) := \left\{ \left[\sum_1^m \lambda_i v_i \right] : (\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0 \right\}$$

Remark 1.7.32. Of course one should ask if this is well-defined. Suppose we are given $\alpha_1, \ldots, \alpha_m \in K \setminus 0$. Then for any $(\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0$,

$$\left[\sum_1^m \lambda_i(\alpha_i v_i) \right] = \left[\sum_1^m (\lambda_i \alpha_i) v_i \right] \in \left\{ \left[\sum_1^m \lambda_i v_i \right] : (\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0 \right\}$$

and conversely

$$\left[\sum_1^m \lambda v_i\right] = \left[\sum_1^m (\lambda_i \alpha_i^{-1})\alpha_i v_i\right] \in \left\{\left[\sum_1^m \lambda_i(\alpha_i v_i)\right] : (\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0\right\}.$$

A further thing to note with this construction, is that if $v_1, \ldots, v_m \in \mathbb{A}^{n+1} \setminus 0$ span $\mathbb{A}^{n+1}$ (this can only happen for $m \geq n+1$), then $\text{Span}([v_1], \ldots, [v_m]) = \mathbb{P}^n$.

Lemma 1.7.33. *$v_1, \ldots, v_m \in K^n$ are linearly independent if and only if $[v_i] \notin \text{Span}([v_1], \ldots, \widehat{[v_i]}, \ldots, [v_m])$ for any i.*

Proof. " $\implies$ ": Suppose there is a $[v_i] \in \text{Span}([v_1], \ldots, \widehat{[v_i]}, \ldots, [v_m])$. Then for some $\lambda_1, \ldots, \lambda_m \in K^m$ with $\lambda_i \neq 0$,

$$-\lambda_i v_i = \sum_{j \neq i} \lambda_j v_j \implies \sum_1^m \lambda_j v_j = 0,$$

hence $v_1, \ldots, v_m$ are not linearly independent. " $\impliedby$ ": Suppose there are $(\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0$ such that $\sum_1^m \lambda_i v_i = 0$. Then for some $j \in \{1, \ldots, m\}$, $\lambda_j \neq 0$, hence

$$[v_j] = \left[\lambda_j^{-1} \sum_{i \neq j} \lambda_i v_i\right] \in \text{Span}([v_1], \ldots, \widehat{[v_j]}, \ldots, [v_m])$$

□

1.7.5 Normed Vector Spaces

Definition 1.7.34. An *ordered field* is a field K together with a total ordering $\leq$ on K, satisfying, for every $a, b, c \in K$

1. $a \leq b \implies a + c \leq b + c$
2. $0 \leq a, 0 \leq b \implies 0 \leq ab$

Example 1.7.35. In this example we endow $\mathbb{Q}$ with the structure of ordered field. For $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ we define $\frac{a}{b} \leq \frac{c}{d}$ if $ad \leq cb$ and $b, d > 0$. Note that we can always find a representative of a rational number whose numerator is greater than 0. It is easy to check that this is a partial order on $\mathbb{Q}$. Given any two $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ with $b, d > 0$ we get that $bd > 0$ and $ad \leq bc$ or $ad \geq bc$, since $(\mathbb{Z}, \leq)$ is totally ordered. It follows that $\leq$ is a total ordering. Suppose $\frac{x}{y}, \frac{z}{w}, \frac{v}{u} \in \mathbb{Q}$ with $y, w, u > 0$ are given. Suppose $\frac{x}{y} \leq \frac{z}{w}$. Then $yu, wu > 0$ and

$$wu(xu + vy) \leq u(yzu + vwy) = yu(zu + vw) \implies \frac{x}{y} + \frac{v}{u} = \frac{xu + vy}{yu} \leq \frac{zu + vw}{wu} = \frac{z}{w} + \frac{v}{u}.$$

Suppose instead now that $0 \leq \frac{x}{y}$ and $0 \leq \frac{z}{w}$. Then $0 \leq x$ and $0 \leq z$, hence $0 \leq xz$, meaning $0 \leq \frac{xz}{yw}$.

Definition 1.7.36. On an ordered field K, we define *the absolute value* to be the function $|\cdot| : K \to K_{\geq 0} := \{a \in K : a \geq 0\}$ to be given by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Definition 1.7.37. A *normed vector space* is a vector space V over an ordered field K with a map $\|\cdot\| : V \to K$ satisfying

1. For every $v \in V$, $\|v\| \geq 0$.
2. For every $v \in V$, $\|v\| = 0 \iff v = 0$.
3. For every $v \in V$, $a \in K$, $\|av\| = |a|\|v\|$.
4. For every $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

We call such a function *a norm on V over K*.

Lemma 1.7.38. *On an ordered field K, the absolute value defines a norm on K over K, making K a normed vector space over K.*

Proof. 1. This is trivial, since if $a < 0$, then $-a > 0$.
2. Suppose $|a| = 0$. Then $a \geq 0$, hence $a = |a| = 0$.
3. Let $a, b \in K$. Then if $a, b \geq 0$ we have that $ab \geq 0$, hence $|ab| = ab = |a||b|$. If $a, b < 0$, then $-a, -b > 0$, hence $ab = (-a)(-b) > 0$. It follows that $|ab| = ab = (-a)(-b) = |a||b|$. If $a \geq 0$ and $b < 0$, then $ab \leq 0$, hence $|ab| = -ab = a(-b) = |a||b|$. The case $a < 0$ and $b \geq 0$ is symmetric.
4. Let $a, b \in K$. Observe that in any case $-c, c \leq |c|$ for any $c \in K$. If $a + b \geq 0$, then $|a + b| = a + b \leq |a| + |b|$. In the other case $|a + b| = -a - b \leq |a| + |b|$. □

1.8 Polynomial Rings & Formal Power Series

In this subsection every ring will be commutative, unless we explicitly declare it to not (necessarily) be the case. Really the base ring for a polynomial ring need not be commutative, but for our purposes we do not need to explore the non-commutative case. A motivation to not be too general is that non-commutative polynomial rings do not admit a nice notion of evaluation.

By a polynomial in n variables over a ring R, we mean some expression of the form

$$\sum_{v=(v_1,\ldots,v_n)\in\mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n},$$

where $x_1, \ldots, x_n$ are *variables* and $a_v = 0$ for all but finitely many $a_v \in R$. Thus we want to consider elements of the algebra over R generated by $x_1, \ldots, x_n$, i.e. $R[x_1, \ldots, x_n]$. The term variable is informal, and a goal of this section is be equip *variable* with a mathematical definition. There are some properties that we want these variables to have. For instance we do not want $x_i = x_j$ when $i \neq j$. In general, we want $x_1^{v_1} \cdots x_n^{v_n} \neq x_1^{w_1} \cdots x_n^{w_n}$ whenever $(v_1, \ldots, v_n) \neq (w_1, \ldots, w_n)$. To do this, we first introduce the notion of algebraic (in)dependence

Definition 1.8.1. Let S be an R-algebra. We say a finite sequence of elements $s_1, \ldots, s_n \in Z(S)$ are *algebraically independent* over R if for every finite sequence $(a_v) \in \prod_{v \in \mathbb{N}^n} R$, which is not the sequence $(0) \in \prod_{v \in \mathbb{N}^n} R$, we get that

$$\sum_{v \in \mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n} \neq 0.$$

If a finite sequence of elements in S are not algebraically independent over R, we say that they are *algebraically dependent*.

One quickly sees that the concept over algebraic independence is really just a special case of linear independence.

Lemma 1.8.2. *Let S be an R-algebra. Then a finite sequence in $s_1, \ldots, s_n \in Z(S)$ is algebraically independent over R if and only if $\{s_1^{v_1} \cdots s_n^{v_n}\}_{(v_1, \ldots, v_n) \in \mathbb{N}^n}$ is linearly independent over R.*

We also want that elements of R can be seen as polynomials, so a ring of polynomials should contain its base ring as a subring in a nice canonical way. Before proceeding with actually constructing a polynomial ring that does the job we will present the approach that at first might seem fruitful, but will not quite capture the behaviour we desire. That is to define $R[x_1, \ldots, x_n]$ as the set of functions

$$\mathrm{Pol}(R^n, R) := \left\{ f : R^n \to R : \begin{array}{c} f(x_1, \ldots, x_n) = \sum_{v \in \mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n} \text{ for some} \\ \text{finite sequence } \{a_v\}_{v \in \mathbb{N}^n} \subset R \text{ for all } (x_1, \ldots, x_n) \in R^n \end{array} \right\},$$

i.e. the set of polynomial functions, which is a subring of $\mathrm{Fun}(R^n, R)$, the set of functions from R^n to R. The main issue is that we cannot always distinguish terms of form $x_1^{v_1} \cdots x_n^{v_n}$. For instance, if $\#R < \infty$, then clearly $\#\mathrm{Pol}(R^n, R) < \infty$, but we want the number of distinct terms $x_1^{v_1} \cdots x_n^{v_n}$ to be countably infinite in every case. To be concrete, taking $R := \mathbb{Z}/2\mathbb{Z}$ then $x \mapsto x$ and $x \mapsto x^2$ is the same function, hence $x = x^2 \iff x^2 - x = 0$, thus $\mathrm{Pol}(R^n, R)$ fails to produce the right notion of variable in a lot of cases. Instead we will present an alternative approach.

1.8.1 Defining the Polynomial Ring

In this subsection we give a rigorous construction of the polynomial ring.

Definition 1.8.3. Consider the function

$$| \bullet | : \mathbb{N}^n \to \mathbb{N}$$
$$v = (v_1, \ldots, v_n) \mapsto \sum_1^n v_i,$$

For an n-tuple $v \in \mathbb{N}^n$ we will refer to the quantity $|v|$ as the *modulus* of v.

Remark 1.8.4. One easily sees that a sequence $(a_v) \in \bigoplus_{v \in \mathbb{N}^n} R$ for some ring R is finite if and only if $a_v = 0$ whenever $|v| > N$ for some $N \geq 0$

One recalls that $(\mathbb{N}^n, +)$ is a commutative monoid for every $n \geq 1$, where for $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{N}^n$,

$$v + w := (v_1 + w_1, \ldots, w_n + v_n).$$

We have the following result.

Lemma 1.8.5. *For every $n \geq 1$, the modulus function is additive.*

Proof. Let $v, w \in \mathbb{N}^n$ with $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Then

$$|v + w| = |(v_1 + w_1, \ldots, v_n + w_n)| = \sum_1^n (v_i + w_i) = \sum_1^n v_i + \sum_1^n w_i = |v| + |w|.$$

□

Definition 1.8.6. Let R be a ring and n a positive integer. By a *polynomial in n variables over R*, we mean an element (a_v) in the R-module $\bigoplus_{v \in \mathbb{N}^n} R$. We denote the set of polynomials in n variables over R by $R[\mathbb{N}^n]$, i.e. $R[\mathbb{N}^n] := \bigoplus_{v \in \mathbb{N}^n} R$.

With this definition we already have that $R[\mathbb{N}^n]$ is aR-module, since it is an R-submodule of $\prod_{v \in \mathbb{N}^n} R$. The set $\{e_v : v \in \mathbb{N}^n\}$ where $e_v = (\delta_{vw}) \in \prod_{w \in \mathbb{N}^n} R$ is a basis of $R[\mathbb{N}^n]$. We now aim to equip $R[\mathbb{N}^n]$ with a suitable multiplication. We do this by adding structure of ring on $\prod_{v \in \mathbb{N}^n} R$ and showing that $R[\mathbb{N}^n]$ is a subring. The set $\prod_{v \in \mathbb{N}^n} R$ with this structure of ring is called *the ring of formal power series in n variables over R.*

Lemma 1.8.7. *We define multiplication on $\prod_{v \in \mathbb{N}^n} R$ by*

$$(a_v)(b_v) = \left(\sum_{v,w \in \mathbb{N}^n : v+w=u} a_v b_v \right) \in \prod_{u \in \mathbb{N}^n} R.$$

This multiplication makes $\prod_{v \in \mathbb{N}^n} R$ a commutative ring. $R[\mathbb{N}^n]$ is a subring of $\prod_{v \in \mathbb{N}^n} R$.

Proof. Let $(a_v), (b_v), (c_v) \in \prod_{v\in\mathbb{N}^n} R$. Then

$$\begin{aligned}((a_v)(b_v))(c_v) &= \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w\right)(c_v) = \left(\sum_{r,u\in\mathbb{N}^n:r+u=s}\left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w\right)c_r\right)\\ &= \left(\sum_{r,v,w\in\mathbb{N}^n:r+v+w=s}(a_v b_w)c_r\right) = \left(\sum_{r,v,w\in\mathbb{N}^n:r+v+w=s} a_v(b_w c_r)\right)\\ &= \left(\sum_{u,v\in\mathbb{N}^n u+v=s} a_v\left(\sum_{r,w\in\mathbb{N}^n:r+w=u} b_w c_r\right)\right) = (a_v)\left(\sum_{r,w\in\mathbb{N}^n:r+w=u} b_w c_r\right)\\ &= (a_v)((b_v)(c_v)).\end{aligned}$$

Put $\mathbf{0} := (0,\ldots,0) \in \mathbb{N}^n$. We then define $1 := e_{\mathbf{0}} = (\delta_{\mathbf{0}v})$. Then

$$1(a_v) = \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} \delta_{\mathbf{0}v} a_w\right) = \left(\sum_{w\in\mathbb{N}^n:w=u} a_w\right) = (a_v).$$

Similarly it is easy to check that $(a_v)1 = (a_v)$. Finally we have that

$$\begin{aligned}(a_v)\left(((b_v) + (c_v)\right) &= \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v(b_w + c_w)\right) = \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w + a_v c_w\right)\\ &= \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w + \sum_{v,w\in\mathbb{N}^n:v+w=u} a_v c_w\right)\\ &= \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w\right) + \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v c_w\right)\\ &= (a_v)(b_v) + (a_v)(c_v).\end{aligned}$$

This means $\prod_{v\in\mathbb{N}^n} R$ is a ring with this multiplication. Note also that

$$(a_v)(b_v) = \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w\right) = \left(\sum_{v,w\in\mathbb{N}^n:v+w=u} b_v a_w\right) = (b_v)(a_v).$$

Hence $\prod_{v\in\mathbb{N}^n} R$ is a commutative ring with this multiplication.
To check that $R[\mathbb{N}^n]$ is a subring of $\prod_{v\in\mathbb{N}^n} R$, we need to check that $1 \in R[\mathbb{N}^n]$ and that $R[\mathbb{N}^n]$ is closed under multiplication. Since $\delta_{\mathbf{0},v} = 0$ for every $v \in \mathbb{N}^n \setminus \{\mathbf{0}\}$, it follows that $1 \in R[\mathbb{N}^n]$. Let $(a_v), (b_v) \in R[\mathbb{N}^n]$. We note that for some $N, M \geq 0$, $a_v = 0$ for $v \in \mathbb{N}^n$ with $|v| \geq N$ and $b_w = 0$ for $w \in \mathbb{N}^n$ with $|w| \geq M$. Let $u \in \mathbb{N}^n$ with $|u| \geq N + M$. Consider then $v, w \in \mathbb{N}^n$ such that $v + w$. Then

$$|v| + |w| = |v + w| = |u| \geq N + M \implies |v| \geq N \text{ or } |w| \geq M \implies a_v b_w = 0.$$

Thus $\sum_{v,w\in\mathbb{N}^n:v+w=u} a_v b_w = 0$, meaning $(a_v)(b_v) \in R[\mathbb{N}^n]$. □

Remark 1.8.8. As a notational trick one often denotes an $(a_v) \in \prod_{v\in\mathbb{N}^n}$ by $\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v$, where $\mathbf{x}^v$ is a short-hand notation for $x_1^{v_1}\cdots x_n^{v_n}$. With this choice of notation the elements of $\prod_{v\in\mathbb{N}^n} R$ are seen to act like some sort of power series in n variables with coefficients in R, where we of course "forget" the notion of convergence. The ring of formal power series in n variables is denoted $R[\![x_1,\ldots,x_n]\!]$, but for now we will make no more remarks about this ring and focus on the polynomial ring. We will see that the notation $\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v$ is actually on the nose in the sense of Remark 1.2.19 for the ring of polynomials.

Definition 1.8.9. Let R be a ring and n a positive integer. Consider a $w \in \mathbb{N}^n$. We define the *monomial* associated with w as the polynomial

$$e_v = (\delta_{vw}) \in R[\mathbb{N}^n].$$

For $i \in \{1,\ldots,n\}$ we define *the i'th variable* in $R[\mathbb{N}^n]$ to be monomial associated with the n-tuple of non-negative integers for which the i'th entry is 1 and for which the remaining entries are 0.

Remark 1.8.10. A note on notation: We choose to denote the i'th variable by some letter, say x, subscripted by x_i, i.e. we denote the n variables by $x_1,\ldots,x_n$. With this choice of letter we choose denote a monomial associated with a $v \in \mathbb{N}^n$ by $\mathbf{x}^v$. Later on this notation will be motivated. Had we chosen y as our letter we would get variables $y_1,\ldots,y_n$ and monomials $\mathbf{y}^v$. This remark is not of a mathematical nature and serves only as an excuse to not explicitly state what is meant by a notation a'la $\mathbf{x}^v$ every time we make use of it.

Lemma 1.8.11. *Any element $f = (a_v) \in R[\mathbb{N}^n]$ can be written uniquely as*

$$\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v,$$

Proof. This is obvious since $\{\mathbf{x}^v : v \in \mathbb{N}^n\}$ form a basis of $R[\mathbb{N}^n]$. □

Remark 1.8.12. A further consequence is that for $f = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v, g = \sum_{v\in\mathbb{N}^n} b_v\mathbf{x}^v \in R[\mathbb{N}^n]$

$$fg = \sum_{u\in\mathbb{N}^n} c_u\mathbf{x}^u,$$

where $(c_u) = \left(\sum_{v,w\in\mathbb{N}^n : v+w=u} a_vb_w\right)$. Suppose $f = \mathbf{x}^\nu$ and $g = \mathbf{x}^\mu$. Then

$$fg = (c_u) = \left(\sum_{v,w\in\mathbb{N}^n : v+w=u} \delta_{\nu v}\delta_{\mu w}\right).$$

Note that

$$\delta_{\nu v}\delta_{\mu w} = 0 \iff \delta_{\nu v} = 0 \text{ or } \delta_{\mu w} = 0 \iff \nu = v \text{ or } \mu = w,$$

hence $\delta_{\nu v}\delta_{\mu w} = 1$ if $\nu = v$ and $\mu = w$ and 0 else. Thus $c_u = 1$ when $u = \nu + \mu$ and else it is 0. This means $f = (\delta_{v,\nu+\mu}) = \mathbf{x}^{\nu+\mu}$. It follows that $\mathbf{x}^{\mu} = x_1^{\mu_1} \cdots x_n^{\mu_n}$. Any polynomial can thus uniquely be represented as sum

$$\sum_{v\in\mathbb{N}^n} a_v \mathbf{x}^v = \sum_{v=(v_1,\ldots,v_n)\in\mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n}.$$

One useful fact that one should note is that this means that for some $v_1, \ldots, v_m \in \mathbb{N}^n$, we have that the above is equal to

$$\sum_{1}^{n} a_{v_i} \mathbf{x}^{v_i}.$$

Another way of representing the above, which also may be useful is that for some N, the above is equal to

$$\sum_{v\in\mathbb{N}^n:|v|\leq N} a_v \mathbf{x}^v$$

We record the fact that R is embedded as a subring in a polynomials in the most natural way

Lemma 1.8.13. *$R1$ is a subring of $R[\![x_1, \ldots, x_n]\!]$ contained in $R[x_1, \ldots, x_n]$, ring isomorphic to R. Furthermore, $R1[x_1, \ldots, x_n] = R[x_1, \ldots, x_n]$. Lastly $x_1, \ldots, x_n$ are algebraically independent over R.*

Proof. We consider the map

$$\sigma : R \to R1$$
$$r \mapsto r1 = r$$

This is clearly a surjective ring homomorphism hence $R1$ is subring of $R[x_1, \ldots, x_n]$ whose inverse is

$$\sigma^{-1} : R1 \to R$$
$$r1 \mapsto r$$

We already know that $R1[x_1, \ldots, x_n]$ is a subring of $R[\mathbb{N}^n]$. Furthermore we have already seen in the above remark that any element in $R[x_1, \ldots, x_n]$ can be written as finite linear combination over R of elements in $\{x_1^{v_1} \cdots x_n^{v_n} : (v_1, \ldots, v_n) \in \mathbb{N}^n\}$. The fact that this set constitutes a basis of $R[x_1, \ldots, x_n]$ over R, means that $x_1, \ldots, x_n$ are algebraically independent over R (cf. Lemma 1.8.2). □

Remark 1.8.14. We have now fully justified the existence of a polynomial ring with the properties described in the introduction to this subsection. In summary, we found that the set of finite sequences in R indexed by elements in $\mathbb{N}^n$ could be endowed with the structure we sought after. From now we we will "forget" the underlying structure.

We collect all the data established about the polynomial ring in the following theorem

Theorem 1.8.15. *The rings $R[\![x_1, \ldots, x_n]\!]$ and*

$$R[x_1, \ldots, x_n] = \left\{ \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v : a_v \in R, a_v = 0 \text{ whenever } |v| > N \text{ for some } N \geq 0 \right\},$$

are rings containing R as a subring. $R[x_1, \ldots, x_n]$ is generated by $x_1, \ldots, x_n$ as an R-algebra. Furthermore, $x_1, \ldots, x_n$ are algebraically independent over R.

1.8.2 Specializations of Polynomials

Proposition 1.8.16. *Let R and S be commutative rings and consider a ring homomorphism $\sigma : R \to S$. Then σ induces a well-defined ring homomorphism given by*

$$\begin{aligned} \overline{\sigma} : R[x_1, \ldots, x_n] &\to S[x_1, \ldots, x_n] \\ \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v &\mapsto \sum_{v \in \mathbb{N}^n} \sigma(a_v) \mathbf{x}^v \end{aligned}$$

Proof. Suppose $\sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v = \sum_{v \in \mathbb{N}^n} b_v \mathbf{x}^v \in R[\mathbf{x}]$ then $a_v = b_v$ for every $v \in \mathbb{N}^n$ hence $\sigma(a_v) = \sigma(a_v)$ for every $v \in \mathbb{N}^n$, meaning

$$\overline{\sigma} \left(\sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v \right) = \sum_{v \in \mathbb{N}^n} \sigma(a_v) \mathbf{x}^v = \sum_{v \in \mathbb{N}^n} \sigma(b_v) \mathbf{x}^v = \overline{\sigma} \left(\sum_{v \in \mathbb{N}^n} b_v \mathbf{x}^v \right),$$

we thus conclude that $\overline{\sigma}$ is well-defined.
Consider arbitrary $\sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v, \sum_{v \in \mathbb{N}^n} b_v \mathbf{x}^v \in R[\mathbf{x}]$. Then

$$\begin{aligned} \overline{\sigma} \left(\sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v + \sum_{v \in \mathbb{N}^n} b_v \mathbf{x}^v \right) &= \sum_{v \in \mathbb{N}^n} \sigma(a_v + b_v) \mathbf{x}^v = \sum_{v \in \mathbb{N}^n} \left(\sigma(a_v) + \sigma(b_v) \right) \mathbf{x}^v \\ &= \sum_{v \in \mathbb{N}^n} \sigma(a_v) \mathbf{x}^v + \sum_{v \in \mathbb{N}^n} \sigma(b_v) \mathbf{x}^v \\ &= \overline{\sigma} \left(\sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v \right) + \overline{\sigma} \left(\sum_{v \in \mathbb{N}^n} b_v \mathbf{x}^v \right) \end{aligned}$$

hence $\overline{\sigma}$ is additive. It also multiplicative. Indeed,

$$\begin{aligned}\overline{\sigma}\left(\left[\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right]\left[\sum_{v\in\mathbb{N}^n} b_v\mathbf{x}^v\right]\right) &= \sum_{u\in\mathbb{N}^n}\left[\sum_{v,w\in\mathbb{N}^n:v+w=u}\sigma(a_vb_w)\right]\mathbf{x}^u\\ &= \sum_{u\in\mathbb{N}^n}\left[\sum_{v,w\in\mathbb{N}^n:v+w=u}\sigma(a_v)\sigma(b_w)\right]\mathbf{x}^u\\ &= \left[\sum_{v\in\mathbb{N}^n}\sigma(a_v)\mathbf{x}^v\right]\left[\sum_{v\in\mathbb{N}^n}\sigma(b_v)\mathbf{x}^v\right]\\ &= \overline{\sigma}\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right)\overline{\sigma}\left(\sum_{v\in\mathbb{N}^n} b_v\mathbf{x}^v\right).\end{aligned}$$

Lastly, $\overline{\sigma}(1) = \sigma(1) = 1$. □

Definition 1.8.17. For a ring extension $R \supset K$ where K is a field, given a ring map $\sigma : R \to K$, we call $\overline{\sigma}$ a *specialization of R in K*.

Lemma 1.8.18. *Let rings R, S, T be given and consider $\sigma \in \mathrm{Hom}(R,S)$, $\tau \in \mathrm{Hom}(S,T)$. Then $\overline{\tau\circ\sigma} = \overline{\tau}\circ\overline{\sigma}$. We also have that $\overline{id_R} = id_{R[x_1,\ldots,x_n]}$. In other words*

$$(R,\sigma) \mapsto (R[x_1,\ldots,x_n], \overline{\sigma})$$

is a covariant endofunctor on CRSing *for every $n \geq 1$.*

Proof. Indeed, for $\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v \in R[\mathbf{x}]$

$$\overline{\tau\circ\sigma}\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right) = \sum_{v\in\mathbb{N}^n}\tau(\sigma(a_v))\mathbf{x}^v = \overline{\tau}\left(\sum_{v\in\mathbb{N}^n}\sigma(a_v)\mathbf{x}^v\right) = \overline{\tau}\circ\overline{\sigma}\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right),$$

and lastly

$$\overline{\mathrm{id}_R}\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right) = \sum_{v\in\mathbb{N}^n}\mathrm{id}_R(a_v)\mathbf{x}^v = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v = \mathrm{id}_{R[\mathbf{x}]}\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right).$$

□

Corollary 1.8.19. *Suppose $R \overset{\sigma}{\simeq} R$. Then $R[x_1,\ldots,x_n] \overset{\overline{\sigma}}{\simeq} S[x_1,\ldots,x_n]$.*

Proof. An immediate consequence of functoriality. □

Example 1.8.20. It is in general not true that if $R[x_1,\ldots,x_n] \simeq S[x_1,\ldots,x_n]$, then $R \simeq S$. Reference example.

1.8.3 The Graded Ring of Polynomials

Definition 1.8.21. For a commutative ring R, we define

$$V_{k,n}(R) := \sum_{v\in\mathbb{N}^n:|v|=k} R\mathbf{x}^v \subset R[x_1,\ldots,x_n] \quad (n\geq 1, k\geq 0).$$

An element of $V_{k,n}(R)$ is called a *homogeneous polynomial (of degree k)*.

Remark 1.8.22. Consider a homogeneous polynomial f of degree $d>0$ in $R[x_1,\ldots,x_n]$. Let $\lambda\in R$. Then

$$f(\lambda x_1,\ldots,\lambda x_n) = \sum_{v\in\mathbb{N}^n:|v|=d} a_v(\lambda x_1)^{v_1}\cdots(\lambda x_n)^{v_n} = \sum_{v\in\mathbb{N}^n:|v|=d} \lambda^d\mathbf{x}^v = \lambda^d f$$

Lemma 1.8.23. *For a commutative ring R and $n\geq 1$,*

$$\{V_{k,n}(R) : k\geq 0\}$$

defines a grading moreover $V_{k,n}(R)$ is an R-module and $V_{0,n}(R)=R$.

Proof. Let $f=\sum_{v\in\mathbb{N}^n:|v|=k} a_v\mathbf{x}^v$ and $g=\sum_{v\in\mathbb{N}^n:|v|=l} b_v\mathbf{x}^v$. Note that whenever $|u|\neq k$ or $|w|\neq l$,

$$a_u b_w = 0,$$

hence

$$fg = \sum_{u,w\in\mathbb{N}^n:|u|=k,|w|=l} a_u b_w \mathbf{x}^{u+w} \in V_{k+l,n}(R).$$

Suppose $f\in V_{k,n}(R)\cap V_{l,n}(R)$ with $k\neq l$. Then

$$\sum_{v\in\mathbb{N}^n:|v|=k} a_v\mathbf{x}^v = f = \sum_{w\in\mathbb{N}^n:|w|=l} b_w\mathbf{x}^w$$

hence for $v\in\mathbb{N}^n$ with $|v|=k$, $a_v=b_v=0$, hence $f=0$. Moreover, for any $g\in R[x_1,\ldots,x_n]$,

$$g=\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v = \sum_{k\in\mathbb{N}}\sum_{v\in\mathbb{N}^n:|v|=k} a_v\mathbf{x}^v \in \sum_{k\in\mathbb{N}} V_{k,n}(R).$$

We thus conclude that $\{V_{k,n}(R)\}$ is a grading of $R[x_1,\ldots,x_n]$.
For $f=\sum_{v\in\mathbb{N}^n:|v|=k} a_v\mathbf{x}^v\in V_{k,n}(R)$, note that $af=\sum_{v\in\mathbb{N}^n:|v|=k} aa_v\mathbf{x}^v\in V_{k,n}(R)$. Let $r\in R$. Then $r=r\cdot 1\in V_{0,n}(R)$. If $r\in V_{0,n}(R)$, then $r=\sum_{v\in\mathbb{N}^n:|v|=0} a_v\mathbf{x}^v = a_0\cdot 1\in R$. □

Definition 1.8.24. Let $f\in R[x_1,\ldots,x_n]\setminus\{0\}$, we define a *leading coefficient* of $f=\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v$, to be a coefficient $a_v\in R\setminus 0$, where $|v|=\deg f$.

Lemma 1.8.25. *If R is an integral domain, then $R[x_1,\ldots,x_n]$ is an integral domain.*

Proof. We proceed by induction in n. Suppose $n = 1$ and let $f, g \in R[x] \setminus 0$. Then $f = \sum_0^d a_i x^i$, $g = \sum_0^{d'} b_i x^i$, for $d, d' \geq 0$, $a_d \neq 0 \neq b_{d'}$. Then

$$fg = \sum_{k=0}^{d+d'} \left(\sum_{0 \leq i \leq d, 0 \leq j \leq d' : i+j=k} a_i b_j \right) x^k.$$

Note that for $i \leq d$ and $j \leq d'$, $i + j = k$ if and only if $i = d$ and $j = d'$, hence $\sum_{0 \leq i \leq d, 0 \leq j \leq d' : i+j=k} a_i b_j = a_d b_{d'} \neq 0$, using that R is a domain. Since $\{x^i : i \in \mathbb{N}\}$ is linearly independent over R it follows that $fg \neq 0$. Suppose $R[x_1, \ldots, x_n]$ is a domain for some $n \geq 1$. Then by the one variable case $R[x_1, \ldots, x_{n+1}] \simeq (R[x_1, \ldots, x_n])[x_{n+1}]$ is a domain. □

Corollary 1.8.26. *The function*

$$\begin{aligned} \deg : R[x_1, \ldots, x_n] \setminus 0 &\to \mathbb{N} \\ f &\mapsto \deg\ f \end{aligned}$$

has the following properties

1. *The degree function is sub-additive for pairs of distinct polynomials, i.e.* $\deg\ f+g \leq \max(\deg\ f, \deg\ g)$ *for every* $f, g \in R[\mathbf{x}] \setminus 0$ *with* $f \neq g$.
2. *For every* $f, g \in R[\mathbf{x}] \setminus 0$, $\deg\ f > \deg\ g \implies \deg\ f + g = \deg\ f$.
3. *The degree function is sub-multiplicative, i.e.* $\deg\ fg \leq \deg\ f + \deg\ g$ *for every* $f, g \in R[\mathbf{x}] \setminus 0$
4. *Suppose* R *is an integral domain. Then* $\deg\ fg = \deg\ f + \deg\ g$ *for every* $f, g \in R[\mathbf{x}] \setminus 0$.

Lemma 1.8.27. *For a commutative ring* R *and a finitely generated* $I \subset R[x_1, \ldots, x_n]$, I *is a homogeneous if and only if* I *is finitely generated by a finite set of homogeneous polynomials.*

Proof. Follows from Lemma 1.6.163 □

Lemma 1.8.28. *Let* $I \subset R[x_1, \ldots, x_n]$ *be a homogeneous ideal. Then* I *is prime if and only if* $fg \in I$ *implies* $f \in I$ *or* $g \in I$ *for every form* $f, g \in R[\mathbf{x}]$.

Proof. This follows from Lemma 1.6.161 □

1.8.4 The Linear Subspaces of Homogeneous Polynomials over a Field

Lemma 1.8.29. *For every $n \geq 1$, $m \geq 1$*

$$\sum_{0}^{m} \binom{k+n}{n} = \binom{m+n+1}{n+1}$$

Proof. One readily verifies the $m = 1$ case. By induction we get that

$$\sum_{0}^{d+1} \binom{k+n}{n} = \binom{m+n+1}{n+1} + \binom{m+n+1}{n} = \binom{(m+1)+n+1}{n+1}.$$

□

Lemma 1.8.30. *For every $n \geq 1$, $d \geq 0$, the set*

$$\Delta_{n,d} := \left\{ v \in \mathbb{N}^n : \sum_{1}^{n} v_i = d \right\}.$$

is of size $\binom{d+n-1}{n-1}$.

Proof. Fix $n \geq 1$. In the case $d = 0$, then clearly $\Delta_{n,0} = \{\mathbf{0}\}$, hence $\#\Delta = 1 = \binom{0+n-1}{n-1}$. Suppose the statement is true for some $d \geq 0$. Then for $n = 1$, we see that $\#\Delta_{1,d+1} = 1 = \binom{d+1+(1-1)}{0}$. So for arbitrary $n \geq 1$,

$$\Delta_{n+1,d+1} = \bigcup_{0}^{d+1} \Delta_j,$$

where

$$\Delta_j := \left\{ v \in \mathbb{N}^{n+1} : v_n = j, \sum_{0}^{n+1} v_i = d+1 \right\}.$$

Note that these are pairwise disjoint sets and that each for j, Δ_j is in bijection with

$$\left\{ v \in \mathbb{N}^n : \sum_{1}^{n} v_i = d+1-j \right\},$$

hence by induction $\#\Delta_j = \binom{d+1-j+n-1}{n-1}$ for $j = 0, \ldots, d+1$. We thus have that

$$\#\Delta_{n+1,d+1} = \sum_{0}^{d+1} \#\Delta_j = \sum_{0}^{d+1} \binom{j+n-1}{n-1} = \binom{d+n+1}{n} = \binom{(d+1)+(n+1)-1}{(n+1)-1}.$$

□

Proposition 1.8.31. *For a field K, the dimension of $V_K(d,n)$ is $\binom{d+n-1}{n-1}$.*

Proof. With the notation of the above lemma $\{\mathbf{x}^v \in K[x_1, \ldots, x_n] : v \in \Delta_{n,d}\}$ forms a basis of $V_K(d, n)$, hence by the prior lemma

$$\dim_K \ V_K(d, n) = \#\Delta_{n,d} = \binom{d + n - 1}{n - 1}$$

□

Example 1.8.32. $\dim \ V_K(d, 1) = 1$, $\dim \ V_K(d, 2) = d + 1$, $\dim \ V_K(d, 3) = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$

Proposition 1.8.33. *For each $n \geq 1$, $d \geq 0$,*

$$\dim \ K_{\leq d}[x_1, \ldots, x_n] = \binom{d + n}{d}$$

Proof. One readily verifies that $K_{\leq d}[x_1, \ldots, x_n] = \sum_0^d V_K(d, n)$, hence by Proposition 1.8.31 and Lemma 1.8.29.

$$\begin{aligned}\dim \ K_{\leq d}[x_1, \ldots, x_n] &= \sum_0^d \dim \ V_K(d, n) = \sum_0^d \#\Delta_{n,j} = \sum_0^d \binom{j + n - 1}{n - 1} \\ &= \binom{d + n}{n} = \frac{(d + n)!}{d!n!} = \binom{d + n}{d}.\end{aligned}$$

□

Example 1.8.34. We in particular get for $d \geq 1$ that $\dim \ K_{\leq d-1}[x, y] = \binom{d+1}{d-1} = \frac{(d+1)!}{(d+1-d+1)!(d-1)!} = \frac{d(d+1)}{2} = \sum_1^d i$.

1.8.5 Evaluation & Roots of Polynomials

Definition 1.8.35. Let $S \supset R$ be a ring extension. We define *evaluation* to be the map

$$\begin{gathered}\mathrm{ev} : S^n \times R[x_1, \ldots, x_n] \\ ((s_1, \ldots, s_n), f) = \left((s_1, \ldots, s_n), \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v\right) \mapsto \sum_{v \in \mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n}\end{gathered}$$

Let $s_1, \ldots, s_n \in S$. We define *evaluation in* $(s_1, \ldots, s_n) \in S^n$ as the map

$$\begin{gathered}\mathrm{ev}_{s_1,\ldots,s_n} : R[x_1, \ldots, x_n] \to S \\ f = \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v \mapsto \mathrm{ev}((s_1, \ldots, s_n), f) = \sum_{v \in \mathbb{N}^n} a_v s_1^{v_1} \cdots s_n^{v_n}\end{gathered}$$

For an $f \in R[x_1, \ldots, x_n]$ we define $f(s_1, \ldots, s_n) := \mathrm{ev}_{s_1,\ldots,s_n}(f)$.

Lemma 1.8.36. *Evaluation is a well-defined map. Moreover, the map* $\mathrm{ev}_{s_1,\dots,s_n}$ *is a well-defined ring homomorphism such that* $\mathrm{ev}_{s_1,\dots,s_n}(r) = r$ *for every* $r \in R$*, therefore it is also an* R*-module homomorphism. In other words, evaluation in an element is an* R*-algebra homomorphism.*

Proof. **The map is well-defined:** Let $(s_1,\dots,s_n) := (t_1,\dots,t_n) \in S^n$ and $f = \sum_{v\in\mathbb{N}^n} a_v \mathbf{x}^v$, $g = \sum_{v\in\mathbb{N}^n} b_v \mathbf{x}^v$ be polynomials in $R[x_1,\dots,x_n]$ such that $f = g$. Note that for a polynomial $h := \sum_{v\in\mathbb{N}^n} c_v \mathbf{x}^v$ that

$$\mathrm{ev}((s_1,\dots,s_n),h) = \sum_{v\in\mathbb{N}^n} c_v s_1^{v_1}\cdots s_n^{v_n} = \sum_{v\in\mathbb{N}^n} c_v t_1^{v_1}\cdots t_n^{v_n} = \mathrm{ev}((t_1,\dots,t_n),h).$$

By Theorem 1.8.15 it follows that $a_v = b_v$ for every $v \in \mathbb{N}^n$. Thus in particular

$$\begin{aligned}\mathrm{ev}(s_1,\dots,s_n),f) &= \sum_{v\in\mathbb{N}^n} a_v s_1^{v_1}\cdots s_n^{v_n} = \sum_{v\in\mathbb{N}^n} b_v s_1^{v_1}\cdots s_n^{v_n} = \mathrm{ev}((s_1,\dots,s_n),g)\\ &= \mathrm{ev}((t_1,\dots,t_n),g),\end{aligned}$$

which means the evaluation map is well defined.

evaluation is a ring homomorphism: Let $f = \sum_{v\in\mathbb{N}^n} a_v \mathbf{x}^v$ and $g = \sum_{v\in\mathbb{N}^n} b_v \mathbf{x}^v$ be polynomials in $R[x_1,\dots,x_n]$ and $s_1,\dots,s_n \in S$. The map is additive

$$\begin{aligned}\mathrm{ev}_{s_1,\dots,s_n}(f+g) &= \sum_{v\in\mathbb{N}^n} (a_v+b_v) s_1^{v_1}\cdots s_n^{v_n} = \sum_{v\in\mathbb{N}^n} a_v s_1^{v_1}\cdots s_n^{v_n} + \sum_{v\in\mathbb{N}^n} b_v s_1^{v_1}\cdots s_n^{v_n}\\ &= \mathrm{ev}_{s_1,\dots,s_n}(f) + \mathrm{ev}_{s_1,\dots,s_n}(g).\end{aligned}$$

The map is multiplicative:

$$\begin{aligned}\mathrm{ev}_{s_1,\dots,s_n}(fg) &= \sum_{v,w\in\mathbb{N}^n} a_v b_w s_1^{v_1+w_1}\cdots s_n^{v_n+w_n} = \sum_{v,w\in\mathbb{N}^n} a_v s_1^{v_1}\cdots s_n^{v_n} b_v s_1^{w_1}\cdots s_n^{w_n}\\ &= \left(\sum_{v\in\mathbb{N}^n} a_v s_1^{v_1}\cdots s_n^{v_n}\right)\left(\sum_{v\in\mathbb{N}^n} b_v s_1^{w_1}\cdots s_n^{w_n}\right) = \mathrm{ev}_{s_1,\dots,s_n}(f)\mathrm{ev}_{s_1,\dots,s_n}(g).\end{aligned}$$

Evaluation fixes R**:** Let $r \in R$. Then

$$\mathrm{ev}_{s_1,\dots,s_n}(r) = \mathrm{ev}_{s_1,\dots,s_n}\left(r\mathbf{x}^{(0,\dots,0)}\right) = r s_1^0\cdots s_n^0 = r.$$

□

Remark 1.8.37. Given a commutative R-algebra S and elements $s_1,\dots,s_n \in S$, we note that the R-algebra generated by these elements, i.e. $R[s_1,\dots,s_n]$ is given as the image of $\mathrm{ev}_{s_1,\dots,s_n} : R[x_1,\dots,x_n] \to S$.

Lemma 1.8.38. *Let R be a ring and $J \subset R[y_1, \ldots, y_m]$ be an ideal. Consider $\overline{f_i} := f_i + J \in R[\mathbf{y}]/J$. Then for each $f \in R[\mathbf{x}]$*

$$\mathrm{ev}_{\overline{f_1},\ldots,\overline{f_n}}(f) = \mathrm{ev}_{f_1,\ldots,f_n}(f) + J$$

Proof. This is a simple matter of using the definition addition and multiplication in the quotient ring. Indeed we can write $f = \sum_1^k a_{v_i}\mathbf{x}^{v_i}$ for suitable distinct $v_1, \ldots, v_k \in \mathbb{N}^n$ and $a_{v_i} \in R$. Then

$$\begin{aligned}\mathrm{ev}_{\overline{f_1},\ldots,\overline{f_n}}(f) &= \sum_1^k a_{v_i}\left(f_1 + J\right)^{v_{i1}} \cdots \left(f_n + J\right)^{v_{in}} = \sum_1^k a_{v_i}\left(f_1^{v_{i1}} + J\right) \cdots \left(f_n^{v_{in}} + J\right) \\ &= \left[\sum_1^k a_{v_i} f_1^{v_{i1}} \cdots f_n^{v_{in}}\right] + J = \mathrm{ev}_{f_1,\ldots,f_n}(f) + J\end{aligned}$$

□

Proposition 1.8.39. *Let S be a commutative R-algebra. Let $\sigma : R[x_1, \ldots, x_n] \to S$ be an R-algebra homomorphism. Then $\sigma = \mathrm{ev}_{\sigma(x_1),\ldots,\sigma(x_n)}$. Hence any element of $\mathrm{Hom}^{R-alg}(R[x_1, \ldots, x_n], S)$ is uniquely determined by it's behavior on the variables of $R[x_1, \ldots, x_n]$.*

Proof. Let $f = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v \in R[x_1, \ldots, x_n]$. Then using the multiplicativity and additivity of σ we have that

$$\begin{aligned}\sigma(f) &= \sigma\left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right) = \sum_{v\in\mathbb{N}^n} \sigma(a_v)\sigma\left(x_1^{v_1} \cdots x_n^{v_n}\right) = \sum_{v\in\mathbb{N}^n} a_v\sigma\left(x_1^{v_1}\right) \cdots \sigma\left(x_n^{v_n}\right) \\ &= \mathrm{ev}_{\sigma(x_1),\ldots,\sigma(x_n)}(f)\end{aligned}$$

□

Corollary 1.8.40. *Let S be a commutative R-algebra. Then*

$$\mathrm{Hom}^{R-Alg}(R[x_1, \ldots, x_n], S) = \left\{\mathrm{ev}_{s_1,\ldots,s_n} : (s_1, \ldots, s_n) \in S^n\right\}.$$

Corollary 1.8.41. *Let $I \subset R[x_1, \ldots, x_n]$, $J \subset R[y_1, \ldots, y_n]$ be ideals. Then an R-algebra map $\sigma : R[x_1, \ldots, x_n]/I \to R[y_1, \ldots, y_m]/J$, is induced by some $\mathrm{ev}_{a_1,\ldots,a_n}$ with $\mathrm{ev}_{a_1,\ldots,a_n}(I) \subset J$.*

Lemma 1.8.42. *Let $S \supset R$ be a ring extension and $s_1, \ldots, s_n$ be algebraically independent over R. When $R[x_1, \ldots, x_n]$ denotes the polynomial over R in n variables then $R[s_1, \ldots, s_n] \simeq R[\mathbf{x}]$.*

Proof. $\mathrm{ev}_{s_1,\ldots,s_n} : R[\mathbf{x}] \to R[s_1,\ldots,s_n]$ defines a surjective ring homomorphism. Let $f \in R[\mathbf{x}]$. By the definition of algebraic independence

$$\mathrm{ev}_{s_1,\ldots,s_n}(f) = 0 \iff f = 0.$$

$\mathrm{ev}_{s_1,\ldots,s_n}$ is therefor injective, implying $R[s_1,\ldots,s_n] \simeq R[\mathbf{x}]$ □

Corollary 1.8.43. *Consider the ring extension $R[x_1,\ldots,x_n,y_1,\ldots,y_m] \supset R$. Then the subring of $R[\mathbf{x},\mathbf{y}]$ generated by $x_1,\ldots,x_n$ is isomorphic to the polynomial ring in n variables.*

Corollary 1.8.44. *Consider the ring extension $R[x_1,\ldots,x_n,y_1,\ldots,y_m] \supset R$. Then $R[\mathbf{x},\mathbf{y}] = R[\mathbf{x}][\mathbf{y}]$. Furthermore $R[\mathbf{x},\mathbf{y}] \simeq R[z_1,\ldots,z_n][w_1,\ldots,w_m]$.*

Definition 1.8.45. Let $S \supset R$ be a commutative ring extension. Let $f \in R[x_1,\ldots,x_n]$, we say that $(s_1,\ldots,s_n) \in S^n$ is a *zero (over S) of f* if $f(s_1,\ldots,s_n) = 0$. If $f \in R[x]$ and $s \in S$ is a zero of f we call it a *root (in S)*.

Definition 1.8.46. Let $S \supset R$ be a commutative ring extension. Given a polynomial $f \in R[x_1,\ldots,x_n]$, we denote the set of zeroes over S of f by

$$V_S(f).$$

The zeroes over R are denoted

$$V(f).$$

The above definitions are central to the classical treatment of algebraic geometry, since the geometric objects considered are build from zero sets of polynomials over a field K.

Proposition 1.8.47. *Let S be an R-algebra. Let $f \in R[x_1,\ldots,x_n] \subset S[x_1,\ldots,x_n]$ and $(s_1,\ldots,s_n) \in S^n$, set $I := \langle x_1 - s_1,\ldots,x_n - s_n\rangle \subset S[x_1,\ldots,x_n]$. Then $(s_1,\ldots,s_n)$ is a zero of f if and only if $f \in I$. We call I the **point ideal of** $(s_1,\ldots,s_n)$.*

Proof. Write $f = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v$. " $\Longrightarrow$ ": Suppose $(s_1,\ldots,s_n)$ is a zero of f. Then, since $x_i + I = s_i + I$ for each i,

$$\begin{aligned} f + I &= \left(\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v\right) + I = \sum_{v\in\mathbb{N}^n} a_v(x_1+I)^{v_1}\cdots(x_n+I)^{v_n} \\ &= \sum_{v\in\mathbb{N}^n} a_v(s_1+I)^{v_1}\cdots(s_n+I)^{v_n} \\ &= \left(\sum_{v\in\mathbb{N}^n} a_v s_1^{v_1}\cdots s_n^{v_n}\right) + I = f(s_1,\ldots,s_n) + I = 0 + I, \end{aligned}$$

thus $f \in I$

" $\Longleftarrow$ ": Suppose $f \in I$. Then there are $\lambda_1, \ldots, \lambda_n \in S[x_1, \ldots, x_n]$ such that $f = \sum_1^n \lambda_i \cdot (x_i - s_i)$. It then follows that

$$f(s_1, \ldots, s_n) = \sum_1^n \lambda_i(s_1, \ldots, s_n)(s_i - s_i) = 0,$$

hence $(s_1, \ldots, s_n)$ is a zero of f. □

Corollary 1.8.48. *Let $(r_1, \ldots, r_n) \in R^n$. Then*

$$R[x_1, \ldots, x_n]/\langle x_1 - r_1, \ldots, x_n - r_n\rangle \simeq R.$$

Explicitly, this isomorpism is induced by $\mathrm{ev}_{r_1,\ldots,r_n}$. *In particular, $\langle x_1 - r_1, \ldots, x_n - r_n\rangle$ is maximal if and only if R is a field.*

Corollary 1.8.49. *Let $S \supset R$ be a commutative ring extension and consider $f \in R[x]$ and $a \in S$. Then a is a root of f if and only if $x - a \mid f$ in $S[x]$.*

The following theorem is useful when one wants to eliminate certain variables in a finitely generated R-algebra.

Corollary 1.8.50. *Let $f_1, \ldots, f_m, g_1, \ldots, g_l \in R[x_1, \ldots, x_n]$, where R is a commutative ring. Set $I := \langle g_1, \ldots, g_l\rangle \subset R[\mathbf{x}]$ and $J := \langle g_1, \ldots, g_l, y_1 - f_1, \ldots, y_m - f_m\rangle \subset R[\mathbf{x}, y_1, \ldots, y_m]$. Then $R[\mathbf{x}, \mathbf{y}]/J \simeq R[\mathbf{x}]/I$*

Proof. Consider the surjective ring homomorphism

$$\sigma := \mathrm{ev}_{\mathbf{x}, f_1, \ldots, f_m} : R[\mathbf{x}, \mathbf{y}] \to R[\mathbf{x}]/I$$
$$h \mapsto h(\mathbf{x}, f_1, \ldots, f_m) + I$$

Clearly $J \subset \ker\ \sigma$. Let $h \in \ker\ \sigma$. Then $h(x_1, \ldots, x_n, f_1, \ldots, f_m) \in I$ and hence is also an element of J. It follows that

$$h + J = h(\mathbf{x}, f_1, \ldots, f_m) + J = 0 + J \implies h \in J.$$

We thus see by the isomorphism theorem that

$$R[\mathbf{x}, \mathbf{y}]/J \simeq R[\mathbf{x}]/I$$

□

Lemma 1.8.51. *Let $S \supset R$ be an integral domain extension. Let $f, g \in R[x_1, \ldots, x_n]$ and $v \in S^n$. Then v is a zero of fg if and only if v is a zero of f or g. In other words $V_S(fg) = V_S(f) \cup V_S(g)$.*

Proof. Since R is an integral domain,

$$0 = (fg)(v) = f(v)g(v) \iff f(v) = 0 \text{ or } g(v) = 0.$$

□

Proposition 1.8.52. *Let R be an integral domain. Consider $f \in R[x] \setminus \{0\}$ with $d :=$ deg f. Then there at most d roots of f in R.*

Proof. We proceed by induction in d. Let $d = 1$. If f has no roots we are done. Suppose it does have a root $a \in R$. Then Corollary 1.8.49 tells us that $f = q(x - a)$, for some $q \in R[x]$. Since $f \neq 0$, we have that $q \neq 0$. By Lemma 1.8.26 4. it follows that

$$1 = \deg\ f = \deg\ q(x-a) = \deg\ q + \deg x - a = q + 1 \implies \deg\ q = 0,$$

hence q is a non-zero constant. It follows from 1.8.51 that f has exactly 1 root.
Now consider a polynomial $f \in R[x]$ of degree $d + 1$ for some $d \geq 1$. If f has no roots, we are done. Suppose then that f has a root $a \in R$. Then by Corollary 1.8.49 $(x - a) \mid f$, hence $f = g \cdot (x - a)$ for some $g \in R[x]$, again since $f \neq 0$, we have that $g \neq 0$, by Lemma 1.8.26 4. it follows that

$$d + 1 = \deg\ f = \deg\ g + \deg\ x - a = \deg\ g + 1 \implies \deg\ g = d$$

it follows by induction hypothesis that g has at most d roots. By Lemma 1.8.51 $V(f) = V(g) \cup V(x-a)$, hence $\#V(f) = \#\left(V(g) \cup V(x-a)\right) \leq \#V(g) + \#V(x-a) \leq d+1$ □

Lemma 1.8.53. *Let R be an integral domain. Consider $f \in R[x_1, \ldots, x_n] \setminus 0$ and $f_1, \ldots, f_n \in R[y_1, \ldots, y_m] \setminus 0$ with $d_i :=$ deg f_i. Then*

$$\deg\ f(f_1, \ldots, f_n) \leq \deg\ f(x_1^{d_1}, \ldots, x_n^{d_n})$$

Proof. Write $f = \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v$. Let $v \in \mathbb{N}^n$ such that $a_v \neq 0$ and set $M_v = \mathbf{x}^v$. Then

$$\deg\ M_v(x_1^{d_1}, \ldots, x_n^{d_n}) = \sum_1^n v_i d_i = \sum_1^n \deg\ f_i^{v_i} = \deg f_1^{v_1} \cdots {f_n}^{v_n} = \deg\ M_v(f_1, \ldots, f_n).$$

Note that the map $(v_1, \ldots, v_n) \mapsto (v_1 d_1, \ldots, v_n d_n)$ is injective hence,

$$\begin{aligned}\deg\ f(f_1, \ldots, f_n) &\leq \max_{v \in \mathbb{N}^n : a_v \neq 0} \deg\ M_v(f_1, \ldots, f_n) = \max_{v \in \mathbb{N}^n : a_v \neq 0} \deg\ M_v(x_1^{d_1}, \ldots, x_n^{d_n}) \\ &= \deg\ f(x_1^{d_1}, \ldots, x_n^{d_n}).\end{aligned}$$

□

Remark 1.8.54. The above result doesn't always hold with equality. take for instance $f = x_1x_2 - x_3$, take $f_1 = y_1$, $f_2 = y_2$ and $f_3 = -y_1y_3$. Then $f(f_1, f_2, f_3) = 0$, while $f(x_1, x_2, x_3^2) = x_1x_2 - x_3^2$.

Lemma 1.8.55. *Let $f = \sum_0^d a_i x^i \in K[x] \setminus 0$ and set $I = \langle f \rangle$. Then $K[x]/I$ is a vector space of dimension d with basis $\{x^i + I : i \in \{0, \dots, d-1\}\}$.*

Proof. One finds that

$$0 = \left[\sum_0^d a_i x^i\right] + I \implies x^d = -\sum_0^{d-1} \left(a_d^{-1} a_i x^i + I\right),$$

so $\{x^i + I : i \in \{0, \dots, d-1\}\}$ generates $K[x]/I$ over K. Suppose $g + I = \left[\sum_0^{d-1} b_i x^i\right] + I = 0$. Then $g \in I$. This means either $g = 0$ or deg $g \geq d$, hence $g = 0$ and $a_i = 0$ for $i \in \{1, \dots, d-1\}$. So $\{x^i + I : i \in \{0, \dots, d-1\}\}$ is linearly independent over K and is thus a basis for $K[x]/I$, which means $\dim_K K[x]/I = d$. □

1.8.6 Polynomials over Infinite Rings

Proposition 1.8.56. *Let R be an infinite integral domain an $f \in R[x_1, \dots, x_n]$. Then $f = 0$ if and only if $f(v) = 0$ for every $v \in K^n$.*

Proof. " $\implies$ ": This is trivial.
"$\impliedby$":We prove that if $f \neq 0$, then there is a $v \in R^n$ such that $f(v) \neq 0$. We prove this by induction in n.
Base case: Consider first the case $n = 1$. Since $f \neq 0$, the number of roots is bounded by the non-negative integer deg f by Proposition 1.8.52. Then since $\#R = \infty$, there is an $a \in R$ such that $f(a) \neq 0$.
Induction hypothesis: Suppose that there is an $n \geq 1$ s.t. if $h \in R[x_1, \dots, x_n] \setminus 0$, then there is a $v \in R^n$ such that $f(v) \neq 0$.
Induction Step: Let $f \in R[x_1, \dots, x_{n+1}] \setminus 0$. We can write

$$f = \sum_0^d f_i x_{n+1}^i,$$

for some $d \geq 0$ and suitable $f_0, \dots, f_d \in R[x_1, \dots, x_n]$ where $f_j \neq 0$ for some $j \in \{0, \dots, d\}$. By the induction hypothesis, there is a $(v_1, \dots, v_n) \in R^n$ such that $f_j(v) \neq 0$. Then

$$R[x_{n+1}] \ni f' := f(v_1, \dots, v_n, x_{n+1}) = \sum_0^d f_i(v_1, \dots, v_n) x_{n+1}^i \neq 0.$$

By the base case there is a $v_{n+1} \in R$ such that $f'(v_{n+1}) \neq 0$. Hence upon putting $v = (v_1, \ldots, v_n, v_{n+1})$ we get that

$$f(v) = f'(v_{n+1}) \neq 0.$$

□

Lemma 1.8.57. *Given an integral domain R, there are infinitely many irreducible polynomials in $R[x]$ and consequently this is also the case in $R[x_1, \ldots, x_n]$.*

Proof. When R is infinite, each $\alpha \in R$, uniquely corresponds to an irreducible polynomial $(x - \alpha)$.
Suppose R is finite. Then R is a finite field. Consider the polynomial

$$f := \prod_{\alpha \in R} (x - \alpha)$$

Then

$$f^n + 1$$

has no roots in R, and is therefore irreducible for each $n \geq 0$. □

1.8.7 The Hilbert Basis Theorem

Theorem 1.8.58. *(Hilbert Basis Theorem) Let R be a noetherian ring. Then $R[x]$ is noetherian. Furthermore $R[x_1, \ldots, x_n]$ is noetherian.*

Proof. We prove the contrapositive. Suppose That $R[x]$ is not noetherian, or equivalently by Theorem 1.4.59 suppose there is an ideal $I \subset R[x]$ that is not finitely generated. Let $d_1 = \min\{\deg\ f : f \in I\}$. Let $f_1 \in I$ such that $\deg\ f_1 = d_1$. We then let $I_1 = R[x]f_1$ and recursively define $I_n = \sum_1^n R[x]f_i$, where $f_n \in I \setminus I_{n-1}$ where $\deg\ f_n = d_n = \min\{\deg\ f : f \in I \setminus I_{n-1}\}$. Note that since $I \setminus I_n \supset I \setminus I_{n+1}$, we get $d_n \leq d_{n+1}$ for each $n \geq 1$. For each n we can write

$$f_n = \sum_{i=0}^{d_n} a_i^{(n)} x^i,$$

for suitable $a_i^{(n)} \in R$. Set $a(n) = a_{d_n}^{(n)}$ We then have an ascending chain $J_1 \subset J_2 \subset \ldots$ in R where $J_n = \sum_1^n Ra(i)$. Suppose for a contradiction that $J_n = J_{n+1}$ for some $n \geq 1$. Then

$$a(n+1) = \sum_1^n b_i a(i),$$

for suitable $b_1, \ldots, b_n \in R$. Put

$$g = f_{n+1} - \underbrace{\sum_{i=1}^n \alpha_i x^{d_{n+1} - d_i} f_i}_{h}.$$

Then $g \in I$, $h \in I_n$ and $f_{n+1} = g + h \in I \setminus I_n$, thus $g \in I \setminus I_n$. However, upon further inspection, we find

$$\begin{aligned} g &= a(n+1)x^{d_{n+1}} - \sum_{i=1}^{n} \alpha_i x^{d_{n+1}-d_i} \sum_{j=1}^{d_i} a_j^{(i)} x^j + \underbrace{\sum_{i=1}^{d_{n+1}-1} a_i^{(n+1)} x^i}_{r} \\ &= a(n+1)x^{d_{n+1}} - \sum_{i=1}^{n} \alpha_i a(i) x^{d_{n+1}-d_i} x^{d_i} - \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{d_i-1} a_j^{(i)} x^i}_{r'} + r \\ &= a(n+1)x^{d_{n+1}} - \left(\sum_{i=1}^{n} \alpha_i a(i) \right) x^{d_{n+1}} + r' + r = a(n+1)x^{d_{n+1}} - a(n+1)x^{d_{n+1}} + r' + r \\ &= \sum_{i=1}^{n+1} \sum_{j=1}^{d_i-1} a_j^{(i)} x^i. \end{aligned}$$

Thus $\deg\ g = \max\{d_1 - 1, \ldots, d_{n+1} - 1\} = d_{n+1} - 1 < d_{n+1} = \min\{\deg\ f : f \in I \setminus I_n\}$, leading to a contradiction. This means that $J_1 \subset J_2 \subset \ldots$ is a non-stabilizing ascending chain hence R is not noetherian.

By induction $R[x_1, \ldots, x_n] \simeq (R[x_1, \ldots, x_{n-1}])\,[x_n]$ is noetherian. □

Corollary 1.8.59. *Let K be a field. Then $K[x_1, \ldots, x_n]$ is noetherian.*

1.8.8 Polynomials over Fields

Definition 1.8.60. A field K is called algebraically closed if every non-constant $f \in K[x] \setminus 0$ has a root $a \in K$.

Lemma 1.8.61. *An algebraically closed field K is infinite.*

Proof. Let $A := \{\alpha_1, \ldots, \alpha_n\} \subset K$. Consider the polynomial

$$f := 1 + \prod_{1}^{n} (x - \alpha_i) \in K[x]$$

Then f has a root $\beta \in K$. Note that $f(\alpha_i) = 1 \neq 0$, hence $\beta \notin A$. It follows that K is not finite. □

Lemma 1.8.62. *Let K be a field and $f = \sum_0^d a_i x^i \in K[x]$ an irreducible polynomial. Set $I := \langle f \rangle$. Then $F := K[x]/I$ is a field and $x + I$ is a root of $g := \sum_0^d a_i y^i \in F[y]$.*

Proof. Corollary 1.8.81 (a later result), Lemma 1.6.62 and Lemma 1.6.63 shows that I is maximal. Then $K[x]/I$ is a field by Proposition 1.6.19. Secondly,

$$g(x + I) = \sum_{0}^{d} a_i (x + I) = \left(\sum_{0}^{d} a_i x \right) + I = 0 + I.$$

□

1.8.9 More on Power Series

Lemma 1.8.63. *Let R be any commutative ring. Then*

$$R[\![x_1,\ldots,x_n,y_1,\ldots,y_m]\!] \simeq R[\![x_1,\ldots,x_n]\!][\![y_1,\ldots,y_m]\!] := (R[\![x_1,\ldots,x_n]\!])[\![y_1,\ldots,y_m]\!].$$

Proof. Consider the map $\sigma : R[\![\mathbf{x},\mathbf{y}]\!] \to R[\![\mathbf{x}]\!][\![\mathbf{y}]\!], \sum_{v\in\mathbb{N}^n, w\in\mathbb{N}^m} a_v \mathbf{x}^v \mathbf{y}^w \mapsto \sum_{w\in\mathbb{N}^n} \left(\sum_{v\in\mathbb{N}^n} a_v \mathbf{x}^v\right) \mathbf{y}^w$. Note that σ is actually just currying of functions $\mathbb{N}^m \times \mathbb{N}^n \to R$. This is trivially a ring homomorphism. The inverse is given by uncurrying. □

Lemma 1.8.64. *If R is an integral domain, so is $R[\![x_1,\ldots,x_n]\!]$.*

Proof. In the case $n = 1$. Consider $f = \sum_{i\in\mathbb{N}} a_i x^i, g = \sum_{i\in\mathbb{N}} b_i x^i \in R[\![x]\!]$. Let $k, l \geq 0$ be the smallest integers such that $a_i \neq 0$, $b_i \neq 0$. Consider $i, j \in \mathbb{N}$ such that $i + j = k + l$. If $i > k$ then $j < l$ and vice versa, hence

$$\sum_{i,j\in\mathbb{N}:i+j=k+l} a_i b_j = a_k b_l \neq 0.$$

By the prior lemma $R[\![x_1,\ldots,x_{n+1}]\!] \simeq R[\![x_1,\ldots,x_n]\!][\![x_{n+1}]\!]$, which by induction and the case $n = 1$ is an integral domain. □

1.8.10 Formal Power Series & DVRs

Lemma 1.8.65. *Suppose R is an integral domain. Then*

$$R[\![x]\!]^* = \left\{ \sum_{i\in\mathbb{N}} a_i x^i \in R[\![x]\!] : a_i \in R^* \right\}$$

Proof. Note that

$$\left(\sum_{i\geq 0} \alpha_i x^i\right)\left(\sum_{i\geq 0} \beta_i x^i\right) = \sum_{i\geq 0}\sum_{j=0}^{i} a_j b_{i-j}$$

Let $s = \sum_{i\in\mathbb{N}} a_i x^i$ be an element of the right-hand side. Set $b_0 := a_0^{-1}$ and $b_k := -a_0^{-1} \sum_{j=1}^{k} a_j b_{k-j}$ for $k \geq 1$. Define $t = \sum_{i\in\mathbb{N}^n} b_i x^i$. We prove by induction that $\sum_0^n a_j b_{n-j} = 0$. Indeed,

$$\sum_{j=0}^{n} a_j b_{n-j} = -\left(a_0 a_0^{-1} \sum_{j=1}^{n} a_j b_{n-j}\right) + \sum_{j=1}^{n} a_j b_{n-j} = 0.$$

It thus follows that

$$\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{i\geq 0} b_i x^i\right) = 1.$$

Conversely, if $\sum_{i\geq 0} a_i x^i \in R[\![x]\!]^a st$. Then

$$\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{i\geq 0} b_i x^i\right)$$

for some $\sum_{i\geq 0} b_i x^i \in R[\![x]\!]$. In particular, $a_0 b_0 = 1$, hence $a_0 \in R^*$. □

Lemma 1.8.66. *For an integral domain R, $x \in R[\![x]\!]$ is irreducible.*

Proof. x is a non-zero, non-unit. Suppose $x = ab$ for $a, b \in R[\![x]\!]$. Then $a_0 b_0 = 0$, hence $a_0 = 0$ or $b_0 = 0$. Furthermore, $a_0 b_1 + a_1 b_0 = 1$, hence $a_0 b_1 = 1$ or $a_1 b_0 = 1$, hence $a_0 \in R^*$ or $b_0 \in R^*$, hence either a or b is a unit in $R[\![x]\!]$. We thus have that x is irreducible in $R[\![x]\!]$ □

Proposition 1.8.67. *The ring of power series $K[\![x]\!]$ is a DVR with uniformizing parameter x when K is a field.*

Proof. x is irreducible by the above lemma. Let $t \in K[\![x]\!]$. Put
$n := \max(\{k \geq 1 : x^k \mid t\})$. Note for $h \in K[\![x]\!]$, $x \mid h$ if and only if h is not a unit, hence $t = sx^n$, where s a unit. uniqueness of this representation follows from the maximality of n and the irreducibility of x. It thus follows that $K[\![x]\!]$ is a DVR with uniformizing parameter x by Proposition 1.6.121. □

Definition 1.8.68. For an integral domain R, we take $R(\!(x_1, \ldots, x_n)\!)$ to mean $Q(R[\![x_1, \ldots, x_n]\!])$.

Proposition 1.8.69. *Consider the setup and statement of Proposition 1.6.133. Then to each $z \in R$ there is a unique (possibly infinite) sequence $(\lambda_i) \in \prod_{i\in\mathbb{N}} K$. In other words the map*

$$\sigma : R \to K[\![x]\!]$$
$$z \mapsto \sum_{i\in\mathbb{N}} \lambda_i x^i$$

is a well-defined map. It is furthermore an injective ring K-algebra homomorphism. It extends to a homomorphism of $L = Q(R)$ onto $K(\!(x)\!)$.

Proof. clearly map fixes K. Let $z, w \in R$ be given with associated power series $\sum_{i\in\mathbb{N}} \lambda_i x^i$ resp. $\sum_{i\in\mathbb{N}} \mu_i x^i$. Then for any $n \geq 0$, $z = \sum_0^n \lambda_i t^i + z_n t^{n+1}$, $w = \sum_0^n \mu_i t^i + w_n t^{n+1}$ for suitable unique $z_{n+1}, w_{n+1} \in R$, hence

$$z + w = \sum_0^n (\lambda_i + \mu_i) t^i + (w_n + z_n) \implies \sigma(z + w) = \sum_{i\in\mathbb{N}} (\lambda_i + \mu_i) x^i$$
$$= \sum_{i\in\mathbb{N}^n} \lambda_i x^i + \sum_{i\in\mathbb{N}^n} \mu_i x^i = \sigma(z) + \sigma(w).$$

Let $n \geq 0$. Then

$$zw = \sum_{0}^{2n} \left(\sum_{i,j \in \mathbb{N}: i+j=h} \lambda_i \mu_j \right) t^h + \underbrace{\sum_{0}^{n} (\lambda_i w_k + \mu_i z_k) t^{i+n+1} + w_n z_n t^{n+2}}_{r}.$$

Since $\text{ord}(r) \geq n+1$, it follows that the n'th coefficient of the formal power series of zw is equal to $\sum_{i+j=n} \lambda_i \mu_i$, hence

$$\sigma(zw) = \sum_{h \in \mathbb{N}} \left(\sum_{i,j \in \mathbb{N}: i+j=h} \lambda_i \mu_j \right) x^h = \sigma(z)\sigma(w).$$

Injectivity follows from the uniqueness of the coefficients in the power series. Hence $\ker \sigma = 0$, hence Lemma **??** implies that

$$\overline{\sigma} : L \to K(\!(x)\!), \frac{z}{w} \mapsto \frac{\sigma(z)}{\sigma(w)}$$

is the unique extension of σ to a K-algebra homomorphism between the fraction fields of R and $K[\![x]\!]$. □

Remark 1.8.70. The unique formal power series $\sum_{i \in \mathbb{N}} \lambda_i \in K[\![x]\!]$ associated with $z \in R$ is called *the power series expansion of z in terms of t*,

1.8.11 Term Orders & a Polynomial Division Algorithms

Definition 1.8.71. A *term order* is a total order $\leq$ on $\mathbb{N}^n$ such that

1. $0 \leq v$ for every $v \in \mathbb{N}^n$,
2. for every $v_1, v_2, v \in \mathbb{N}^n, v_1 \leq v_2 \implies v_1 + v \leq v_2 + v$.

Example 1.8.72.

1. A simple example of a term order is $\leq$ on $\mathbb{N}$.
2. *The lexicographic term order*, denoted $\leq_{\text{lex}}$, on $\mathbb{N}^n$ for $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ is defined by $v \leq_{\text{lex}} w$ if $v = w$ or there is an $i \in \{1, \ldots, n\}$ such that $v_j = w_j$ for $j < i$ and $v_i < w_i$. For example $\left(2, 10^6, 10^{10^6}\right) \leq_{\text{lex}} (3, 1, 1)$ since $2 < 3$. This is indeed a term order: We first check that it is a total order. By definition it is reflexive. Let $v, w, u \in \mathbb{N}^n$.
   Note that if $v \neq w$, then there is a minimal i such that $v_i \neq w_i$, hence either $v <_{\text{lex}} w$ or $w <_{\text{lex}} v$. Hence in general $v \leq_{\text{lex}} w$ or $w \leq_{\text{lex}} v$.
   If there is an i such $v_i < w_i$ and $v_j = w_j$ for $j < i$ then $v \neq w$. Hence if $v \leq_{\text{lex}} w$ and $w \leq_{\text{lex}} v$, then necessarily $v = w$.

Suppose $v \leq_{\text{lex}} w$ and $w \leq_{\text{lex}} u$. We check by cases that $v \leq_{\text{lex}} u$.
Case 1: Suppose first $v = w$ and $w = u$. Then $v = u$, implying $v \leq_{\text{lex}} u$.
Case 2: Suppose $v = w$ and that there is an i such that $w_i < u_i$ and $w_j = u_j$ for every $j < i$. Then $v_i = w_i < u_i$ and $v_j = w_j = u_j$ for $j < i$ hence $v \leq_{\text{lex}} u$.
Case 3: Suppose there are $h, i \in \{1, \ldots, n\}$ such that $v_h < w_h$, $w_i < u_i$ and $v_j = w_j$, $w_k = u_k$ for $h < j$, $i < k$. If $h \leq i$, then $v_h < w_h \leq u_h$ and $v_j = w_j = u_j$ for $j < h$. If $i < h$, then $v_i = w_i < u_i$ and $v_j = w_j = u_j$ for $j < i$. In any case $v \leq_{\text{lex}} u$.
Case 4: Suppose there is an i such that $v_i < w_i$ and $v_j = w_j$ for every $j < i$ and $w = u$. Then $v_i < w_i = u_i$ and $v_j = w_j = u_j$ for every $j < i$, hence $v \leq_{\text{lex}} u$.
In conclusion $\leq_{\text{lex}}$ is a total order. Note that $0 \leq v_i$ for every $i \in \{1, \ldots, n\}$. Hence either $0 = v_i$ for every i or there is an i such that $0 < v_i$, meaning $0 \leq_{\text{lex}} v$. Suppose $v \leq_{\text{lex}} w$. If $v = w$ then, $v + u = w + u$, hence $v + u \leq_{\text{lex}} w + u$. Suppose there is an i such that $v_i < w_i$ and $v_j = w_j$ for each $j < i$. Then $v_i + u_i < w_i + u_i$ and $v_j + u_j = w_j + u_j$ for every $j < i$, which implies $v + u \leq_{\text{lex}} w + u$.

For $v \in \mathbb{N}^n$ define

$$v + \mathbb{N}^n := \left\{v + w : w \in \mathbb{N}^n\right\}.$$

Theorem 1.8.73. *(Dickson's Lemma)*
Let $S \subset \mathbb{N}^n$ be non-empty. Then there are vectors $v_1, \ldots, v_m \in S$ such that

$$S \subset \bigcup_1^m (v_i + \mathbb{N}^n)$$

Proof. We proceed by induction in n. For $n = 1$, S has a minimal element s by the well ordering of the natural numbers, hence any element of S can be written as $s + t$ for some $t \in \mathbb{N}$. Suppose Dickson's lemma is true for some $n \geq 1$. Let S be some non-empty subset of $\mathbb{N}^{n+1}$. Consider the surjection

$$\pi : \mathbb{N}^{n+1} \to \mathbb{N}^n$$
$$(v_1, \ldots, v_n, v_{n+1}) \mapsto (v_1, \ldots, v_n)$$

Consider the set

$$S' := \pi(S) = \{(x_1, \ldots, x_n) \in \mathbb{N}^n : (x_1, \ldots, x_n, x_{n+1}) \in S \text{ for some } x_{n+1} \in \mathbb{N}\}.$$

By induction there are $s_1 = (s_{11}, \ldots, s_{1,n+1}), \ldots, s_m = (s_{m1}, \ldots, s_{m,n+1}) \in S$ such that upon defining $s_i' := (s_{i1}, \ldots, s_{in}) \in S'$

$$S' \subset \bigcup_1^m (s_i' + \mathbb{N}^n).$$

Let $s_{\max} = \max_{i\in\{1,\dots,m\}} s_{i,(n+1)}$. Define

$$S_i := \{v = (v_1, \dots, v_{n+1}) \in S : v_1 = i\} \quad (i \in \{0, \dots, s_{\max}\})$$

and put

$$S_{\max} := \{v = (v_1, \dots, v_{n+1}) \in S : v_{n+1} \geq s_{\max}\}.$$

Note that $S_{\max} \subset \bigcup_1^m (s_i + \mathbb{N}^{n+1})$. Indeed, if $x \in S_{\max}$, then $(x_1, \dots, x_n) \in \bigcup_1^m (s'_i + \mathbb{N}^n)$. In particular, for some $i \in \{1, \dots, m\}$ $x = (s_{i1} + v_1, \dots, s_{in} + v_n) \in s'_i + \mathbb{N}^n$. Since $x_{n+1} \geq s_{\max}$, it follows that $x_{n+1} = s_{i,n+1} + v_{n+1}$, and thus that

$$x = (s_{i1} + v_1, \dots, s_{in} + v_n, s_{i,n+1} + v_{n+1}) \in s_i + \mathbb{N}^{n+1} \subset \bigcup_1^m \left(s_j + \mathbb{N}^{n+1}\right).$$

We furthermore have that $S = S_{\max} \cup \bigcup_0^{s_{\max}-1} S_i$. Again using induction there are $s_1^{(i)}, \dots, s_{m_i}^{(i)} \in \pi(S_i)$ such that

$$\pi(S_i) \subset \bigcup_{j=1}^{m_i} \left(s_j^{(i)} + \mathbb{N}^n\right)$$

Then

$$S_i \subset \bigcup_{j=1}^{m_i} \left(\left(s_{j1}^{(i)}, \dots, s_{jn}^{(i)}, i\right) + \mathbb{N}^{n+1}\right).$$

It thus follows that

$$S \subset \bigcup_1^m \left(s_j + \mathbb{N}^{n+1}\right) \cup \bigcup_{i=0}^{s_{\max}} \bigcup_{j=1}^{m_i} \left(\left(s_{j1}^{(i)}, \dots, s_{jn}^{(i)}, i\right) + \mathbb{N}^{n+1}\right).$$

□

Corollary 1.8.74. *A term ordering $\leq$ on $\mathbb{N}^n$ is a well-ordering.*

Proof. Let $S \subset \mathbb{N}^n$ be a non-empty subset. By Dickson's lemma there are $s_1, \dots, s_m \in S$ such that $S \subset \bigcup_1^m (s_i + \mathbb{N}^n)$. Since $\leq$ is a total order, we can define $s_{\min} := \min_{i\in\{1,\dots,m\}} s_i$. Let $s \in S$. For some $j \in \{1, \dots, m\}$, $s = s_j + v$ for some $v \in \mathbb{N}^n$. Now we have the following implications using properties of term orders,

$$0 \leq v \text{ and } s_{\min} \leq s_j \implies s_{\min} \leq s_{\min} + v \leq s_j + v = s,$$

hence $s_{\min}$ is a least element of S, implying $\leq$ is a well-ordering. □

A term order on $\mathbb{N}^n$ defines a total order on the monomials in $R[x_1, \dots, x_n]$ by defining $x^v \leq x^w$ if $v \leq w$. This total order will have the property that $x^{v_1} \leq x^{v_2} \implies x^{v_1+v} \leq x^{v_2+v}$ and $1 \leq x^v$. From this definition we gain a way of comparing polynomials using initials terms with respect to a term order.

Definition 1.8.75. Let $f = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v \in R[x_1,\ldots,x_n] \setminus 0$ and $\leq$ a term order on $\mathbb{N}^n$. We define the *initial term of f with respect to* $\leq$ to be the monomial

$$\text{in}_{\leq} f := \max_{v\in\mathbb{N}^n : a_v \neq 0} a_v\mathbf{x}^v.$$

Lemma 1.8.76. *Let $f, g \in R[x_1,\ldots,x_n] \setminus 0$. Then one finds that*

1. $(\text{in}_{\leq} f + g) \leq \max(\text{in}_{\leq} f, \text{in}_{\leq} g)$
2. *If* $\text{in}_{\leq} f < \text{in}_{\leq} g$, *then* $(\text{in}_{\leq} f + g) = \text{in}_{\leq} g$.
3. *If the leading terms of f and g are equal then* $\text{in}_{\leq}(f-g) < \text{in}_{\leq} f = \text{in}_{\leq} g$.
4. $\text{in}_{\leq} fg \leq (lm_{\leq} f)(lm_{\leq} g)$.
5. *Suppose R is an integral domain. Then* $\text{in}_{\leq} fg = (\text{in}_{\leq} f)(\text{in}_{\leq} g)$.

Proof. Write $f = \sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v$ and $g = \sum_{v\in\mathbb{N}^N} b_v\mathbf{x}^v$. Let $w, u \in \mathbb{N}^n$ be given such that $a_v\mathbf{x}^w = \text{in}_{\leq} f$ and $b_w\mathbf{x}^u = \text{in}_{\leq} g$. The proofs here are very similar Lemma 1.8.26 in some respects. 1. Let $v \in \mathbb{N}^n$ be given such that $v > \max(w,u)$. Then $a_v, b_v = 0$ hence $a_v + b_v = 0$. It thus follows that

$$(\text{in}_{\leq} f + g) = \max_{v\in\mathbb{N}^n : a_v+b_v\neq 0} (a_v + b_v)\mathbf{x}^v \leq \max(\text{in}_{\leq} f, \text{in}_{\leq} g).$$

2. When $\text{in}_{\leq} f < \text{in}_{\leq} g$, $(\text{in}_{\leq} f + g) \leq \max(\text{in}_{\leq} f, \text{in}_{\leq} g)$. Note that $a_u + b_u = b_u \neq 0$, hence $(\text{in}_{\leq} f + g) = \text{in}_{\leq} g$.
3. Since leading terms of f and g are equal $w = u$ and $a_w = b_w$. Hence if $v \in \mathbb{N}^n$ is given such that $v \geq w$, we have that $a_v - b_v = 0$, thus it follows that

$$\text{in}_{\leq}(f-g) = \max_{v\in\mathbb{N}^n : a_v-b_v\neq 0} (a_v - b_v)\mathbf{x}^v < \text{in}_{\leq} f.$$

4. & 5. Let $v_1, v_2 \in \mathbb{N}^n$ such that $v_1 + v_2 = w + u$ and $v_1 \neq w$. If $v_1 > w$ or $v_1 < w$. Hence

$$a_{v_1} = 0 \implies a_{v_1} b_{v_2} = 0.$$

In the other case $v_2 > u$, because otherwise $v_1 + v_2 < u + w$. Hence

$$b_{v_2} = 0 \implies a_{v_1} b_{v_2} = 0.$$

It thus follows that

$$\sum_{v_1,v_2\in\mathbb{N}^n : v_1+v_2=w+u} a_{v_1} b_{v_2} = a_w b_u.$$

Let $v > u + w$. Let $v_1, v_2 \in \mathbb{N}^n$ such that $v_1 + v_2 = v$. Then $a_{v_1} b_{v_2} = 0$, implying $\sum_{v_1,v_2 \in \mathbb{N}^n : v_1+v_2=v} a_{v_1} b_{v_2} = 0$. We thus have that

$$\operatorname{in}_{\leq} fg = \max_{v \in \mathbb{N}^n : \sum_{v_1,v_2 \in \mathbb{N}^n : v_1+v_2=v} a_{v_1} b_{v_2} \neq 0} \left[\sum_{v_1,v_2 \in \mathbb{N}^n : v_1+v_2=v} a_{v_1} b_{v_2} \right] \mathbf{x}^v = \mathbf{x}^{w+u} = (\operatorname{in}_{\leq} f)(\operatorname{in}_{\leq} g).$$

If R is an integral domain we get that

$$\operatorname{in}_{\leq} fg = \left[\sum_{v_1,v_2 \in \mathbb{N}^n : v_1+v_2=w+u} a_{v_1} b_{v_2} \right] \mathbf{x}^{w+u} = (a_w \mathbf{x}^w)(b_u \mathbf{x}^u) = (\operatorname{in}_{\leq} f)(\operatorname{in}_{\leq} g).$$

□

The upshot of introducing this tool of bookkeeping is that it allows to do polynomial division. For $\mathbf{x}^v \mid \mathbf{x}^w$ we define $\frac{a\mathbf{x}^w}{b\mathbf{x}^v} := \frac{a}{b}\mathbf{x}^{w-v}$.

Theorem 1.8.77. *Let R be an integral domain. Let $f, f_1, \ldots, f_m \in R[x_1, \ldots, x_n] \setminus 0$. Put $F = \{f_1, \ldots, f_m\}$. Then there are $\lambda_1, \ldots, \lambda_m, f^F \in R[x_1, \ldots, x_n]$ such that*

$$f = \left[\sum_1^m \lambda_i f_i \right] + f^F,$$

and $\operatorname{in}_{\leq} \lambda_i f_i \leq \operatorname{in}_{\leq} f$ for every $i \in \{1, \ldots, m\}$ with $\lambda_i \neq 0$ and $f^F = 0$ or $\operatorname{in}_{\leq} f_i \nmid t$ for every i where t is a term of f^F.

Proof. We aim to provide a division algorithm that produces the desired the $\lambda_1, \ldots, \lambda_m, f^F$. Define $\lambda_i^{(0)} := 0$ for every $i \in \{1, \ldots, m\}$, $r^{(0)} = 0$ and $s^{(0)} = f$. We note that

$$f = \left[\sum_1^m \lambda_i^{(0)} f_i \right] + \left(r^{(0)} + s^{(0)} \right).$$

We want to recursively define $\lambda_1^{(j)}, \ldots, \lambda_m^{(j)}, r^{(j)}, s^{(j)} \in R[\mathbf{x}]$ such that

$$f = \left[\sum_1^m \lambda_i^{(j)} f_i \right] + \left((r^{(j)} + s^{(j)} \right) \tag{2}$$

for every j and have that $s^{(N)} = 0$ at some N such that putting $\lambda_i = \lambda_i^{(N)}$, $f^F = s^{(N)}$ these polynomials will have the remaining desired properties. For $j \geq 0$ if $s^{(j)} = 0$ put $N = j$ and terminate, otherwise if there is an $i \in \{1, \ldots, m\}$ such $\operatorname{in}_{\leq} f_i \mid \operatorname{in}_{\leq} s^{(j)}$, and pick the smallest such. We then define

$$\begin{cases} s^{(j+1)} := s^{(j)} - \frac{\operatorname{in}_{\leq} s^{(j)}}{\operatorname{in}_{\leq} f_i} f_i, \\ \lambda_i^{(j+1)} := \lambda_i^{(j)} + \frac{\operatorname{in}_{\leq} s^{(j)}}{\operatorname{in}_{\leq} f_i}, \\ \lambda_k^{(j+1)} = \lambda_k^{(j)} \text{ for } k \neq i, \\ r^{(j+1)} := r^{(j)}. \end{cases} \tag{3}$$

We note that indeed the identity (2) is fulfilled for $j+1$ since it is obtained by adding and subtracting $\frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} f_i$. If no such i exists we instead define

$$\begin{cases} r^{(j+1)} := r^{(j)} + \text{in}_\leq\, s^{(j)}, \\ s^{(j+1)} := s^{(j)} - \text{in}_\leq\, s^{(j)}, \\ \lambda_i^{(j+1)} := \lambda_i^{(j)}. \end{cases} \tag{4}$$

Again clearly (2) is still true for $j+1$, since $r^{(j+1)} + s^{(j+1)} = r^{(j)} + s^{(j)}$.
We now show that the above algorithm terminates. Let $j \geq 0$ such that $s^{(j+1)} \neq 0$.
Consider that we land in case (3). We denote $\text{in}_\leq\, s^{(j)} = \alpha_v \mathbf{x}^v$ and $\text{in}_\leq\, f_i = \beta_w \mathbf{x}^w$, where i is the minimal index for which the initial term of f_i divides the initial term of $s^{(j)}$. Then

$$\text{in}_\leq \left(\frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} f_i \right) = \frac{a_v}{b_w} \mathbf{x}^{v-w} b_w \mathbf{x}^w = a_v \mathbf{x}^v = \text{in}_\leq\, s^{(j)}$$

we have thus have that

$$\text{in}_\leq\, s^{(j+1)} = \text{in}_\leq \left(s^{(j)} - \frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} f_i \right) < \text{in}_\leq\, s^{(j)}.$$

Landing in case (4) we have that

$$\text{in}_\leq\, s^{(j+1)} = \text{in}_\leq \left(s^{(j)} - \text{in}_\leq\, s^{(j)} \right) < \text{in}_\leq\, s^{(j)}.$$

Then sequence of non-zero $s^{(j)}$ is thus a strictly decreasing sequence. Let S denote the set of these elements. Since $s^{(0)} = f \neq 0$, $S \neq \emptyset$. This means S has a minimal element $s^{(N-1)}$, since a term order is a well-ordering. Then $s^{(N)} = 0$, for otherwise $s^{(N)} < s^{(N-1)}$.
As advertised we put $\lambda_i := \lambda_i^{(N)}$ for $i \in \{1, \ldots, m\}$ and $f^F := r^{(N)}$. For each $j \geq 0$ for each $i \in \{1, \ldots, m\}$ for which $a_i^{(j)} = 0$ and $a_i^{(j+1)} \neq 0$ we have that

$$\text{in}_\leq\, \lambda_i^{(j+1)} f_i = \text{in}_\leq \left(\left(\lambda_i^{(j)} + \frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} \right) f_i \right) = \text{in}_\leq \left(\frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} f_i \right) = \text{in}_\leq\, s^{(j)} \leq \text{in}_\leq\, f$$

It thus follows by induction in the j for which $\lambda_i^{(j)} \neq 0$ that

$$\begin{cases} \text{in}_\leq\, \lambda_i^{(j+1)} f_i = \text{in}_\leq\, \lambda_i^{(j)} f_i \leq \text{in}_\leq\, f, \\ \text{or} \\ \text{in}_\leq\, \lambda_i^{(j+1)} f_i = \text{in}_\leq \left(\left(\lambda_i^{(j)} + \frac{\text{in}_\leq\, s^{(j)}}{\text{in}_\leq\, f_i} \right) f_i \right) \leq \max \left(\lambda_i^{(j)} f_i, \text{in}_\leq\, s^{(j)} \right) \leq \text{in}_\leq\, f. \end{cases}$$

It thus follows that if $\lambda_i \neq 0$,

$$\text{in}_\leq\, \lambda_i f_i \leq \text{in}_\leq\, f.$$

Note lastly that each $r^{(j)}$ is 0 or a sum of terms not divisible by any $\text{in}_{\leq} f_i$, since only case where $r^{(j+1)}$ is different from $r^{(j)}$ is in case (4), where the no initial of an f_i divides $\text{in}_{\leq} s^{(j)}$, thus

$$f^F = \sum_1^k \text{in}_{\leq} s^{(j_i)}$$

where no initial term of an f_i divides $\text{in}_{\leq} s^{(j_i)}$. □

A special case of the above is the case of $R[x]$ where our only choice of term ordering is the usual ordering on $\mathbb{N}$.

Corollary 1.8.78. *Let R be an integral domain. Let $f, g \in R[x_1, \ldots, x_n]$ where the leading coefficient of g is invertible. Then there are $q, r \in R[x_1, \ldots, x_n]$ with $r = 0$ or* deg $r <$ deg g *such that*

$$f = qg + r.$$

In particular, $R[x]$ is a Euclidean domain.

Proof. For the existence, note that by the above theorem, there are $q, r \in R[x]$ with

$$f = qg + r$$

and $r = 0$ or $\text{in}_{\leq} g \nmid r$. The second condition is equivalent to $r = 0$ or deg $r <$ deg g, since $x^k \mid x^l$ if and only if $k \leq l$.
To prove uniqueness, note that if $qg + r = q'g + r'$, then $(q - q')g = r - r'$, then $r - r' = 0$ since otherwise

$$\text{deg } g \leq \text{deg } r - r' \leq \max(\text{deg } r, \text{deg } r')$$

which would imply deg $g \leq$ deg r or deg $g \leq$ deg r'. Since R is an integral domain, $(q - q')g = 0$ implies $q = q'$. □

Remark 1.8.79. When there is no restrictions on the ring R, we get that there $q, r \in R[\mathbf{x}]$ such that

$$f = qg + r$$

with $r = 0$ or no term of r is divisible by the leading term of g. In general, this is insufficient to get that $R[x]$ is a Euclidean domain, which indeed is not the case in general.

Corollary 1.8.80. *When K is a field $K[x]$ is a Euclidean domain.*

Corollary 1.8.81. *When K is a field $K[x]$ is a PID.*

Proof. Since a Euclidean domain is a PID. □

Corollary 1.8.82. *When K is a field $K[x]$ is a UFD.*

Proof. Since a PID is a UFD. □

1.8.12 Gröbner Bases and Buchbergers Algorithm

For the exposition on Gröbner bases we fix a field K.

Definition 1.8.83. Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Let $\leq$ be a term ordering on $K[\mathbf{x}]$. A finite set of polynomials $G \subset K[\mathbf{x}] \setminus 0$ is called a *Gröbner basis* for I with respect to $\leq$, if $G \subset I$ and for every $f \in I \setminus 0$ there is a $g \in G$ such that $\text{in}_{\leq} g \mid \text{in}_{\leq} f$.
A finite set $G = \{f_1, \ldots, f_m\} \subset K[\mathbf{x}] \setminus 0$ is called a Gröbner basis with respect to $\leq$ if it is a Gröbner basis for $\langle f_1, \ldots, f_m \rangle$ with respect to $\leq$.

Proposition 1.8.84. *Let $G = \{f_1, \ldots, f_m\} \subset I$ be a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ with respect to a term order $\leq$. For $f \in K[\mathbf{x}]$,*

$$f \in I \iff f^G = 0$$

Proof. " $\Longleftarrow$ ": If $f^G = 0$ there are $\lambda_1, \ldots, \lambda_m \in K[x_1, \ldots, x_n]$ such that $f = \sum_1^m \lambda_i f_i \in I$ using the division algorithm.
" $\Longrightarrow$ ": Suppose $f \in I$. Suppose for a contradiction that $f^G \neq 0$. Using the division algorithm with respect to $\leq$ we obtain $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$f = \left[\sum_1^m \lambda_i f_i\right] + f^G \implies f^G = f - \sum_1^m \lambda_i f_i \in I.$$

Then since G is a Gröbner basis there is some $i \in \{1, \ldots, m\}$ such that $\text{in}_{\leq} f_i \mid f^G$, but since $f^G \neq 0$ this is not possible. □

As a corollary we obtain that every Gröbner basis of an ideal with respect to some term order will be a generating set for said ideal.

Corollary 1.8.85. *Let $I \subset K[x_1, \ldots, x_n]$ be an ideal and $G \subset I$ a Gröbner basis for I with respect to some term order $\leq$. Then $I = \langle G \rangle$.*

Proof. By definition $G \subset I \implies \langle G \rangle \subset I$. Let $f \in I$. By the above proposition, $f^G = 0$, meaning there are $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$f = \sum_1^m \lambda_i f_i \in \langle G \rangle$$

□

A somewhat curious consequence of the introduction of Gröbner bases is that it provides us with a rather simple way to prove the Hilbert basis theorem over fields. I.e. one can prove that any polynomial ideal over a field has a Gröbner basis, which by the above corollary constitutes a finite generating set.

Corollary 1.8.86. *(Hilbert's basis theorem over fields) Let $I \subset K[x_1, \ldots, x_n]$ be a non-zero ideal and $\leq$ a term order. Then there is a Gröbner basis $G = \{f_1, \ldots, f_m\} \subset I$ for I with respect to $\leq$, hence $I = \langle f_1, \ldots, f_m \rangle$ by the prior corollary.*

Proof. Put $S = \{v \in \mathbb{N}^n : \mathbf{x}^v = \text{in}_{\leq} f \text{ for some } f \in I\}$. Clearly $S \neq \emptyset$, hence by Dickson's lemma we may find $v_1, \ldots, v_m \in S$ such that

$$S \subset \bigcup_1^m (v_i + \mathbb{N}^n)$$

Let $f_i \in I$ be given such that $\text{in}_{\leq} f_i = \mathbf{x}^{v_i}$ for $i \in \{1, \ldots, m\}$ and put $G = \{f_1, \ldots, f_m\} \subset I$. Let $f \in I \setminus 0$, and pick $v \in \mathbb{N}^n$ such that $a_v \mathbf{x}^v = \text{in}_{\leq} f$. Since, $v \in S$, $v = v_j + w$ for some $j \in \{1, \ldots, m\}$ and $w \in \mathbb{N}^n$. Then

$$\text{in}_{\leq} f = a_v \mathbf{x}^v = a_v \mathbf{x}^{v_j + w} = (a_v \mathbf{x}^w)\, \mathbf{x}^{v_j} = (a_v \mathbf{x}^w)\, \text{in}_{\leq} f_j \implies \text{in}_{\leq} f_j \mid \text{in}_{\leq} f.$$

This verifies that G is a Gröbner basis for I. □

The machinery of Gröbner bases provides a way to perform the division algorithm with respect to a term order in a fashion that ensures uniqueness of remainders and the indifference of the order of the divisor polynomials.

Theorem 1.8.87. *Let $\leq$ be a term order on $K[x_1, \ldots, x_n]$ and $G = \{f_1, \ldots, f_m\} \subset K[\mathbf{x}] \setminus 0$ a Gröbner basis with respect to $\leq$. Let $f \in K[\mathbf{x}] \setminus 0$. Then any polynomial r satisfying the properties of f^G obtained from the division algorithm of f by $f_1, \ldots, f_m$ is equal to f^G. Furthermore, the remainder outputted by the division algorithm remains unchanged after a permutation of $f_1, \ldots, f_m$.*

Proof. Let $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$f = \left[\sum_1^m \lambda_i f_i \right] + f^G.$$

Suppose there is an $r \in K[\mathbf{x}]$ with $r = 0$ or r is not divisible by the initial term of any f_i such that there are $\lambda_1', \ldots, \lambda_m' \in K[\mathbf{x}]$ satisfying

$$f = \left[\sum_1^m \lambda_i' f_i \right] + r.$$

Then

$$f^G - r = \sum_1^m (\lambda_i - \lambda_i') f_i \in I.$$

Suppose for a contradiction $f^G - r \neq 0$. Then there is a $j \in \{1, \ldots, m\}$ such that $\text{in}_{\leq} f_j \mid \text{in}_{\leq} \left(f^G - r\right)$ implying $\text{in}_{\leq} f_j \mid \text{in}_{\leq} f^G$ or $\text{in}_{\leq} f_j \mid \text{in}_{\leq} r$ leading to a contradiction. It follows that $f^G = r$.
Let $\omega \in \mathcal{S}(m)$ be a permutation. Let $\lambda'_1, \ldots, \lambda'_m, (f^G)' \in K[\mathbf{x}]$ be the outcome of the division with respect to $\leq$ of f with $f_{\omega(1)}, \ldots, f_{\omega(m)}$. Then by uniqueness of the remainder $f^G = (f^G)'$. □

We now seek a way to compute a Gröbner basis for a given ideal. The introduction of S-polynomials and Buchberger's S-criterion will lead us to Buchberger's algorithm for computing Gröbner bases.

Definition 1.8.88. Let $f \in K[x_1, \ldots, x_n]$ and $F = \{f_1, \ldots, f_m\} \subset K[\mathbf{x}] \setminus 0$. We say that f *reduces to zero modulo* F if there are $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$f = \sum_1^m \lambda_i f_i$$

and $\text{in}_{\leq} \lambda_i f_i \leq \text{in}_{\leq} f$ for $i \in \{1, \ldots, m\}$ with $a_i f_i \neq 0$. This will be denoted $f \to_F 0$.

Note that this definition does not depend on a term order, however this definition leads us to a reformulation of Proposition 1.8.84. Before formulating this consider the following lemmas

Lemma 1.8.89. *Let $F = \{f_1, \ldots, f_m\} \subset K[\mathbf{x}] \setminus 0$ and let $f \in I := \langle F \rangle$ be non-zero with initial term $a_v \mathbf{x}^v$. Consider $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that*

$$f = \sum_1^m \lambda_i f_i.$$

For each $i \in \{1, \ldots, m\}$ where $\lambda_i \neq 0$, pick $v_i, w_i \in \mathbb{N}^n$ such that $b_i \mathbf{x}^{v_i} = \text{in}_{\leq} \lambda_i$ and $c_i \mathbf{x}^{w_i} = \text{in}_{\leq} f_i$. set

$$\kappa = \max \left\{ v_i + w_i \in \mathbb{N}^n : i \in \{1, \ldots, m\}, \lambda_i \neq 0 \right\}.$$

Then $v \leq \kappa$ and the following statements hold

1. $v = \kappa \iff \text{in}_{\leq} f_i \lambda_i \leq \text{in}_{\leq} f$ *for every* $i \in \{1, \ldots, m\}$ *such that* $\lambda_i \neq 0$.
2. $v = \kappa \implies \text{in}_{\leq} f_i \mid \text{in}_{\leq} f$ *for some* $i \in \{1, \ldots, m\}$

Proof. Forgetting briefly that we assumed $f = \sum_1^m \lambda_i f_i$, if $v > \kappa$, then

$$\text{in}_{\leq} f > \max_{i \in \{1,\ldots,m\} : \lambda_i \neq 0} \text{in}_{\leq} \lambda_i f_i = \text{in}_{\leq} \left(\sum_1^m \lambda_i f_i \right) \implies f \neq \sum_1^m \lambda_i f_i.$$

It thus follows that since we assumed $f = \sum_1^m \lambda_i f_i$, we get the bound $v \leq \kappa$.
1. For every $i \in \{1, \ldots, m\}$ such that $\lambda_i f_i \neq 0$ we have that

$$b_i c_i \mathbf{x}^{v_i + w_i} = \text{in}_{\leq} \lambda_i \leq \text{in}_{\leq} f = a_v \mathbf{x}^v \iff \mathbf{x}^{v_i+w_i} \leq \mathbf{x}^v \iff v_i + w_i \leq v$$

hence

$$\kappa = v \iff \text{in}_{\leq} \lambda_i f_i \leq \text{in}_{\leq} f.$$

2. Suppose $v = \kappa$. Then $v = v_i + w_i$and hence $b_i c_i = a_v$ for some $i \in \{1, \ldots, m\}$. WLOG we may then write

$$\text{in}_{\leq} f = a_v \mathbf{x}^v = \left[\sum_1^l b_i c_i\right] \mathbf{x}^{v_1} \mathbf{x}^{w_1} = \left(\left[\sum_1^l b_i \frac{c_i}{c_1}\right] \mathbf{x}^{v_1}\right) c_1 \mathbf{x}^{w_1} = \left(\left[\sum_1^l b_i \frac{c_i}{c_1}\right] \mathbf{x}^{v_1}\right) \text{in}_{\leq} f_1,$$

implying $\text{in}_{\leq} f_1 \mid \text{in}_{\leq} f$ for some $l \leq m$. □

Lemma 1.8.90. *Let $F = \{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n] \backslash 0$ and set $I = \langle F \rangle$. The following statements hold*

1. *If $f \to_F 0$ for every $f \in I$ then F is a Gröbner basis.*
2. *If F is a Gröbner basis then for $f \in I \setminus 0$,*

$$f^F = 0 \iff f \to_F 0$$

Proof. 1. Let $f \in I \setminus 0$. Then there are $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$f = \sum_1^m \lambda_i f_i,$$

and $\text{in}_{\leq} \lambda_i f_i \leq \text{in}_{\leq} f$ for every $i \in \{1, \ldots, m\}$ with $\lambda_i \neq 0$. Then by the above lemma

$$\kappa := \max \{\text{in}_{\leq} \lambda_i f_i : i \in \{1, \ldots, m\}, \lambda_i \neq 0\} = \text{in}_{\leq} f$$

and by the same lemma we then have that $\text{in}_{\leq} f_i \mid \text{in}_{\leq} f$ for some $i \in \{1, \ldots, m\}$, hence F is a Gröbner basis.
2. " $\Longrightarrow$ ": If $f^F = 0$ then there are $\lambda_1, \ldots, \lambda_i \in K[\mathbf{x}]$ such that

$$f = \sum_1^m \lambda_i f_i$$

and $\text{in}_{\leq} \lambda_i f_i \leq \text{in}_{\leq} f$ for $i \in \{1, \ldots, m\}$ with $\lambda_i \neq 0$ hence by definition $f \to_F 0$.
" $\Longleftarrow$ ": This follows from Proposition 1.8.84. □

We now introduce S-polynomials

Definition 1.8.91. Let $f, g \in K[x_1, \ldots, x_n] \backslash 0$. Pick $w \in \mathbb{N}^n$ such that $\mathbf{x}^w = \operatorname{lcm}(\operatorname{in}_\leq f, \operatorname{in}_\leq g)$. We define *the S-polynomial* or *the syzygy* of f and g to be

$$S(f,g) := \frac{\mathbf{x}^w}{\operatorname{in}_\leq f} f - \frac{\mathbf{x}^w}{\operatorname{in}_\leq g} g.$$

Remark 1.8.92. Recall that $w = \left(\max\left(w_1^f, w_1^g\right), \ldots, \max\left(w_n^f, w_n^g\right)\right)$, where $a_{w^f}\mathbf{x}^{w^f} = \operatorname{in}_\leq f$ and $b_{w^g}\mathbf{x}^{w^g} = \operatorname{in}_\leq g$.

Note this simple fact about S-polynomials

Lemma 1.8.93. *Let $f, g \in K[x_1, \ldots, x_n] \backslash 0$. Pick $w \in \mathbb{N}^n$ such that $\mathbf{x}^w = \operatorname{lcm}(\operatorname{in}_\leq f, \operatorname{in}_\leq g)$. Then $\operatorname{in}_\leq S(f,g) < \mathbf{x}^w$. In other words, the initial term of $\frac{\mathbf{x}^w}{\operatorname{in}_\leq f} f$ cancels with the initial term of $-\frac{\mathbf{x}^w}{\operatorname{in}_\leq g} g$.*

Proof. Indeed, note that

$$\begin{aligned}\operatorname{in}_\leq \frac{\mathbf{x}^w}{\operatorname{in}_\leq f} f - \operatorname{in}_\leq \frac{\mathbf{x}^w}{\operatorname{in}_\leq g} g &= \left(\operatorname{in}_\leq \frac{\mathbf{x}^w}{\operatorname{in}_\leq f}\right)(\operatorname{in}_\leq f) - \left(\operatorname{in}_\leq \frac{\mathbf{x}^w}{\operatorname{in}_\leq g}\right)(\operatorname{in}_\leq g) \\ &= \frac{\mathbf{x}^w}{\operatorname{in}_\leq f}\operatorname{in}_\leq f - \frac{\mathbf{x}^w}{\operatorname{in}_\leq g}\operatorname{in}_\leq g = \mathbf{x}^w - \mathbf{x}^w = 0\end{aligned}$$

hence the result follows from Lemma 1.8.76 3. □

These polynomials will be the backbone of the criterion for checking that a generating set is a Gröbner basis that Lemma 1.8.90 1. provides more practical. To be precise, we can reduce this criterion to just check that a finite set of S-polynomials reduce to zero modulo F.

Lemma 1.8.94. *Let $F = \{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n] \setminus 0$ and let $1 \leq l \leq m$ and $\sum_1^m \lambda_i f_i \in \langle F \rangle$ with $b_i\mathbf{x}^{v_i} = \operatorname{in}_\leq \lambda_i$, $c_i\mathbf{x}^{w_i} = \operatorname{in}_\leq f_i$ be given such that*

$$b_i c_i x^{v_i + w_i} = \operatorname{in}_\leq \lambda_i f_i = \kappa := \max_{j \in \{1,\ldots,m\}} \operatorname{in}_\leq \lambda_j f_i$$

for every $i \in \{1, \ldots, l\}$ and $\sum_1^l b_i c_i \neq 0$. Define

$$\mu_{\lambda_1,\ldots,\lambda_m,F} := \sum_1^m (\operatorname{in}_\leq \lambda_i) f_i.$$

Then $\mu_{\lambda_1,\ldots,\lambda_m,F} \in I := \langle S(f_1, f_2), S(f_2, f_3), \ldots, S(f_{l-1}, f_l) \rangle$, and $\operatorname{in}_\leq \mu_{\lambda_1,\ldots,\lambda_m,F} < \kappa$.

Proof. Put $g_i := \mathbf{x}^{v_i}\frac{f_i}{c_i}$ for $i \in \{1,\dots,l\}$. Then

$$\mu_{\lambda_1,\dots,\lambda_m,F} = \sum_1^l b_i c_i \left(\mathbf{x}^{v_i+w_i} + \dots\right) = \sum_1^l b_i c_i g_i$$
$$= \left[\sum_{j=1}^{l-1}\left[\sum_{i=1}^{j} b_i c_i\right](g_j - g_{j+1})\right] + \underbrace{\left[\sum_1^l b_i c_i\right]}_{=0} g_l.$$

Put $x^{u_{ij}} := \operatorname{lcm}(\mathbf{x}^{w_i}, \mathbf{x}^{w_j})$ for $i,j \in \{1,\dots,l\}$ and note that

$$g_i - g_j = \frac{\mathbf{x}^{v_i}}{c_i}f_i - \frac{\mathbf{x}^{v_j}}{c_j}f_j = \frac{\mathbf{x}^{v_i+w_i}}{c_i\mathbf{x}^{w_i}}f_i - \frac{\mathbf{x}^{v_j+w_j}}{c_j\mathbf{x}^{w_j}}f_j \overset{(*)}{=} \mathbf{x}^{\xi_{ij}}\left(\frac{\mathbf{x}^{u_{ij}}}{\operatorname{in}_\leq f_i}f_i - \frac{\mathbf{x}^{u_{ij}}}{\operatorname{in}_\leq f_j}f_j\right) = \mathbf{x}^{\xi_{ij}}S(f_i,f_j),$$

where we in step $(*)$ use that $u_{ij} < w_i + v_i = w_j + v_j$ to see that

$$v_i + w_i = v_j + w_j \implies \underbrace{v_i + w_i - u_{ij}}_{\xi_{ij}} = v_j + w_j - u_{ij}.$$

Upon setting $\xi_i := \xi_{i,i+1}$ we find

$$\mu_{\lambda_1,\dots,\lambda_l,F} = \sum_1^{l-1} \mathbf{x}^{\xi_i} S(f_i, f_{i+1}) \in I.$$

Set $u_i = u_{i(i+1)}$. Then additionally we have that

$$\operatorname{in}_\leq \mu_{\lambda_1,\dots,\lambda_m,F} = \max_{i\in\{1,\dots,l-1\}} x^{\xi_i}\operatorname{in}_\leq S(f_i,f_{i+1}) < \max_{i\in\{1,\dots,l-1\}} \mathbf{x}^{\xi_i+u_i} = \max_{i\in\{1,\dots,l-1\}} \mathbf{x}^{v_i+w_i} = \kappa.$$

□

Theorem 1.8.95. *Let $F = \{f_1,\dots,f_m\} \subset K[x_1,\dots,x_n]\setminus 0$. If $S(f_i,f_j) \to_F 0$ for every $i,j \in \{1,\dots,m\}$, then $f \to_F 0$ for every $f \in I := \langle F\rangle$ meaning F is a Gröbner basis (cf. Lemma 1.8.90).*

Proof. Let $f = \sum_1^m \lambda_i f_i \in I$. If $\operatorname{in}_\leq \lambda_i f_i \leq \operatorname{in}_\leq f$ for every i, we are done.
Suppose this is not the case. We aim to re-express f as an element of I. We do this via a **right-hand side initial term reduction** (this is non-standard terminology), which we will describe now. WLOG we may assume, adopting the notation from Lemma 1.8.94, that

$$b_i c_i x^{v_i+w_i} = \operatorname{in}_\leq \lambda_i f_i = \kappa.$$

Then we have that $\operatorname{in}_\leq f < \kappa$ by Lemma 1.8.89, hence necessarily $\sum_1^l b_i c_i = 0$, which implies $\mu_{\lambda_1,\dots,\lambda_m,F} \in \langle S(f_1,f_2), S(f_2,f_3),\dots,S(f_{l-1},f_l)\rangle$ and $\operatorname{in}_\leq \mu_{\lambda_1,\dots,\lambda_m} < \kappa$. By assumption there are $\psi_1^{(i)},\dots,\psi_m^{(i)} \in K[\mathbf{x}]$ such that $S(f_i,f_{i+1}) = \sum_{j=1}^m \psi_j^{(i)} f_j$ with $\operatorname{in}_\leq \psi_j^{(i)} f_j \leq$

$\text{in}_{\leq} S(f_i, f_{i+1})$ for every $i \in \{1, \ldots, l-1\}$ and $j \in \{1, \ldots, m\}$. This means that

$$\mu_{\lambda_1,\ldots,\lambda_m,F} = \sum_{j=1}^{m} \underbrace{\left[\sum_{i=1}^{l-1} \mathbf{x}^{\xi_i} \psi_j^{(i)}\right]}_{\chi_j} f_j$$

with

$$\text{in}_{\leq} \chi_j f_j = \max_{i\in\{1,\ldots,l-1\}} \mathbf{x}^{\xi_i} \left(\text{in}_{\leq} \psi_j^{(i)}\right) (\text{in}_{\leq} f_j) \leq \max_{i\in\{1,\ldots,l-1\}} \mathbf{x}^{\xi_i} \text{in}_{\leq} S(f_i, f_{i+1}) = \text{in}_{\leq} \mu_{\lambda_1,\ldots,\lambda_m,F} < \kappa.$$

Now note that

$$f = \mu_{\lambda_1,\ldots,\lambda_m,F} + \sum_{1}^{l} (\lambda_i - \text{in}_{\leq} \lambda_i) f_i + \sum_{l+1}^{m} \lambda_i f_i,$$

and that every term on the right-hand side of the above expression is strictly smaller then κ. We now obtain another expression for f: Upon putting $\delta_j = 1$ if $j \leq l$ and $\delta_j = 0$ otherwise we have

$$f = \sum_{j=1}^{m} \underbrace{(\chi_j + \lambda_j - \delta_j \text{in}_{\leq} \lambda_j)}_{\lambda'_j} f_j.$$

This is exactly the right-hand side initial term reduction we wanted to describe. If

$$\text{in}_{\leq} f = \kappa' := \max_{i\in\{1,\ldots,m\}} \lambda'_i f_i,$$

we have $\text{in}_{\leq} \lambda'_i f_i \leq \text{in}_{\leq} f$ for every i and we are done. Otherwise we perform another right-hand side initial term reduction. Note that $\kappa' < \kappa$. If we follow the algorithm of terminating if the right-hand side expression leads to concluding $f \to_F 0$ or otherwise performing a right-hand side reduction we see that by the well-ordering of term orders we can only perform a finite number of iterations of right-hand side reductions, hence this algorithm will have to terminate. We thus conclude that $f \to_F 0$. □

from this theorem we readily collect Buchberger's criterion for checking that a generating set is a Gröbner basis

Corollary 1.8.96. *(Buchberger's S-criterion) Let $F \subset K[x_1, \ldots, x_n] \setminus 0$. Then F is a Gröbner basis if and only if $S(f_i, f_j) \to_F 0$ or equivalently $S(f_i, f_j)^F = 0$ (cf. 1.8.90) for every $i, j \in \{1, \ldots, m\}$.*

Proof. If F is a Gröbner basis then $S(f_i, f_j)^F = 0$ for every $i, j \in \{1, \ldots, m\}$ since $S(f_i, f_j) \in I := \langle F \rangle$ by Proposition 1.8.84, hence $S(f_i, f_j) \to_F 0$ by Lemma 1.8.90. If conversely $S(f_i, f_j) \to_F 0$ for every i and j, it follows from Theorem 1.8.95 that F is a Gröbner basis and hence that $S(f_i, f_j)^F = 0$ by Proposition 1.8.84. □

This leads to Buchberger's algorithm for finding a Gröbner basis for an ideal $I = \langle f_1, \ldots, f_m \rangle \subset K[x_1, \ldots, x_n] \setminus 0$, which we will discuss in the following remark

Remark 1.8.97. (Buchberger's algorithm) We now describe an algorithm for computing a Gröbner basis given an arbitrary generating set for an ideal I. Let $F_0 = \left\{ f_1^{(0)}, \ldots, f_{m(0)}^{(0)} \right\} \subset I$ where $m(0) \geq 1$ be a generating set for I. For $i \geq 0$ if $S\left((f_j^{(i)}, f_k^{(i)} \right)^{F_i} = 0$ for every $f_j^{(i)}, f_k^{(i)} \in F_i = \left\{ f_1^{(i)}, \ldots, f_{m(i)}^{(i)} \right\}$ put $F_{i+1} = F_i$ or simply terminate, for then F_i is a Gröbner basis. Otherwise if there are $f_j^{(i)}, f_k^{(i)} \in F_i$ such that $S\left(f_j^{(i)}, f_k^{(i)} \right)^{F_i} \neq 0$ put

$$F_{i+1} = F_i \cup \left\{ S\left(f_j^{(i)}, f_k^{(i)} \right)^{F_i} \right\}.$$

Note that $S\left(f_j^{(i)}, f_k^{(i)} \right)^{F} = \sum_1^m \lambda_i f_i - S\left(f_j^{(i)}, f_k^{(i)} \right) \in I$, hence $\langle F_{i+1} \rangle = I$. The claim is that the ascending chain $F_0 \subset F_1 \subset \ldots$ will in fact stabilize, hence there we produce a Gröbner basis at some point. We check this claim in the next theorem.

Lemma 1.8.98. *Let $\{t_i\}_{i \geq 0} \subset K[x_1, \ldots, x_n]$ be some sequence of which an element is either a term or 0, i.e. $t_i = a_i \mathbf{x}^{v_i}$ for some $a_i \in K$, $v_i \in \mathbb{N}^n$. Then for some $N \geq 0$ for every $i \geq N$, $t_j \mid t_i$ for some $j < N$*

Proof. If $a_i = 0$ for every $i \geq 0$ then the statement is trivial. Suppose this is not the case and put $S = \{v_i : i \geq 0, a_i \neq 0\} \subset \mathbb{N}^n$, then by Dickson's lemma there are $v_{i(1)}, \ldots, v_{i(k)} \in S$ such that

$$S \subset \bigcup_{j=1}^{k} \left(v_{i(j)} + \mathbb{N}^n \right).$$

Set $N = \max_{j \in \{1,\ldots,k\}} i(j)$ and let $i \geq N$. Then $v_i = v_{i(j)} + w$ for some $j \in \{1, \ldots, k\}$, $w \in \mathbb{N}^n$, hence

$$t_i = a_i \mathbf{x}^{v_i} = a_i \mathbf{x}^{v_{i(j)}+w} = \left(a_{i(j)} \mathbf{x}^{v_{i(j)}} \right) \left(\frac{a_i}{a_{i(j)}} \mathbf{x}^w \right) = t_{i(j)} \left(\frac{a_i}{a_{i(j)}} \mathbf{x}^w \right) \implies t_{i(j)} \mid t_i$$

□

Theorem 1.8.99. *Buchberger's algorithm terminates and outputs a Gröbner basis.*

Proof. Buchberger's gives rise to an infinite sequence of polynomials in the following way: start with the initial elements of $F_0 = \{f_1, \ldots, f_m\}$. For $i \geq m+1$ if F_{i-m} is the union of F_{i-m-1} and $\left\{ S(f_j, f_k)^{F_{i-m-1}} \right\}$ for some $j, k \in \{1, \ldots, i-1\}$ then put $f_i = S(f_j, f_k)^{F_{i-m-1}}$ otherwise put $f_i = 0$. We then put $t_i = \text{in}_{\leq} f_i$ if $f_i \neq 0$ or $t_i = 0$ otherwise for every $i \geq 0$. By the above lemma there is an $N \geq 0$ such that for every $l \geq N$, $t_h \mid t_l$ for some $h < N$.

For each $i \geq m$, if $f_i = S(f_j, f_k)^{F_{i-m-1}}$, then any term of f_i is not divisible by $\text{in}_\leq f_q$ for any $q \in \{1, \ldots, i-1\}$. Consider then $l \geq \max(m, N)$ if $t_h \mid t_l$ for $h < N$, then t_l cannot be a term of some $S(f_j, f_k)^{F_{l-m-1}}$, hence $t_l = 0$, implying $f_l = 0$ and hence that F_{l-m} satisfies Buchberger's criterion. □

The below proposition will give an easy criterion for checking whether two polynomials in a generating reduce modulo said generating set.

Proposition 1.8.100. *Let $\leq$ be a term order on $K[x_1, \ldots, x_n]$. Let $f, g \in K[\mathbf{x}] \backslash 0$. Suppose* $\gcd(f, g) = 1$*, then* $S(f, g) \to_{\{f,g\}} 0$.

Proof. By assumption,

$$\text{lcm}(\text{in}_\leq f, \text{in}_\leq g) = (\text{in}_\leq f)\,(\text{in}_\leq g) = \text{in}_\leq fg.$$

Put $r = f - \text{in}_\leq f$ and $s = g - \text{in}_\leq g$. Then $\text{in}_\leq r < \text{in}_\leq f$ or $\text{in}_\leq r = 0$ and $\text{in}_\leq s < \text{in}_\leq g$ or $\text{in}_\leq s = 0$. Then

$$S(f, g) = (\text{in}_\leq g)\, f - (\text{in}_\leq f)\, g = (g - s)f - (f - r)g = rg - sf.$$

Suppose $r = 0$, then $S(f, g) = -sf$, implying $\text{in}_\leq f \leq \text{in}_\leq S(f, g)$ and hence $S(f, g) \to_{\{f,g\}} 0$. Suppose $\text{in}_\leq r < \text{in}_\leq f$. Suppose for a contradiction

$$(\text{in}_\leq r)\,(\text{in}_\leq g) = (\text{in}_\leq s)\,(\text{in}_\leq f)\,.$$

Then $\text{in}_\leq f \mid \text{in}_\leq r$ (since $\text{in}_\leq g \nmid \text{in}_\leq f$), but then $\text{in}_\leq f \leq \text{in}_\leq r$ leading to a contradiction. We then find that

$$\text{in}_\leq S(f, g) = \text{in}_\leq (rg - sf) = \max(\text{in}_\leq rg, \text{in}_\leq sf)$$

hence by Lemma 1.8.89, $S(f, g) \to_{\{f,g\}} 0$. □

Definition 1.8.101. A Gröbner basis $G = \{f_1, \ldots, f_m\} \subset K[x_1, \ldots, x_n] \setminus 0$ is called *minimal*, if

1. $\text{in}_\leq f_i \nmid \text{in}_\leq f_j$ for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$.
2. $\text{in}_\leq f_i = \mathbf{x}^{v_i}$ for some $v_i \in \mathbb{N}^n$ for every $i \in \{1, \ldots, m\}$.

G is *reduced* if it is minimal and if every term in f_i is not divisible by $\text{in}_\leq f_j$ for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$.

Remark 1.8.102. We describe an algorithm for computing a minimal Gröbner basis for a non-zero ideal $I \subset K[x_1, \ldots, x_n]$. Let $G = \{f_1, \ldots, f_m\} \subset I \setminus 0$ be a Gröbner basis. Let a_i be the leading coefficient for f_i for each i. Then define

$$G_0 := \left\{a_1^{-1} f_1, \ldots, a_m^{-1} f_m\right\}$$

For $k \geq 0$, if there are some $f, g \in G_k \setminus 0$ with $f \neq g$ such that that $\text{in}_{\leq} g \mid \text{in}_{\leq} f$, define $G_{k+1} := G_k \setminus \{f\}$, otherwise terminate.

Lemma 1.8.103. *Every ideal $0 \neq I \subset K[x_1, \ldots, x_n]$ has a minimal Gröbner basis.*

Proof. We prove that the algorithm described above always produces a minimal Gröbner basis. Let $G = \{f_1, \ldots, f_m\} \subset I \setminus 0$ be a Gröbner basis for I. We prove the statement by induction in m. For $m = 1$, $G = \{g\}$ for some $g \in I \setminus 0$. Let $a \in K \setminus 0$ be the leading coefficient of g. Then $G_0 = \{a^{-1} g\}$ defines a minimal Gröbner basis. Suppose the algorithm always terminates with a minimal Gröbner basis with mm elements for some $m \geq 1$. Let $G = \{f_1, \ldots, f_{m+1}\}$ be a Gröbner basis and assume WLOG that the coefficient of the polynomials in G are all 1, i.e. that $G_0 = G$. Then if there are no $i, j \in \{1, \ldots, m{+}1\}$ with $i \neq j$ such that $\text{in}_{\leq} f_j \mid \text{in}_{\leq} f_i$, G_0 we terminate and indeed G_0 is a minimal Gröbner basis. Otherwise we put $G_1 = G_0 \setminus \{f_i\}$. Let $f \in I$, then for some $l \in \{1, \ldots, m+1\}$, $\text{in}_{\leq} f_l \mid \text{in}_{\leq} f$. If $l = i$, then $\text{in}_{\leq} f_j \mid \text{in}_{\leq} f$, hence in any case the initial term of f is divisible by the initial term of some polynomial in G_1, implying G_1 is a Gröbner basis. Since G_1 has m elements it follows by induction that G_k is a minimal Gröbner basis. □

Proposition 1.8.104. *Every ideal $0 \neq I \subset K[x_1, \ldots, x_n]$ has a unique reduced Gröbner basis.*

Proof. **Uniqueness:** Consider two reduced Gröbner bases $G = \{f_1, \ldots, f_m\}$ and $G' = \{f'_1, \ldots, f'_{m'}\}$. We first check that the cardinality of these Gröbner bases match. Let $i \in \{1, \ldots, m\}$, then for some $\tau(i) \in \{1, \ldots, m'\}$, $\text{in}_{\leq} f'_{\tau(i)} \mid \text{in}_{\leq} f_i$. Let $j \in \{1, \ldots, m'\}$ then for some $\omega(j) \in \{1, \ldots, m\}$, $\text{in}_{\leq} f_{\omega(j)}$ $\text{in}_{\leq} f'_j$. Then we have that

$$\text{in}_{\leq} f_{\omega(\tau(i))} \mid \text{in}_{\leq} f'_{\tau(i)} \text{ and } \text{in}_{\leq} f'_{\tau(i)} \mid \text{in}_{\leq} f_i \implies \text{in}_{\leq} f_{\omega(\tau(i))} \mid \text{in}_{\leq} f_i$$

by minimality of the Gröbner bases $i = \omega(\tau(i))$. A similar argument shows that $\tau(\omega(j))$ for every $j \in \{1, \ldots, m'\}$, thus τ is a bijection, implying $m = m'$. We proceed by checking that the sets are equal. Note that the above argument also shows that $\text{in}_{\leq} f'_{\tau(i)} = \text{in}_{\leq} f_i$ for every $i \in \{1, \ldots, m\}$ since the coefficient of every initial term is 1.
Let $i \in \{1, \ldots, m\}$. Since $\text{in}_{\leq} f_i = \text{in}_{\leq} f'_{\tau(i)}$, either $f_i = f_{\tau(i)}$ or $\text{in}_{\leq} \left(f_i - f'_{\tau(i)}\right) < \text{in}_{\leq} f_i$. We will see that the second case implies $f_i = f_{\tau(i)}$ as well. In this case no term in $f_i - f'_{\tau(i)}$

is divisible by $\text{in}_{\leq} f_i$. Any term in $f_i - f'_{\tau(i)}$ is a term in $f'_{\tau(i)}$ subtracted from f_i, where at least one of these terms in non-zero. Then by the Gröbner bases being reduced, $\text{in}_{\leq} f_j$ does not divide such a term for any $j \in \{1, \ldots, m\} \setminus \{i\}$. Then $f_i - f'_{\tau(i)} = \left(f_i - f'_{\tau(i)}\right)^G$, but since $f_i - g_i \in I$, this must imply that $f_i = f'_{\tau(i)}$ by Proposition 1.8.84.

Existence: By the prior lemma there is a minimal for Gröbner basis $G = \{f_1, \ldots, f_m\}$ for I. Define $g_i := f_i^{\{g_1,\ldots,g_{i-1},f_i,\ldots,f_m\}\setminus\{f_i\}}$ for every $i \in \{1, \ldots, m\}$. Since $\text{in}_{\leq} f_i$ is divisible by any $\text{in}_{\leq} f_j$ for $j \in \{i+1, \ldots, m-1\}$ we see that g_i is of the form $\text{in}_{\leq} f_i + \ldots$. Thus if $f \in I$, there is some g_j such that $\text{in}_{\leq} g_j = \text{in}_{\leq} f_j \mid \text{in}_{\leq} f$, meaning that each set $\{g_1, \ldots, g_{\{}k-1\}, f_k, \ldots, f_m\}$ and in particular $G' := \{g_1, \ldots, g_m\}$ is a Gröbner basis. The g_i's being residues following the division algorithm by a set of polynomials with initial terms coming from $\{f_1, \ldots, f_m\} \setminus \{f_i\}$ implies that no term in g_i is divisible by any $\text{in}_{\leq} f_j$ for $j \neq i$, thus G' is a reduced Gröbner basis. □

Remark 1.8.105. Note that the existence proof above is of an algorithmic nature. I.e. given an ideal, use Buchberger's algorithm to produce a Gröbner basis. Then use the already presented algorithm for producing a minimal gröbner basis. Then apply the division algorithmic in the way we described above to produce the elements of the reduced Gröbner basis.

Theorem 1.8.106. *Let G be a Gröbner basis for an ideal $I \subset K[x_1, \ldots, x_n]$ with respect to the lexicographic term order with $x_1 < \cdots < x_n$. Then $G \cap K[x_1, \ldots, x_i] \subset K[x_1, \ldots, x_i]$ is a Gröbner basis for the ideal $I \cap K[x_1, \ldots, x_i] \subset K[x_1, \ldots, x_i]$ with respect to the lexicographic term order with $x_1 < \cdots < x_i$ for every $i \in \{1, \ldots, i\}$.*

Proof. Let $G' = G \cap K[x_1, \ldots, x_i]$. Let $f \in I' = I \cap K[x_1, \ldots, i]$. For some $g \in G$, $\text{in}_{\leq} g \mid \text{in}_{\leq} f$. Let $t = a\mathbf{x}^v$ be a term of g and write $b\mathbf{x}^w = \text{in}_{\leq} f$. Then $a\mathbf{x}^v \leq b\mathbf{x}^w$, or in other words $v \leq w$. Let $u \in \mathbb{N}^n$. If $u_j \neq 0$ for $j \in \{i+1, \ldots, m\}$, then $w <_{\text{lex}} u$. Hence we conclude that $v_j = 0$ for every $j \in \{i+1, \ldots, m\}$, implying $t \in K[x_1, \ldots, x_i]$ and ultimately that $g \in K[x_1, \ldots, x_i]$. □

1.8.13 Polynomials over UFD's

We aim to prove that polynomials over unique factorization domains are unique factorization domains. For this reason we fix a UFD R (unless something else is explicitly stated).

Lemma 1.8.107. *Let R be an integral domain. If $p \in R$ is prime then $p \in R[x]$ is prime. Therefor, by induction, if $p \in R$ is prime, then $p \in R[x_1, \ldots, x_n]$ is prime.*

Proof. $p \neq 0$ and $p \notin R[x]^* = R^*$. Let $f = \sum_0^k a_i x^i, g = \sum_0^h b_i x^i \in R[x]$ such that $p \nmid f, g$. Set $s := \max\{i \in \{1, \ldots, k\} : p \nmid a_i\}$ and $t := \max\{i \in \{1, \ldots, h\} : o \nmid b_i\}$. Note that

if $i > s$ or $j > r$, then $p \mid a_i$ or $p \mid b_j$. This means that since $i + j = s + t$ implies $i \geq s$ or $j \geq t$, one finds that p divides every term in $\sum_{i=1}^{h} \sum_{j=1}^{k} \sum_{i+j=s+t} a_i b_j$ other than $a_s b_t$, hence

$$p \nmid \sum_{i=1,\ldots,h, j=1,\ldots,k i+j,=s+t} a_i b_j \implies p \nmid fg.$$

□

Definition 1.8.108. A polynomial $f = \sum_0^n a_i x^i \in R[x]$ is said to be *primitive* if $\gcd(a_0, \ldots, a_n : a_i \neq 0) = 1$.

Lemma 1.8.109. *Let R be a UFD, $K := Q(R)$ and consider a monic $f = x^d + \sum_0^{d-1} \frac{a_i}{b_i} x^i$. Let c be the least common multiple of the b_i's. If $\gcd(a_i, b_i) = 1$ whenever $a_i \neq 0$ then $g := cf$ is primitive.*

Proof. Take any prime divisor p of c. Pick i where p has maximal multiplicity among all b_i's. Then $p \nmid \frac{c}{b_i}$ and $p \nmid a_i$, since $c = \prod_{p \in \mathcal{P}} p^{\max(\nu_p(b_0), \ldots, \nu_p(b_0))}$. Then $p \nmid c\frac{a_i}{b_i}$. So for any divisor of c there is some i for which this is not a divisor of $c\frac{a_i}{b_i}$, implying $\gcd(c, ca_i/b_i : a_i \neq 0) = 1$, hence cf is primitive. □

Lemma 1.8.110. *(Gauss' Lemma) If $f, g \in R[x]$ is primitive, then fg is primitive. This property extends to multivariable polynomials by induction.*

Proof. Suppose fg is not primitive then the greatest common divisor of the coefficients of fg is divisible by some prime $p \in R$. This means $p \mid fg$, hence $p \mid f$ or $p \mid g$ by Lemma 1.8.107, hence the p divides all of the coefficients f or g, hence f primitive or g is primitive. □

Lemma 1.8.111. *Let $f, g \in R[x]$. If f is primitive and $f \mid g$ in $Q(R)[x]$, then $f \mid g$ in $R[x]$.*

Proof. By assumption we can find an $h \in Q(R)[x]$ such that $g = hf$. If $h = 0$, then $g = 0$ and we are done. Suppose $h \neq 0$. Then for some $c \in R \setminus 0$, $ch \in R[x]$. For some $d \in R \setminus 0$ and primitive $h' \in R[x]$, $ch = dh'$, implying $cg = chf = dh'f$. Note $h'f$ is primitive by the prior lemma, hence d is the greatest common divisor for the coefficients of $dh'f$. This implies d is the greatest common divisor of the coefficients of cg. Since $c \mid cg$, it follows that $c \mid d$, hence $\frac{d}{c} \in R$. This implies that

$$g = \frac{d}{c} h'f \in R[x],$$

which means $f \mid g$ in $R[x]$. □

In the following lemma we classify all the irreducible polynomials in $R[x]$.

Lemma 1.8.112. *1. Let $f \in R[x]$ be primitive. If f is irreducible in $Q(R)[x]$, then f is prime in $R[x]$.*

2. Any non-zero, non-unit element in $R[x]$ can be written as a product of irreducible elements.

3. The irreducible elements in $R[x]$ are the primes in R and the polynomials described in 1.

Proof. 1. Let $a, b \in R[x]$ such that $f \mid ab$ in $R[x]$. Using Corollary 1.8.82 we get that f is prime by Proposition 1.6.73 in $K[x]$, hence $f \mid a$ or $f \mid b$ in $K[x]$, hence by Lemma 1.8.111 $f \mid a$ or $f \mid b$ in $R[x]$.
2. Let $f \in R[x]$ be a non-zero, non-unit element. If deg $f = 0$ it is an element in R, which is a UFD, hence f has a factorization into irreducibles. If deg $f > 0$, then there primes/irreducibles $g_1, \ldots, g_m \in Q(R)[x]$ such that $f = \prod_1^m g_i$. For suitable $a_1, \ldots, a_m \in K$ and primitive $f_1, \ldots, f_m \in R[x]$ such that $g_i = a_i f_i$ for each $i \in \{1, \ldots, m\}$. Since g_i is irreducible in $Q(R)[x]$ for each i, so is f_i in $Q(R)[x]$. Set $a := \prod_1^m a_i$. Then $f = a \prod_1^m f_i$, hence $\prod_1^m f_i \mid f$ in $Q(R)[x]$, and hence also in $R[x]$ by Lemma 1.8.111. This means $a \in R$. If a is a unit, set $f_1' = af_1$. Then we get a factorization into irreducibles,

$$f = f_1' \prod_2^m f_i.$$

If a is not a unit, we write $a = \prod_1^l p_i$ for primes/irreducibles in R, getting a factorization intio irreducibles

$$f = \left(\prod_1^l p_i \right) \left(\prod_1^m f_i \right).$$

3. We have already established that the two types of elements in question are irreducible in $R[x]$ (cf. 1 and Lemma 1.8.107). Let f be irreducible in $R[x]$. If deg $f = 0$, then $f \in R$, hence f is prime as R is a UFD. If deg $f > 0$, then we saw in 2. that f can be written as a product of primes in R and primitive polynomials in $R[x]$ irreducible in $Q(R)[x]$. Writing $f = af_1 \cdots f_m$ for $f_1, \ldots, f_m$ polynomials being of the second type of irreducibles. We see that $m = 1$ and a is a unit for otherwise this would contradict the irreducibility of f. □

Theorem 1.8.113. *$R[x]$ is a UFD. By induction $R[x_1, \ldots, x_n]$ is a UFD.*

Proof. Every non-zero non-unit element is a product irreducible elements in $R[x]$ by Lemma 1.8.112 2. Let f be an irreducible element in $R[x]$. Then either f is a prime in R

or is primitive in $R[x]$ and irreducible in $Q(R)[x]$ by Lemma 1.8.112 3. In the second case f is also prime by Lemma 1.8.112 1. It follows by Proposition 1.6.73 that $R[x]$ is a UFD. Since $R[x_1, \ldots, x_n] \simeq (R[x_1, \ldots, x_{n-1}])[x_n]$ it follows by induction that $R[x_1, \ldots, x_n]$ is a UFD. □

Proposition 1.8.114. *Let $f \in R[x]$ such that f is monic and* deg $f \in \{2, 3\}$. *Then f is irreducible if and only if f has no roots.*

Proof. " $\implies$ ": Suppose f has a root $\alpha \in R$. Then $f \in \langle x - \alpha \rangle$, hence $f = (x - \alpha)g$ for some $g \in R[x]$. In either case of deg $f = 2$ or deg $f = 3$, we have that deg $g \geq 1$, implying g is not a unit. This means f is reducible.
" $\impliedby$ ": Suppose f is reducible. Since the irreducible non-constant polynomials in $R[x]$ are monic, there is, In any case of f having degree 2 or 3, a polynomial $g = (x - \alpha) \in R[x]$, such that

$$f = gh,$$

for some $h \in R[x]$. It hence follows that f has a root α. □

Corollary 1.8.115. *The polynomial $x^2 - a \in R[x]$ is irreducible if and only if a is not a square.*

1.8.14 Eisenstein's Criterion

1.8.15 Dehomogenization & Homogenization of Polynomials

Definition 1.8.116. Let $f \in R[x_1, \ldots, x_n]$. We define *the dehomogenization of f at x_i* to be the polynomial

$$f_{*,i} := f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n).$$

Remark 1.8.117. We write f_* when it is implicitly understood with respect to what variable dehomogenization is applied.

Definition 1.8.118. Let $f \in R[x_1, \ldots, x_n]$, write $f = \sum_0^d f_i$ (cf. the above lemma). Then *the homogenization of f* is the polynomial

$$f^* := \sum_0^d x_{n+1}^{d-i} f_i$$

Remark 1.8.119. An alternative definition is that $f^* = x_{n+1}^d f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$ (in e.g. $Q(R[x_{n+1}])[x_1, \ldots, x_{n+1}]$). Indeed,

$$\begin{aligned} x_{n+1}^d f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) &= x_{n+1}^d \sum_{v \in \mathbb{N}^n} a_v \frac{x_1^{v_1}}{x_{n+1}^{v_1}} \cdots \frac{x_n^{v_n}}{x_{n+1}^{v_n}} \\ &= x_{n+1}^d \sum_0^d \sum_{v \in \mathbb{N}^n : |v| = i} a_v \frac{x_1^{v_1} \cdots x_n^{v_n}}{x_{n+1}^{v_1 + \cdots + v_n}} \\ &= \sum_0^d \sum_{v \in \mathbb{N}^n : |v| = i} x_{n+1}^{d-i} a_v \mathbf{x}^v = \sum_0^d x_{n+1}^{d-i} f_i. \end{aligned}$$

Note also $x_{n+1}^{d-i} f_i$ is homogeneous of degree $d-i+i = d$, hence $f^* = x_{n+1}^d f_0 + x_{n+1}^{d-1} f_1 + \cdots + f_d$ is homogeneous of degree d. Note that $f^* = 0$ if and only if $f = 0$. Indeed if $f \neq 0$, Then $x_{n+1}^{d-i} f_i \neq 0$, furthermore any monomial in $x_{n+1}^{d-i} f_i$ is different from any monomial in $x_{n+1}^{d-j} f_j$ since their x_{n+1}-degree is $d - i$ resp. $d - j$.

We observe the following facts about homogenization and de-homogenization.

Proposition 1.8.120. *Let $f, g \in R[x_1, \ldots, x_n]$ and $F \in R[x_1, \ldots, x_{n+1}]$ be homogeneous.*

1. $x_{n+1}^{\deg f + \deg g - \deg(f+g)}(f+g)^* = x_{n+1}^{\deg g} f^* + x_{n+1}^{\deg g} g^*$. *Suppose additionally that R is an integral domain. Then $(fg)^* = f^* g^*$.*
2. $(fg)_{*,i} = f_{*,i} g_{*,i}$ & $(f+g)_{*,i} = F_{*,i} + G_{*,i}$.
3. $(f^*)_* = f$.
4. *Let* $r := \max\left(\left\{j \geq 0 : x_{n+1}^j \mid F\right\}\right)$. *Then* $x_{n+1}^r (F_*)^* = F$

Proof. 1. For the first identity one finds that

$$\begin{aligned} x_{n+1}^{\deg f + \deg g - \deg(f+g)}(f+g)^* &= x_{n+1}^{\deg f + \deg g - \deg(f+g) + \deg(f+g)} \left[f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) + g\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \right] \\ &= x_{n+1}^{\deg g} \left[x_{n+1}^{\deg f} f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \right] + x_{n+1}^{\deg f} \left[x_{n+1}^{\deg g} g\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \right] \\ &= x_{n+1}^{\deg g} f^* + x_{n+1}^{\deg f} g^*. \end{aligned}$$

For the second statement, see that

$$\begin{aligned} (fg)^* &= x_{n+1}^{\deg f + \deg g} f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) g\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \\ &= x_{n+1}^{\deg f} f\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) x_{n+1}^{\deg g} g\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) = f^* g^*. \end{aligned}$$

2. This follows from evaluation being a ring homomorphism.

3. Indeed

$$(f^*)_* = \left(\sum_0^d x_{n+1}^{d-i} f_i\right)_* = \sum_0^d f_i = f.$$

4. Write $F = \sum_{v\in\mathbb{N}^{n+1}} a_v \mathbf{x}^v$. Note that if $x_{n+1}^j \mid F$, then $F = x_{n+1}^j Q$ for some $Q \in R[x_1, \ldots, x_{n+1}]$. By Lemma 1.6.158 Q is homogeneous of degree $d-j$ where $d := \deg\ F$. Then x_{n+1}^j divides every term F. Set

$$d' = \deg\ F(x_1, \ldots, x_n, 1) = \max_{v\in\mathbb{N}^{n+1}: a_v\neq 0} \sum_1^n v_i = \max_{v\in\mathbb{N}^{n+1}: a_v\neq 0} d - v_{n+1}.$$

For some $w \in \mathbb{N}^{n+1}$ with $a_w \neq 0$, $w_{n+1} = r$. Then $d' = d - w_{n+1} = d - r$, since if there were a $u \in \mathbb{N}^{n+1}$ with $a_u \neq 0$ and $u_{n+1} < r$ then $x_{n+1}^r \nmid a_u x_{n+1}^{u_{n+1}}$. As F is homogeneous $d = \sum_1^{n+1} v_i$ for every $v \in \mathbb{N}^n$ with $a_v \neq 0$, hence $d' = -r + \sum_1^{n+1} v_i$. We get

$$\begin{aligned} x_{n+1}^r (F_*)^* &= \sum_{v\in\mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n} x_{n+1}^{d'+r-\sum_0^n v_i} \\ &= \sum_{v\in\mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n} x_{n+1}^{\sum_1^{n+1} v_i - r + r - \sum_0^n v_i} \\ &= \sum_{v\in\mathbb{N}^n} a_v x_1^{v_1} \cdots x_n^{v_n} x_{n+1}^{v_{n+1}} = F. \end{aligned}$$

□

Corollary 1.8.121. *Suppose R is an integral domain and consider $F \in R[x_1, \ldots, x_{n+1}]$ a homogeneous polynomial. Then a factorization of F determines a factorization of F_* up to a factor x_{n+1}^r. If $R = K$ an algebraically closed field and $F \in K[x, y]$ then F factors into a product of linear factors.*

Proof. Indeed, $F = x_{n+1}^r Q$ for some homogeneous $Q \in R[x_1, \ldots, x_n, x_{n+1}]$. Then suppose $Q = q_1 \cdots q_l$ for some $q_1, \ldots, q_l \in R[x_1, \ldots, x_n, x_{n+1}]$. Then $F_* = q_{1*} \cdots q_{l*}$. Conversely, if $F_* = q_1 \cdots q_l$, then $F = x_{n+1}^* q_1^* \cdots q_l^*$.

For the second statement we can again write $F = y^r Q$ for some homogeneous $Q \in K[x, y]$. Then $Q_* = a \prod_1^l (x - a_i)^{r_i}$ for some $a, a_1, \ldots, a_l \in K$ where $a \neq 0$. Then $F = y^r (F_*)^* = ay^r \prod_1^l (x - a_i y)^{r_i}$. □

Proposition 1.8.122. *Let R be an integral domain. Consider $f, g \in R[x_1, \ldots, x_n]$ homogeneous of degree d respectively degree $d+1$ with $\gcd(f, g) = 1$. Then $f + g$ is irreducible.*

Proof. We proof that if $f + g$ is reducible, then f and g has common factor. Let $a, b \in R[\mathbf{x}]$ with $\deg\ a, \deg\ b > 1$ such that $f + g = ab$. We can write $a = \sum_m^M a_m$ and $b = \sum_l^L b_j$

where $m, l > 1$ and $a_m, \ldots, a_M, b_l, \ldots, b_L \in R[\mathbf{x}]$ are homogeneous with degree being the index such that $a_m, a_M, b_l, b_L \neq 0$. Note that $d = \deg\ a_m b_l = m + l$. Note also that $d + 1 = \deg\ a_M b_L = L + M$, hence (WLOG) $L = l + 1$ and $M = m$. We thus find that $ab = a_m b_l + a_m b_{l+1}$. Then $f = a_m b_l$ and $g = a_m b_{l+1}$, hence f, g has a common factor a_m. □

Definition 1.8.123. Let $I \subset R[x_1, \ldots, x_{n+1}$ and $J \subset R[x_1, \ldots, x_n]$. We define the *dehomogenization of I at $i \in \{1, \ldots, n+1\}$* and the *homogenization of J* to be

$$I_{*,i} = \{f_{*,i} : f \in I\} \text{ resp. } J^* := \langle \{f^* : f \in J\} \rangle \subset R[x_1, \ldots, x_{n+1}].$$

We furthermore define $I_* := I_{*,n+1}$

Remark 1.8.124. Note that an ideal is homogeneous if and only if it is (not necessarily finitely) generated by a set of homogeneous polynomials. We therefor note that the homogenization of an ideal is a homogeneous ideal.

Lemma 1.8.125. *$I_{*,i}$ is an ideal. $I = \langle f_1, \ldots, f_m \rangle$, then $I_{*,i} = \langle (f_1)_{*,i}, \ldots, (f_m)_{*,i} \rangle$. If $J = \langle f \rangle$, then $J^* = \langle f^* \rangle$.*

Proof. The first statement is a trivial consequence of evaluation being a ring homomorphism. The second statement is a matter of checking the definition. □

Example 1.8.126. Consider $I := \langle y - x^2, z - x^3 \rangle R[x, y, z]$, Note that $f := z - xy = z - x^3 - x(y - x^2) \in I$, hence $f^* = zw - xy \in I^*$, however $f^* \notin J := \langle (y - x^2)^*, (z - x^3)^* \rangle = \langle yw - x^2, zw^2 \rangle$, since any term containing z in a polynomial in J has w-degree ≥ 2 or y-degree ≥ 1 or x-degree ≥ 2, therefor no polynomial in J can contain the term zw. We thus see that for a finitely generated ideal $I = \langle f_1, \ldots, f_m \rangle$, while trivially $\langle f_1^*, \ldots, f_m^* \rangle \subset I^*$, it is not necessarily the case that this holds with equality.

Definition 1.8.127. Let K be any field and $f \in K[x_1, \ldots, x_n] \setminus 0$. A point $[v] \in \mathbb{P}^n$ is said to be a *zero of f* if $f(\lambda v) = 0$ for every $\lambda \in K \setminus 0$. We thus write $f([v]) = 0$.

Remark 1.8.128. For a fixed $[v] \in \mathbb{P}^n$ we thus get a well-defined evaluation function on the space of polynomials for which $[v]$ is a zero, mapping to 0. If f is homogeneous of degree d, then if $v \in K \setminus 0$ is a zero of f, $[v]$ is a zero of f. Indeed, for any non-zero $\lambda \in K \setminus 0$ and an $s = (s_1, \ldots, s_{n+1}) \in S^{n+1}$ where $S \supset K$ is a K-algebra. Then

$$f(\lambda s) = \sum_{v \in \mathbb{N}^n} a_v \lambda^{v_1} s_1^{v_1} \cdots \lambda^{v_{n+1}} s_{n+1}^{v_{n+1}} = \lambda^{\sum_1^{n+1} v_i} \sum_{v \in \mathbb{N}^n} a_v s_1^{v_1} \cdots s_{n+1}^{v_{n+1}} = \lambda^d f(s).$$

In particular, if v is a zero of f, then

$$f(\lambda v) = \lambda^d f(v) = 0 \implies f([v]) = 0.$$

Note that if $[v] \in \mathbb{P}^n$ is a zero of f, g then $(f+g)([v]) = f([v]) + g([v]) = 0$ and $(fg)([v]) = f([v])g([v]) = 0$, hence $[v]$ is a zero of $f + g$ and fg.

Lemma 1.8.129. *Let K be an infinite field. Consider $f = \sum_0^d f_i \in K[x_1, \ldots, x_{n+1}]$ where f_i is homogeneous of degree i. Let $[v] \in \mathbb{P}^n$ be a zero of f. Then $[v]$ is a zero of f_i for each i.*

Proof. Fix $v \in [v]$ and consider

$$g := f(tv_1, \ldots, tv_{n+1}) = \sum_0^f t^i f_i(v) \in K[t].$$

Then $g(\lambda) = 0$ for every $\lambda \in K \setminus 0$, hence $g = 0$. This implies that $f_i(tv_1, \ldots, tv_{n+1}) = t^i f_i(v) = 0$ for each i, meaning $f_i(\lambda v) = 0$ for each $\lambda \in K \setminus 0$. We thus conclude that $f_i([v]) = 0$. □

Lemma 1.8.130. *If $I \subset R[x_1, \ldots, x_n]$ is prime, then $I^* \subset R[x_1, \ldots, x_{n+1}]$ is prime*

Proof. Let $a, b \in R[x_1, \ldots, x_{n+1}]$ such that $ab \in I^*$. Then $a_* b_* = (ab)_* \in I$, hence $a_* \in I$ or $b_* \in I$. Suppose $a_* \in I$. Then $(a_*)^* \in I^*$, meaning for a suitable $r \geq 0$, $a = x_{n+1}^r (a_*)^* \in I^*$. The other case follows by symmetry. □

Remark 1.8.131. Consider $V_R(d, n, I) = \{\alpha \in R[\mathbf{x}]/I : \alpha \text{ homogeneous of degree } d\}$ is an R-submodule of $R[\mathbf{x}]/I$ finitely generated by $\{\mathbf{x}^v + I : |v| = d\}$. In particular $V_K(d, n, I)$ is a finite dimensional vector space for fields K. In general, it takes some work to say anything about the dimension of this vector space, below we give an example in the case $n = 3$ and $d > n$ This is exercise 4.10 in Fulton. Do at some point.

Example 1.8.132.

Lemma 1.8.133. *Let $f \in K[x_1, \ldots, x_n]$ be a non-zero form of degree d for a field K. Then f_* is non-zero.*

Proof. $0 \neq f = x_{n+1}^r (f_*)^*$ for some $r \geq 0$, hence $(f_*)^* \neq 0$, so by Remark 1.8.119, $f_* \neq 0$. □

1.8.16 Multi- and Bihomogeneous Polynomials

Definition 1.8.134. Let $\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n_i\}$ be algebraically independent variables over a ring R. A polynomial in $R[\mathbf{x}]$ is called an *m-homogeneous polynomial* or an *m-form* of *m-degree* $(d_1, \ldots, d_m) \in \mathbb{N}^m$, if it is a form of degree d_i when seen as an element in $R[x_{kj} : k \neq i][x_{i1}, \ldots, x_{in_i}]$ for each $i \in \{1, \ldots, m\}$. When $m = 2$, a 2-form is called a *bihomogeneous polynomial* or a *biform* of *bidegree* (d_1, d_2).

Lemma 1.8.135. *With the same notation as above a polynomial $f \in K[\mathbf{x}] \setminus 0$ of degree d has a unique decomposition*

$$f = \sum_{(i_1,\ldots,i_m):\sum_1^m i_j \leq d} f_{i_1,\ldots,i_m},$$

where $f_{i_1,\ldots,i_m}$ is an m-form of m-degree $(i_1, \ldots, i_m)$ such that for some $(i_1, \ldots, i_m)$ with $\sum_1^m i_j = d$,

$$f_{i_1,\ldots,i_m} \neq 0.$$

Proof. Write

$$f = \sum_{(i_1,\ldots,i_m):\sum_1^m i_j \leq d} \underbrace{\sum_{\substack{\mathbf{v} \in \prod_{k=1}^m \mathbb{N}^{n_k}: \\ \forall k, |v_k| = i_k}} a_{\mathbf{v}} {\mathbf{x_1}}^{v_1} \cdots {\mathbf{x_m}}^{v_m}}_{f_{i_1,\ldots,i_m}},$$

each monomial in $f_{i_1,\ldots,i_m}$ is a homogeneous of degree i_k in $R[x_{kj} : k \neq i][x_{i1}, \ldots, x_{in_i}]$ for each k, hence $f_{i_1,\ldots,i_m}$ is m-homogeneous of m-degree $(i_1, \ldots, i_m)$. Uniqueness follows from $\{\mathbf{x}^{\mathbf{v}}\}$ being linearly independent over R. □

Definition 1.8.136. For an $f \in K[\mathbf{x_1}, \ldots, \mathbf{x_m}]$, where K is a field, we say that $([v_1], \ldots, [v_m]) \in \mathbb{P}^{n_1} \times \mathbb{P}^{n_m}$ is *a zero of f* if for every $(\lambda_1, \ldots, \lambda_m) \in K^m$ with $\lambda_i \neq 0$,

$$f(\lambda_1 v_1, \ldots, \lambda_m v_m) = 0.$$

When $([v_1], \ldots, [v_m])$ is a zero of f. We write

$$f([v_1], \ldots, [v_m]) = 0,$$

otherwise we write

$$f([v_1], \ldots, [v_m]) \neq 0$$

Remark 1.8.137. If f were an m-form of m-degree $(d_1, \ldots, d_m)$ note that $f(\lambda_1 v_1, \ldots, \lambda_m v_m) = \left(\prod_1^m \lambda_i^{d_i}\right) f(v_1, \ldots, v_m)$. Hence if $(v_1, \ldots, v_m)$ is a zero of f, then so is $([v_1], \ldots, [v_m]) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.

Definition 1.8.138. An ideal $I \subset R[\mathbf{x_1}, \ldots, \mathbf{x_m}]$ is called m-homogeneous if for each

$$f = \sum_{i_1,\ldots,i_m} f_{i_1,\ldots,i_m} \in I,$$

$f_{i_1,\ldots,i_m} \in I$

Remark 1.8.139. The above is equivalent to I being a homogeneous ideal in $R[\mathbf{x_1}, \ldots, \widehat{\mathbf{x_k}}, \ldots, \mathbf{x_m}]$ for each k. Here $\widehat{\ }$ denotes removale of these variables. Any result proven about homogeneous ideals therefor naturally generalizes to m-homogeneous ideals.

1.8.17 Differentiation of Polynomials

Definition 1.8.140. We define *differentiation (with respect to x)* in polynomial ring $R[x]$ as the R-module map

$$D_x : R[x] \to R[x]$$

mapping 1 to 0 and x^n to x^{n-1} for $n \geq 1$.For a polynomial $f \in R[x]$, we call the polynomial $D_x f$ the *derivative of f (with respect to x)*, which we may denote by f'.

Remark 1.8.141. One easily checks that $D_x fg = (D_x f)g + f(D_x g)$, i.e. that the Leibniz rule is satisfied. It also satisfies the chain rule, i.e.

$$D_x(f(g)) = (D_x g) \cdot (D_x f)(g).$$

By definition over an integral domain of characteristic 0, $\deg f' = \deg f - 1$ when $\deg f \geq 1$. In positive characteristic this may not be the case. For instance, when $\operatorname{char} R = p > 0$, we get that $D_x x^p = px^{p-1} = 0$. We can also come up with pathological examples over commutative rings of characteristic 0. Indeed take $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ with the usual structure of product ring. Then $D_x(1,0)x^2 = (2,2)(1,0)x = (0,2)(1,0)x = 0$.

Definition 1.8.142. In a polynomial ring $R[x_1, \ldots, x_n]$, *the partial derivative of f with respect to x_i* is the polynomial $D_{x_i} f$, where

$$D_{x_i} : R[x_1, \ldots, x_{i-1}, \widehat{x_i}, x_{i+1}, x_n][x_i] \to R[x_1, \ldots, x_{i-1}, \widehat{x_i}, x_{i+1}, x_n][x_i],$$

is differentiation with respect to x_i. We sometimes denote it by $\frac{\partial f}{\partial x_i} := \frac{\partial}{\partial x_i} f := f_{x_i}$.

Lemma 1.8.143. *Let T be translation of one variable, x_1 say, by some element $a \in K$. Then $D_{x_i} T = T D_{x_i}$. Hence in general translation commutes with partial derivatives.*

Proof. Indeed,

$$(D_{x_1}T)f = D_{x_1}f(x_1{+}a, x_2, \ldots, x_n) = (D_{x_1}(x_1{+}a))\cdot(D_{x_1}f)(x_1{+}a, x_2, \ldots, x_n) = (TD_{x_1})f.$$

(Here we implicitly use that differentiation commutes with permutation of variables). □

Lemma 1.8.144. *(Euler's theorem) Let R be an integral domain, $f \in R[x_1, \ldots, x_n] \setminus 0$ be homogeneous of degree $d > 0$. Then*

$$\sum_1^n x_i \frac{\partial f}{\partial x_i} = df.$$

Proof. Let $a_v\mathbf{x}^v$ be a term of f. Then

$$\frac{\partial}{\partial x_i} a_v\mathbf{x}^v = \begin{cases} v_i a_v \mathbf{x}^{v-e_i} & \text{if } v_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_1^n x_i \frac{\partial}{\partial x_i} a_v\mathbf{x}^v = \sum_{i:v_i>0} v_i a_v x_i \mathbf{x}^{v-e_i} = \left(\sum_{i:v_i>0} v_i\right) a_v x^v = d a_v \mathbf{x}^v.$$

It thus follows that

$$\sum_1^n x_i \frac{\partial f}{\partial x_i} = \sum_1^n x_i \sum_{v\in\mathbb{N}^n} \frac{\partial}{\partial x_i} a_v\mathbf{x}^v = \sum_{v\in\mathbb{N}^n}\sum_1^n x_i \frac{\partial}{\partial x_i} a_v\mathbf{x}^v = d\sum_{v\in\mathbb{N}^n} a_v\mathbf{x}^v = df.$$

□

We provide a lemma which alleviate some ugly cases in characteristic $p > 0$ cases

Lemma 1.8.145. *Let R be an integral domain of characteristic $p > 0$. Let $f \in R[x_1, \ldots, x_n]$ be a non-constant polynomial such that $f_{x_i} = 0$ for each i. Then $f = g(x_1^p, \ldots, x_n^p)$ for some g.*

Proof. For any $k \in \mathbb{Z}$, let k_R denote the image of k in R. Consider the case where $n = 1$. Write $f = \sum_0^d a_i x^i$. Then $0 = f_x = \sum_1^d i_R a_i x^{i-1}$. For each i, we then get that $i_R a_i = 0$, hence $a_i = 0$ or $i_R = 0$. If $i < p$, then $i_R \neq 0$, hence $a_i = 0$. We furthermore have that if $a_i \neq 0$ then $p \mid i$ It thus follows that $f = \sum_1^k a_{jp} x^{jp}$, where k is chosen such that $d = kp$. Hence picking $g = \sum_1^k a_{jp} x^j$ we are done. We prove the general case by induction using the fact that char $R[x_1, \ldots, x_n] = p$, $R[x_1, \ldots, x_{n+1}] \simeq R[x_1, \ldots, x_n][x_{n+1}]$ in conjunction with the validity of the 1-variable case. □

1.9 Ring Extensions and Algebras over Rings

We proceed with considerations of the theory of algebras over rings in conjunction with some of the theory modules already developed. We could have noted this earlier, but it is more relevant to note it now.

1.9.1 Finitely Generated Ring Extensions

Definition 1.9.1. A ring extension $S \supset R$ is said to be *module-finite (over R)* if S is finitely generated as an R-module.

Definition 1.9.2. A ring extension $S \supset R$ is a said to be *finitely generated R-algebra* or *ring-finite* if $S = R[s_1, \ldots, s_n]$ for suitable $s_1, \ldots, s_n \in S$.

Proposition 1.9.3. *Let $S \supset R$ be a ring-finite ring extension. Then*

$$S \simeq R[x_1, \ldots, x_n]/I,$$

for some $n \geq 1$ and some ideal $I \subset R[x_1, \ldots, x_n]$.

Proof. For suitable $s_1, \ldots, s_n \in S$, $S = R[s_1, \ldots, s_n]$, hence

$$\text{ev}_{s_1,\ldots,s_n} : R[x_1, \ldots, x_n] \to S$$

is a surjective R-algebra homomorphism. Then by the first isomorphism theorem

$$S \simeq K[\mathbf{x}]/\ker\ \text{ev}_{s_1,\ldots,s_n}.$$

□

Proposition 1.9.4. *Let $S \supset R$ is a ring extension that is also a finitely generated R-module. Then S is a finitely generated R-algebra.*

Proof. For suitable $s_1, \ldots, s_n \in S$, $S = \sum_1^n Rs_i$. We prove that $S = R[s_1, \ldots, s_n]$. We already know that $S \supset R[s_1, \ldots, s_n]$. Let $s \in S$. Then $s = \sum_1^n r_i s_i$ for suitable $r_1, \ldots, r_n \in R$, hence $s \in R[s_1, \ldots, s_n]$. □

Example 1.9.5. The converse implication of the above proposition is clearly not true. Consider for instance $R[x_1, \ldots, x_n] \supset R$. given $f_1, \ldots, f_m \in R[\mathbf{x}]$. If these are all 0, clearly $R[\mathbf{x}] \supsetneq \sum_1^m Rf_i$. Otherwise putting $D = \max_{i\in\{1,\ldots,m\}} \{\deg\ f_i\}$, we see that for $r_1, \ldots, r_m \in R$,

$$\deg \sum_1^m r_i f_i \leq D < D + 1 = \deg\ x_1^{D+1},$$

hence $x_1^{D+1} \notin \sum_1^m Rf_i$, hence $R[\mathbf{x}] \supsetneq \sum_1^m Rf_i$.

Definition 1.9.6. A ring extension $L \supset K$ is called a *field extension (over K)* if both L and K are fields.

Let $S \supset R$ be a ring extension where S is an integral domain. For $s_1, \ldots, s_n \in S$, $R[s_1, \ldots, s_n]$ is also an integral domain. We denote the fraction field of $R[s_1, \ldots, s_n]$ by $R(s_1, \ldots, s_n)$

Definition 1.9.7. A field extension $L \supset K$ is said to be *finite*, if there exist $a_1, \ldots, a_n$ such that $L = K(a_1, \ldots, a_n)$.

Lemma 1.9.8. *Let K be a field. Consider $K[x_1, \ldots, x_n]$ as a subring of $K[\mathbf{x}, y_1, \ldots, y_m]$ Then $K(x_1, \ldots, x_n)(y_1, \ldots, y_m) = K(x_1, \ldots, x_n, y_1, \ldots, y_m)$.*

Proof. Clearly $K(\mathbf{x}) \subset K(\mathbf{x}, \mathbf{y})$. To be very precise, this means $K(\mathbf{x}, \mathbf{y}) \supset K(\mathbf{x})$ is a ring extension. Hence $K(\mathbf{x}, \mathbf{y}) \supset K(\mathbf{x})[\mathbf{y}]$. This means $K(\mathbf{x}, \mathbf{y}) = Q(K(\mathbf{x}, \mathbf{y})) \supset Q(K(\mathbf{x})[\mathbf{y}]) = K(\mathbf{x})(\mathbf{y})$. □

Lemma 1.9.9. *Consider $L := K(x_1, \ldots, x_n)$ and $R := K\left[\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\right]$ for $\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m} \in K(\mathbf{x})$. Then there is a $b \in K[\mathbf{x}]$ such that $b^d z \in K[\mathbf{x}]$ for every $z \in R$ for some $d \geq 0$.*

Proof. If $\operatorname{lcm}(b_1, \ldots, b_m) = 1$ the statement is trivial. Set $b := \operatorname{lcm}(b_1, \ldots, b_m)$ and assume $\deg b > 0$. Let $z \in R$. If $z = 0$, then $b^0 z \in K[\mathbf{x}]$. Suppose $z \neq 0$. For some $f \in K[y_1, \ldots, y_m] \setminus 0$, $z = f\left(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\right)$. Set $d := \deg f$. Let $v \in \mathbb{N}^m$ with $|v| \leq d$. Then

$$\prod_1^m b_i^{v_i} \mid \prod_1^m b^{v_i} = b^{|v|} \text{ and } b^{|v|} \mid b^d \implies \prod_1^m b_i^{v_i} \mid b^d \implies b^d \prod_1^m \left(\frac{a_i}{b_i}\right)^{v_i} \in K[\mathbf{x}],$$

which implies $b^d z \in K[\mathbf{x}]$. □

Proposition 1.9.10. *Consider $L := K(x_1, \ldots, x_n) \supset K$. Then L is a finite field extension over K, but not a finitely generated K-algebra.*

Proof. The first statement is obvious as $K(x_1, \ldots, x_n)$ is finitely generated as field extension over K by $x_1, \ldots, x_n \in K(x_1, \ldots, x_n)$. To prove the second statement, let $\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m} \in K(\mathbf{x})$. Set $b := \operatorname{lcm}(b_1, \ldots, b_m)$. Then there is a $c \in K[\mathbf{x}]$ such that $c \nmid b^d$ for any $d \geq 0$, since there are infinitely many irreducible polynomials over K (cf. Lemma 1.8.57) and $K[\mathbf{x}]$ is a UFD, hence $b^d \frac{1}{c} \notin K[x]$ for any $d \geq 0$. By lemma 1.9.9, it follows that $\frac{1}{c} \notin K\left[\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\right]$, hence $K(\mathbf{x}) \supsetneq K\left[\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\right]$. This means $K(\mathbf{x})$ is not a finitely generated K-algebra. □

Lemma 1.9.11. *These finiteness conditions are transitive, i.e. the following three statements are true:*

1. *Let $T \supset S$,$S \supset R$ be module-finite. Then $T \supset R$ is module-finite.*

2. *Let $T \supset S$, $S \supset R$ be ring-finite. Then $T \supset R$ is ring-finite.*

3. *Let $M \supset L$, $L \supset K$ be finite field extensions. Then $M \supset K$ is a finite field extension.*

Proof. 1. We can find $s_1, \ldots, s_n \in S$ such that $S = \sum_1^n Rs_i$ and $t_1, \ldots, t_m \in T$ such that $T = \sum_1^m St_i$. Let $t \in T$. Then there are $a_1, \ldots, a_m \in S$ such that $t = \sum_1^m a_i t_i$. For each i, $a_i = \sum_{j=1}^n b_{ij} s_j$ for suitable $b_{ij} \in R$, hence

$$t = \sum_1^m \sum_1^n b_{ij} t_i s_j \in \sum_1^n \sum_1^m R t_i s_j.$$

Hence T is finitely generated as an R-module by the elements of $\{t_i s_j : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$.
2. We can find $s_1, \ldots, s_n \in S$ such that $S = R[s_1, \ldots, s_n]$ and $t_1, \ldots, t_m \in T$ such that $T = S[t_1, \ldots, t_m]$. Let $t \in T$. Then there are $a_v \in S$ such that

$$t = \sum_{v \in \mathbb{N}^m} a_v t_1^{v_1} \cdots t_m^{v_m}.$$

For each $v \in \mathbb{N}^m$,

$$a_v = \sum_{w \in \mathbb{N}^n} b_{vw} s_1^{w_1} \cdots s_n^{w_n} a_v$$

for suitable $b_{vw} \in R$, hence

$$t = \sum_{v \in \mathbb{N}^m} \sum_{w \in \mathbb{N}^n} b_{vw} t_1^{v_1} \cdots t_m^{v_m} s_1^{w_1} \cdots s_n^{w_n} \in R[s_1, \ldots, s_n, t_1, \ldots, t_m].$$

Hence T is finitely generated as an R-algebra by the elements of $s_1, \ldots, s_n, t_1, \ldots, t_m \in T$.
3. There are $\alpha_1, \ldots, \alpha_m \in M$ such that $M = L(\alpha_1, \ldots, \alpha_m)$ and $\beta_1, \ldots, \beta_n \in L$ such that $L = K(\beta_1, \ldots, \beta_n)$. Then

$$M = L(\alpha_1, \ldots, \alpha_m) = K(\beta_1, \ldots, \beta_n)(\alpha_1, \ldots, \alpha_m) = K(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n).$$

□

1.9.2 Integral- & Algebraic Extensions

Definition 1.9.12. Let $S \supset R$ be a ring extension. An element of $s \in S$ is said to be *integral (over R)* if it is algebraically dependent over R, i.e. there is a monic $f \in R[x] \setminus 0$ such that $f(s) = 0$.
Let $L \supset K$ be a field extension. An element of L is *algebraic (over K)* if it is integral over K. An element that is not algebraic over K is transcendental over K.

Remark 1.9.13. In the case R is a field, consider $d = \min(\{k > 0 : \exists f \in R[x] \setminus 0, f(s) = 0\})$. Let f_s be a polynomial of degree d vanishing on s. Consider another polynomial $g \in R[x] \setminus 0$ vanishing on s. We can write $g = qf_s + r$, where $r = 0$ or deg $r < d$. Then

$$r(s) = g(s) - q(s)f_s(s) = 0.$$

By minimality $r = 0$, hence $f_s \mid g$. There f_s is the unique non-zero polynomial vanishing on s of minimal degree. We call this polynomial the *defining polynomial of s over K*. Note that $\ker \text{ ev}_s = \langle f_s \rangle$, hence $R[s] \simeq R[x]/\langle f_s \rangle$. We refer to deg f_s as *the degree of s over R*. We can extend this result to the case where R is a UFD. By the same argument as before $f_s \mid g$ in $Q(R)[x]$, hence since f is primitive, $f_s \mid g$ in $R[x]$ by Lemma 1.8.111

Remark 1.9.14. Note that if $a_1, \ldots, a_n \in L \supset K$ are algebraic over K ($L \supset K$ is a field extension), then

$$K[a_1, \ldots, a_n] = K(a_1, \ldots, a_n).$$

Indeed, in the case $n = 1$, $K[a] \simeq K[x]/\langle f_a \rangle$. Then $K[a]$ is a subfield of $K(a)$, and since $K(a)$ is the smallest subfield containing $K[a]$, $K[a] = K(a)$. By induction $K[a_1, \ldots, a_n] = K(a_1, \ldots, a_n)$, so it is sufficient to prove that $K(a_1, \ldots, a_n)[a_{n+1}] = K(a_1, \ldots, a_n)(a_{n+1})$. Since a_{n+1} is algebraic over K it is algebraic over $K(a_1, \ldots, a_n)$, hence by the base case, $K(a_1, \ldots, a_n)[a_{n+1}] = K(a_1, \ldots, a_n)(a_{n+1})$.

Example 1.9.15. Let R be an integral domain. Then $Q(R) \supset R$ is integral. Indeed consider $\alpha = \frac{a}{b} \in Q(R)$. Then α vanishes on $bx - a \in R[x] \setminus 0$

Lemma 1.9.16. *Let $S \supset R$ be ring extension where S is an integral domain. Furthermore, let $s \in S$. The following are equivalent*

1. *s is integral over R.*
2. *$R[s] \supset R$ is module-finite.*
3. *There is a subring of S containing $R[s]$, R' say, which is finitely generated as an R-module.*

Proof. "1. $\implies$ 2.": We may find $a_0, \ldots, a_{n-1} \in R$ such that

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0.$$

It follows that $s^n \in \sum_1^{n-1} Rs^i$. By a simple induction argument it follows that $s^{n+j} \in \sum_1^{n-1} Rs^i$ for every $j \geq 0$. Let $\sum_1^m b_j s^j \in R[s]$. Then by the considerations prior to this, $\sum_1^m b_j s^j \in \sum_1^{n-1} Rs^i$, hence $R[s] = \sum_1^{n-1} Rs^i$.

"2. $\implies$ 3.": Putting $R' = R[s]$ we have such a subring of S.

"3. $\implies$ 1.": We can write $R' = \sum_1^n a_i t_i$ for suitable $t_1, \ldots, t_n \in R[s] \setminus 0$. Then

$$st_i = \sum_1^n a_{ij} t_i$$

for suitable $a_{ij} \in R$. Note then that

$$s \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = (a_{ij}) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}.$$

This implies that s is a root of the characteristic polynomial

$$\det(x\mathbb{1} - (a_{ij})) \in R[x] \setminus 0$$

which is monic. It thus follows that s is integral over R. □

Lemma 1.9.17. *Consider a tower of ring extensions $T \supset S \supset R$ where T is a domain. Suppose T is integral over S and S is integral over R, then T is integral over R.*

Proof. Let $t \in T$, then there is a monic $f = x^n + \sum_0^{n-1} a_i x^i \in S[x] \setminus 0$ such that $f(t) = 0$. By the above lemma $R' := R[a_1, \ldots, a_n, t] = R[a_1, \ldots, a_n][t] \supset R[a_1, \ldots, a_n]$ is module-finite. By the above lemma we also find that $R[a_1] \supset R$ is module-finite. Recursive usage of the above and the transitivity of module-finiteness thus implies that $R[a_1, \ldots, a_{n-1}] \supset R$ is module-finite and hence that $R' = R[a_1, \ldots, a_{n-1}, t] \supset R$ is module-finite. We have thus found a subring of T containing $R[t]$, which is module-finite over R. Hence using the above lemma yet another time it follows that t is integral over R, hence T is integral over R. □

Proposition 1.9.18. *Let $S \supset R$ be an integral domain and a ring extension. Then*

$$\{s \in S : s \text{ is integral over } R\}$$

is a subring S.

Proof. Let $a, b \in S$ be integral over R. We repeatably use Lemma 1.9.16. Note that b is integral over $R[a]$ hence $R[a, b] \subset R$ is module-finite. Since $a + b, ab \in R[a, b]$, it follows that $R[a, b]$ is a ring contained in S, containing $R[a + b]$ and $R[ab]$ that is module-finite over R, meaning $a + b$ and ab are integral over R. □

Lemma 1.9.19. *If $S \supset R$ is module-finite then $S \supset R$ is integral.*

Proof. Let $s \in S$. The ring S is a subring of S containing $R[s]$ which is finitely generated as an R-module. It thus follows by Lemma 1.9.16 that s is integral over R. Hence $S \supset R$ is integral. □

Lemma 1.9.20. *Let $S \supset R$ be a ring extension with S an integral domain. Then $S \supset R$ is module-finite if and only if $S = R[s_1, \ldots, s_n]$ where $s_i \in S$ is integral over R.*

Proof. " $\Longrightarrow$ ": This follows from Lemma 1.9.19 " $\Longleftarrow$ ": By assumption there are $s_1, \ldots, s_n \in S$ such that $S = R[s_1, \ldots, s_n]$. Since $s_i \in S$ is integral over R, it follows by Lemma 1.9.16 that $R[s_1] \supset R$ is module-finite and by induction $S = R[s_1, \ldots, s_n] \supset R$ is module-finite. □

Lemma 1.9.21. *Let $L \supset K$ be a field extension with K algebraically closed.*

1. *Every $f \in K[x] \setminus 0$ with $n := \deg\ f > 0$ has exactly n roots all in K.*
2. *If $a \in L$ is algebraic over K, then $a \in K$.*
3. *If $L \supset K$ is module-finite, then $L = K$.*

Proof. 1. We proceed by induction in n. For $n = 1$, f has a root $a \in K$, and since f has exactly one root, this is the only root.
Suppose now f has degree $n + 1$, then f has a root $a \in K$, hence $f = (x - a)g$ for a polynomial $g \in K[x] \setminus 0$ with $\deg\ g = n$. By induction g has exactly n roots in K. Using that K is an integral domain it follows that f has $n + 1$ roots.
2. If $a \in L$ is algebraic over L, then there is a polynomial $f \in K[x] \setminus 0$ such that a is a root f. Since $V(f) \subset K$ by 1. it follows that $a \in K$.
3. If $L \supset K$ is module-finite, then it is algebraic by Lemma 1.9.19. Let $a \in L$. Then a is algebraic over K, hence $a \in K$ by 2. We thus get that $L = K$. □

Lemma 1.9.22. *Let K be a field and set $L := Q(K[x]) = K(x)$. Then*

1. *$a \in L$ is integral then $a \in K[x]$.*
2. *There is no $f \in K[x] \setminus 0$ such that for every $a \in L$, $F^n a$ is integral over $K[x]$ for some $n > 0$.*

Proof. 1. We may write $a = \frac{f}{g}$ for $f, g \in K[x]$ with $g \neq 0$ and $\gcd(f, g) = 1$. We can then find $a_0, \ldots, a_{n-1} \in K[x]$ such that

$$\frac{f^n}{g^n} + \sum_0^{n-1} a_i \frac{f^i}{g^i} = 0 \implies f^n = \sum_0^{n-1} a_i g^{n-i} f^i = g \sum_0^{n-1} a_i g^{n-(i+1)} f^i,$$

hence $g \mid f^n$, meaning $g \mid f$, hence $g \in K \setminus 0$. This implies that $a = \frac{f}{g} = g^{-1}f \in K[x]$.
2. By Proposition 1.9.10 $K(x) \supsetneq R := K[z_1, \ldots, z_m]$ for any $z_1, \ldots, z_m \in K(x)$. Recall that we proved this by showing that for any $f \in K[x] \setminus 0$ there is some $c \in K[x]$ such that $f^d \frac{1}{c} \notin K[x]$ for any $d > 0$. By 1. this implies that $f^d \frac{1}{c}$ is not integral over K for a. □

Proposition 1.9.23. *Let $L \supset K$ be a field extension. The set*

$$\{a \in L : a \text{ is integral}/K\}$$

is a subfield of L.

Proof. We already know that it is a subring by Proposition 1.9.18. Let $a \in L \setminus 0$ be integral over K. Then there are $a_0, \ldots, a_{n-1} \in K$ such that

$$0 = a^n + \sum_0^{n-1} a_i a^i,$$

where choosing n minimal implies, $a_0 \neq 0$, hence $a\left(a^{n-1}(-a_0)^{-1}\sum_1^{n-1} a_i a^i\right) = 1$, implying a is a unit. □

Proposition 1.9.24. *Let $L \supset K$ be a module-finite field extension. Consider a subring R of L containing K as a subring. Then R is a field.*

Proof. By Lemma 1.9.19, $L \supset K$ is algebraic. Let $r \in R \setminus 0$. Then r has a multiplicative inverse $r^{-1} \in L \setminus 0$. We can thus find $a_0, a_1, \ldots, a_{n-1} \in K$ such that

$$r^{-n} + \sum_0^{n-1} a_i r^{-i} = 0,$$

This implies that

$$r^{-1} = r^{n-1}r^{-n} = -\sum_0^{n-1} a_i r^{n-1} r^{-i} = -\sum_0^{n-1} a_i r^{n-1-i} \in R,$$

hence R is a subfield of L. □

Theorem 1.9.25. *Let $L \supset K$ be a ring-finite field extension generated by $a_1, \ldots, a_n \in L$. Then $L \supset K$ is module finite and hence also algebraic.*

Proof. We use induction in n. For $n = 1$, suppose $L = K[a]$ for some $a \in L$. Consider $\rho = \text{ev}_a : K[x] \to L$. Since $K[x]$ is a PID (cf. Corollary 1.8.81), we find that $\ker \rho = \langle g \rangle$ for some $g \in K[x]$. Then $K[x]/\langle g \rangle \simeq K[a] = L$. We claim that $g \neq 0$
Proof of the claim: Suppose $K[x] \simeq K[a]$, then $K(x) \simeq K(a)$, but then L is not ring-finite over K, by Proposition 1.9.10 leading to a contradiction.

So we may WLOG assume g is monic. Then a is algebraic over K.
Assume the statement is true for some $n \geq 1$. Suppose $L = K[a_1, \ldots, a_n]$. Set $K' = K(a_1)$. Then by induction $L = K'[a_2, \ldots, a_{n+1}]$ is algebraic. Suppose a_1 is algebraic over K. Then $K' \supset K$ is algebraic, hence $L \supset K$ is algebraic. Suppose for a contradiction that a_1 is not algebraic over K. We note that this implies that $K[a_1] \overset{\sigma}{\simeq} K[x]$. We have identities

$$a_i^{n_i} + \sum_{j=0}^{n_i-1} \alpha_{ij} a_i^j = 0,$$

for each $i \geq 2$ for suitable $n_i \geq 1$, $\alpha_{ij} \in K'$. Let $\alpha \in K[a_1]$ be the common denominator of the α_{ij}. Let $M \geq \max_{i \in \{2,\ldots,n+1\}} n_i$. Then

$$(\alpha a_i)^M + \sum_{j=0}^{n_i-1} \alpha^{M-j} \alpha_{ij} (\alpha a_i)^j = 0,$$

hence $\alpha^M a_i$ is integral over $K[a_1]$. Let $z = \sum_{v \in \mathbb{N}^{n+1}} c_v a_1^{v_1} \cdots a_{n+1}^{v_{n+1}} \in L$. Then taking $N \geq 0$ sufficiently large we get that $\alpha^N z$ is integral over $K[a_1]$ by Proposition 1.9.18. However taking $z \in K(a_1)$, this implies that $\sigma(z) \in K(x)$ is a polynomial such that $\sigma(\alpha_1)^N \sigma(z)$ is integral over $K[x]$, leading to a contradiction by Lemma 1.9.22. □

Corollary 1.9.26. *Let $L \supset K$ be a field extension where K is algebraically closed. Suppose also that there is a surjective K-algebra homomorphism from $K[x_1, \ldots, x_n]$ to L for some $n > 0$. Then $K = L$.*

Proof. Let $\sigma : K[x_1, \ldots, x_n] \to L$ be a surjective K-algebra map. By Corollary 1.8.40 there are $a_1, \ldots, a_n \in L$ such that $\sigma = \mathrm{ev}_{a_1,\ldots,a_n}$. It thus follows that $L = \sigma(K[\mathbf{x}]) = \mathrm{ev}_{a_1,\ldots,a_n}(K[\mathbf{x}]) = K[a_1, \ldots, a_n]$, hence by the above theorem L is module-finite over K. It follows from Lemma 1.9.21 that $L = K$. □

Corollary 1.9.27. *Let K be algebraically closed and $I \subset K[x_1, \ldots, x_n]$ be a maximal ideal. Then $K[x_1, \ldots, x_n]/I = K$ (thinking about K as the canonical embedding of K in $K[\mathbf{x}]/I$). This implies $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.*

Proof. The quotient map $\pi : K[\mathbf{x}] \to K[\mathbf{x}]/I, f \mapsto f + I$ is a canonically a surjective K-algebra homomorphism. It follows by the above corollary that $K[\mathbf{x}]/I = K$. Then for each i, $x_i + I = a_i + I$ for some $a_i \in K$, this means $J := \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subset I$. J is maximal by Corollary 1.8.48, hence $J = I$. □

1.9.3 Field Extensions

Lemma 1.9.28. *Let $L \supset K$ be a field extension and $a \in L$. Then a is module finite if and only if* $\dim_K K[a] < \infty$. *If a is algebraic, let d denote* deg f_a. *Then $\{1, a, \ldots, a^{d-1}\}$ is a*

basis for $K[a]$.

Proof. The first statement follows immediately from Proposition 1.9.16. Set $I = \langle f_a \rangle$. By Lemma 1.8.55, $\{1 + I, x + I, \ldots, x^{d-1} + I\}$ is a basis of $K[x]/I \overset{\overline{\text{ev}_a}}{\simeq} K[a]$, since ev_a is a K-algebra homomorphism, it is in particular a K-linear map, hence $\{1, a, \ldots, a^{d-1}\}$ is a basis of $K[a]$ □

Definition 1.9.29. If $L \supset K$ is a module finite field extension we define define $[L : K] := \dim_K L$ to be *the degree of* L *over* K

Lemma 1.9.30. *Let* $L \supset K$ *be a field extension. Then* $L \supset K$ *is module finite if and only if* $L \supset K$ *is a finite field extension generated by some* $a_1, \ldots, a_n \in L$ *that are algebraic over* K.

Proof. " $\implies$ ": By Lemma 1.9.20 $L = K[a_1, \ldots, a_n]$ for suitable $a_1, \ldots, a_n \in L$ that are algebraic over K, hence $L \subset K[a_1, \ldots, a_n] \subset K(a_1, \ldots, a_n) \subset L$, hence $L = K(a_1, \ldots, a_n)$.
" $\impliedby$ ": Suppose $L = K(a_1, \ldots, a_n)$ for some $a_i \in L$. If $n = 1$, then $L = K(a_1) = K[a_1]$ which is module finite over K due to Proposition 1.9.16. Set $L' = K(a_1, \ldots, a_n)$ which by induction is module finite over K. Note that a_{n+1} is algebraic over L', hence applying Proposition 1.9.16, $L \supset L'$ is module finite. By the transitive property of module finite extensions, it follows that $L \supset K$ is module finite. □

Lemma 1.9.31. *Let* $L \supset K$ *be a field extension and* $f \in K[x]$ *irreducible. Suppose there is an* $a \in L$ *such that* $f(a) = 0$. *Then* $L \simeq K[x]/I$ *where* $I := \langle f \rangle$.

Proof. Consider the K-algebra map

$$\sigma := \text{ev}_a : K[x] \to L$$

This induces an isomorphism

$$\overline{\sigma} : K[x]/\ker\ \sigma \simeq \text{im}\ \sigma$$
$$\mu + \ker\ \sigma \mapsto \mu(a)$$

Since $K[x]$ is a PID, $\ker\ \sigma = \langle f' \rangle$ for some f' and since $f \in \ker\ \sigma$ it follows that $f' \mid f$, hence $\langle f \rangle = \langle f' \rangle$ by the irreducibility of f. Note that $K[x]/I$ is a field (cf. Lemma 1.8.62). Let $z = \frac{g(a)}{h(a)} \in K(a)$. Then since $h(a) \neq 0$, $f \nmid h$, hence $h + I \neq 0$. Then

$$z = \frac{g(a)}{h(a)} = \frac{\sigma(g)}{\sigma(h)} = \frac{\overline{\sigma}(g + I)}{\overline{\sigma}(h + I)} = \overline{\sigma}\left((g + I)(h + I)^{-1}\right) \in \text{im}\ \overline{\sigma} = \text{im}\ \sigma.$$

□

Lemma 1.9.32. *Let $L \supset K$ be a field extension and $f \in K[x]$ an irreducible, monic polynomial. Suppose there is an $a \in L$ such that $f(a) = 0$. Set $L' := K[x]/I \simeq K(a)$, where $I := \langle f \rangle$*

1. *Suppose there is a $g \in k[x]$ that also vanishes on a. Then $f \mid g$.*
2. *identifying K canonically with a subfield of L' and $K(a)$ with L' we find $f = (y - (x + I))f_1$ for some $f_1 \in L[y]$.*

Proof. 1. From the proof of the last lemma we learned that $\mathrm{ev}_a(g) = 0$ if and only if $g \in I$, hence $f \mid g$.
2. Since $x + I$ is a zero of f in L the result follows. □

Theorem 1.9.33. *(Existence Theorem for Splitting Fields) Let K be a field and $f \in K[x]$. There is a field L extending K such that f can be written as a product of linear polynomials over L*

Proof. When $\deg f = 1$ the statement is trivial by taking $K = L$. If f is of of degree $d+1$ for some $d \geq 1$, pick a monic irreducible factor of f, g say. Then over $L' = K[x]/\langle g \rangle$, $g = (y - (x + I))g_1$ for some $g_1 \in L'[y]$. The $f = qg = qg_1(y - (x + I))$ for some $q \in L'[y]$ and the result follows by induction in the degree. □

Definition 1.9.34. The above L is called *the splitting field of f over K*.

Lemma 1.9.35. *Let K be a characteristic 0 field and $f \in K[x]$ irreducible monic. Let L be the splitting field of f over K and write $f = \prod_1^d (x - \alpha_i)$ for suitable $\alpha_i \in L$.*

Proof. Suppose L is a field extension over K, and suppose there is an $\alpha \in L$ such that $(x - \alpha^2) \mid f$. Then $g := Df$ also has α as a root. Then $g \nmid f$, hence by Lemma 1.9.32 f cannot be irreducible. In particular if L was the splitting field, and f has a multiple linear factor, then f is not irreducible. □

1.9.4 Theorem of the Primitive Element

Theorem 1.9.36. *Let K be a characteristic 0 field and $L \supset K$ a module-finite extension. Then there is a $c \in L$ such that $L = K(c)$.*

Proof. Suppose $L = K(a, b)$. Then there are monic irreducible polynomials $f, g \in K[x]$ such that $f(a) = 0$ and $g(b) = 0$. Let S be the splitting field of f and g. Write $f = (x - a) \prod_1^l (x - \alpha_i)$ and $g = (x - b) \prod_1^k (x - \beta_i)$. We may pick $\lambda \neq 0$ such that $c := \lambda a + b \neq \lambda \alpha_i + \beta_j$ for any i, j, since $V_{ij} := V(\alpha_i t + \beta_j - (at + b))$ can have at most finitely many points. So

pick any $\lambda \notin \{0\} \cup \bigcup V_{ij}$. Set $K' := K(c)$ and $h := g(c - \lambda x) \in K'[x]$. Note that $h(a) = g(b) = 0$ and $h(\alpha_i) = g(\lambda a + b - \lambda\alpha_i)$ and since $\lambda a + b - \lambda\alpha_i \neq \beta_j$ for any j, $h(\alpha_i) \neq 0$. Then $\gcd(f, h) = x - a \in K'[x]$, implying $a \in K'$, and so $b = c - \lambda a \in K'$. In conclusion, $L = K(c)$.
Suppose $L = K(a_1, \ldots, a_{n+1})$ for some $n \geq 1$. By induction there are $\lambda_1, \ldots, \lambda_n \in K \setminus 0$, so that upon defining $c = \sum_1^n \lambda_i a_i$, $K(a_1, \ldots, a_n) = K(c)$, hence $L = K(c, a) = K(c')$ by the first case. □

1.9.5 Transcendence Degree & Transcendence Bases

Definition 1.9.37. Let $L \supset K$ be a field extension. We say that L has *transcendence degree d over K* if there is a set $X = \{a_1, \ldots, a_d\} \subset L$ such that X is algebraically independent over K and every other set $Y \subset L$ with more than n elements is algebraically dependent over K. We define

$$\text{trdeg}_K\, L := \text{trdeg}\, L = d.$$

If there is not such d we write $\text{trdeg}_K\, L = \infty$.
A finite field extension over K of transcendence degree n is called *an algebraic function field (over K) in n variables*

Remark 1.9.38. If $\delta < d$ is a positive integer such that there are $b_1, \ldots, b_\delta$ that are algebraically independent, then $a_1, \ldots, a_\delta$ are algebraically independent over K, hence the transcendence degree of L over K is unique.

Definition 1.9.39. Let $L \supset K$ be a field extension. A set $X = \{a_1, \ldots, a_d\} \subset L$ is a *transcendence basis of L over K* if X is algebraically independent over K and $K(a_1, \ldots, a_d) \supset K$ is algebraic.

Remark 1.9.40. When $L \supset K$ is algebraic, then $\emptyset$ is a transcendence basis of L over K.

Lemma 1.9.41. *Let $L \supset K$ be a field extension and $X = \{a_1, \ldots, a_d\} \subset L$ be algebraically dependent. Consider an element $a \in L$. $X \cup \{a\}$ is algebraically dependent over K if and only if a is algebraic over $K(a_1, \ldots, a_d)$. Therefor $a_1, \ldots, a_d \in L$ forms a transcendence basis of L over K if and only if $a_1, \ldots, a_d$ are algebraically independent over K and every $a \in L$ is algebraic over $K(a_1, \ldots, a_d)$.*

Proof. " $\Longrightarrow$ ": Let $f \in K[x_1, \ldots, x_{d+1}] \setminus 0$ be given such that $f(a_1, \ldots, a_d, a) = 0$. Then $f = \sum_0^m f_i x_{d+1}^i$, with $f_m \neq 0$. Since X is algebraically independent over K, this implies $f_m(a_1, \ldots, a_d) \neq 0$. Then

$$g := f_m(a_1, \ldots, a_d)^{-1} f(a_1, \ldots, a_d, x)$$

is non-zero, monic and has a as a root, implying a is algebraic over $K(a_1, \ldots, a_d)$.
" $\Longleftarrow$ ": There is some $f = y^m + \sum_0^{m-1} b_i y^i \in K(a_1, \ldots, a_d)[y] \setminus 0$ such that $f(a) = 0$. Then $g := cf \in K[a_1, \ldots, a_d][y] \setminus 0$, where c is a common denominator of the b_i's, hence

$$g = \sum_0^m g_i(a_1, \ldots, a_d) y^i,$$

for suitable $g_i \in K[x_1, \ldots, x_d]$, with $g_m = c \neq 0$. It follows that

$$h := \sum_0^m g_i y^i \in K[x_1, \ldots, x_d, y],$$

such that $h(a_1, \ldots, a_d, a) = g(a) = 0$, hence $X \cup \{a\}$ is algebraically dependent over K. □

Lemma 1.9.42. *Let $L \supset K$ and $L' \supset K$ be field extensions. Suppose there is a surjective K-algebra homomorphism $\sigma : L \to L'$. Then* $\text{trdeg}_K\, L \geq \text{trdeg}_K\, L'$.

Proof. Let $\alpha \in L'$ and set $n :=$ trdeg L. Pick $\alpha_1, \ldots, \alpha_n \in L'$ and pick $\beta, \beta_1, \ldots, \beta_n \in L$ such that $\sigma(\beta), \sigma(\beta_i) = \alpha$. Pick $f \in K(\beta_1, \ldots, \beta_n)[x] \setminus 0$ be monic such that $f(\beta) = 0$. Let $\overline{\sigma}$ denote the restriction of $L[x] \to L'[x]$, the induced K-algebra map induced by σ to a K-algebra map $K(\beta_1, \ldots, \beta_n) \to K(\alpha_1, \ldots, \alpha_n)$. Then

$$\overline{\sigma}(f)(\alpha) = \overline{\sigma}(f)(\sigma(\beta)) = \sigma(f(\beta)) = \sigma(0) = 0,$$

and since $\overline{\sigma}(f)$ is monic, this shows that α is algebraic over $K(\alpha_1, \ldots, \alpha_n)$. So every sequence of n elements in L' are algebraically dependent over K by the prior lemma. It follows that $\text{trdeg}_K\, L' \leq n = \text{trdeg}_K\, L$. □

Remark 1.9.43. Note that the assumption that X is algebraically independent over K is not necessary to prove " $\Longleftarrow$ ".

Lemma 1.9.44. *Let $L \supset K$ be a finite field extension generated by $a_1, \ldots, a_d \in L$.*

1. *There is a subset of $X := \{a_1, \ldots, a_r\}$ that is a transcendence basis for L over K.*
2. *Let $Y = \{a \in X : a$ is transcendental over $K(b_1, \ldots, b_s)\}$. If $b_1, \ldots, b_s \in L$ are algebraically independent over K, then for some $Z \subset Y$, $\{b_1, \ldots, b_s\} \cup Y$ is a transcendence basis of L over K*

Proof. 1. If no subset of X is algebraically independent over K, then each a_i is algebraic over K, hence L is algebraic over K. This means that $\emptyset$ is a transcendence basis for L over K.

Suppose now that there is a subset of $\{a_1, \ldots, a_r\}$ whose elements are algebraically independent over K. Let $Y \subset \{a_1, \ldots, a_r\}$ be a maximal subset of elements that are algebraically independent over K. After a permutation, we can write $Y = \{a_1, \ldots, a_k\}$ for some $k < r$. Then $a_1, \ldots, a_k, a_i$ are algebraically dependent over K for each $i \in \{k+1, \ldots, r\}$. Then by Lemma 1.9.41, a_i is algebraic over $K(a_1, \ldots a_k)$ for each $i \in \{k+1, \ldots, r\}$. Then L is algebraic over $K(a_1, \ldots, a_k)$, implying Y is a transcendence basis of L over K.

2. We proceed by induction in k: if a_i is algebraic over $K(b_1, \ldots, b_s)$ for every $i \in \{1, \ldots, r\}$, then L is algebraic over $K(b_1, \ldots, b_s)$, hence $\{b_1, \ldots, b_s\}$ is a transcendence basis for L over K by Lemma 1.9.41.

Suppose the statement is true for some $k < n$. Consider WLOG $Y = \{a_1, \ldots, a_{k+1}\}$. By Lemma 1.9.41 $b_1, \ldots, b_s, a_1$ are algebraically independent over K. Every element in $X \setminus Y$ is algebraic over $K(b_1, \ldots, b_s, a_1)$, hence $Y' := \{a \in X : a \text{ is transcendental over } K\} \subset Y$. It thus follows by induction that for some $Z' \subset Y'$,

$$\{b_1, \ldots, b_s\} \cup \underbrace{\{a_1\} \cup Z'}_{=:Z}$$

is a transcendence basis of L over K. □

Theorem 1.9.45. *Let $L \supset K$ be a field extension. L has a transcendence basis $X = \{a_1, \ldots, a_d\}$ if and only if* $\text{trdeg}_K\, L = d$

Proof. " $\Longrightarrow$ ": **To prove this we make the following claim**: L is algebraic over $K(b_1, \ldots, b_k, a'_{k+1}, \ldots, a'_d)$ for any $k \in \{0, \ldots, d\}$ for some subset $\{a'_{k+1}, \ldots, a'_d\} \subset \{a_1, \ldots, a_d\}$ with $d - k$ elements. We prove this using induction in r. For $k = 0$, we have that $L \supset K(a_1, \ldots, a_d)$ is algebraic by the assumption that $\{a_1, \ldots, a_d\}$ is a transcendence basis for L over K. Suppose that the claim holds for some $k \in \{0, \ldots, d-1\}$. Then by induction hypothesis L is algebraic over $M := K(b_1, \ldots, b_k, a'_{r+1}, \ldots, a'_d)$ for a suitable subset $\{a'_{k+1}, \ldots, a'_d\} \subset \{a_1, \ldots, a_d\}$. This means b_{k+1} is algebraic over M, hence by Lemma 1.9.41 $b_1, \ldots, b_{k+1}, a'_{r+1}, \ldots, a_d$ are algebraically dependent over K. By Lemma 1.9.44 there is an integer $r \leq s < d$ and a subset $\{a''_{k+1}, \ldots, a''_{s+1}\} \subset \{a'_{k+1}, \ldots, a'_d\}$ such that $b_1, \ldots, b_{k+1}, a''_{k+2}, \ldots, a''_{s+1}$ are algebraically independent over K and $b_1, \ldots, b_{r+1}, a''_{k+1}, \ldots, a''_{s+1}$ are algebraically dependent over K. Again by Lemma 1.9.41, we have that a''_{k+1} is algebraic over $M' := K(b_1, \ldots, b_{k+1}, a''_{k+2}, \ldots, a''_d)$. Now $L \supset M(b_{r+1})$ is algebraic and $M'(a''_{r+1}) \supset M'$ is algebraic. Since $M(b_{r+1}) = M'(a''_{r+1})$ we find that L is algebraic over M', finishing the proof of the claim. **Application of the claim:** Suppose for a contradiction that there are $b_1, \ldots, b_{d+1} \in L$ that are algebraically independent over K. Then $b_1, \ldots, b_d$ are algebraically independent over K. But then by the

claim $L \supset K(b_1, \ldots, b_d)$ is algebraic, hence b_{d+1} is algebraic over $K(b_1, \ldots, b_d)$, but then $b_1, \ldots, b_{d+1}$ are algebraically dependent over K.
" $\Longleftarrow$ ": There are algebraically independent $a_1, \ldots, a_d \in L$ and for every $a \in L$ $a, a_1, \ldots, a_d$ are algebraically dependent, hence by Lemma 1.9.41 $\{a_1, \ldots, a_d\}$ is a transcendence basis of L over K. □

Remark 1.9.46. Note the claim of " $\Longleftarrow$ ", simply proves that every algebraically independent $b_1, \ldots, b_d$ forms a transcendence basis of L over K

Corollary 1.9.47. *Two transcendence bases have the same cardinality.*

Example 1.9.48. Consider $L = K(x_1, \ldots, x_n) = Q(K[x_1, \ldots, x_n])$.
The elements $x_1, \ldots, x_n \in K[x_1, \ldots, x_n]$ are algebraically independent over K, hence $x_1, \ldots, x_n \in K(x_1, \ldots, x_n)$ are algebraically independent over K. It follows that $\{x_1, \ldots, x_n\}$ is a transcendence basis of $K(\mathbf{x}) \supset K$, hence $\text{trdeg}_K\, K(\mathbf{x}) = n$.

Corollary 1.9.49. *Consider a tower of field extensions $M \supset L \supset K$. Let $X = \{a_1, \ldots, a_n\} \subset L$ be a transcendence basis for L over K. Suppose M is finitely generated L-module (in other words a finite dimensional vector space over L). Then X is transcendence basis for M over K and hence $\text{trdeg}_K\, M = \text{trdeg}_K\, L$.*

Proof. We need to check that M is algebraic over $K(a_1, \ldots, a_n)$. Note that $M \supset L$ being module-finite, implies $M \supset L$ is algebraic (cf. Lemma 1.9.20), hence $M \supset K(a_1, \ldots, a_n)$ is algebraic. □

Lemma 1.9.50. *Let $L \supset K$ and $M \supset K$ be field extension with an injective K-algebra homomorphism $\sigma : L \hookrightarrow M$. If $a_1, \ldots, a_n \in L$ are algebraically independent over K, then so are $\sigma(a_1), \ldots, \sigma(a_n)$. It follows that* $\text{trdeg}_K\, L \leq \text{trdeg}_K\, M$.

Proof. Let $a_1, \ldots, a_n \in L$ be algebraically independent over K. Let $f \in K[x_1, \ldots, x_n] \setminus 0$. Note that $\sigma(k) = k$ for every $k \in K$ and $f(a_1, \ldots, a_n) \neq 0$ by the assumption that $a_1, \ldots, a_n$ are algebraically independent over K. It follows that

$$f(\sigma(a_1), \ldots, \sigma(a_n)) = \sigma(f(a_1, \ldots, a_n)) \neq 0,$$

hence $\sigma(a_1), \ldots, \sigma(a_n)$ are algebraically independent. Hence for any $n \leq \text{trdeg } L$ there are algebraically independent $b_1, \ldots, b_n \in M$, hence $\text{trdeg } L \leq \text{trdeg } M$. □

Remark 1.9.51. Note that trdeg_K is an integer invariant (cf. Example D.0.45) of the category with objects being field extensions over K and morphisms being injective K-algebra homomorphism.

Lemma 1.9.52. *Let $M \supset L \supset K$ be a tower of field extensions.*

$$\text{trdeg}_K\ M = \text{trdeg}_K\ L + \text{trdeg}_L\ M.$$

If L has transcendence basis $X = \{a_1, \ldots, a_d\}$ over K and M has transcendence basis $Y = \{b_1, \ldots, b_\delta\}$ over L, then $X \cup Y$ is a transcendence basis for M over K. It follows that

Proof. If $\text{trdeg}_L\ M = \infty$, then $\text{trdeg}_K\ M = \infty$. Suppose this is not the case. Then we may assume that L has transcendence basis $X = \{a_1, \ldots, a_d\}$ over K and M has transcendence basis $Y = \{b_1, \ldots, b_\delta\}$ over L. Then $M \supset L(b_1, \ldots, b_\delta)$ is algebraic and $L \supset K(a_1, \ldots, a_d)$ is algebraic, hence

$$L(b_1, \ldots, b_\delta) \supset K(a_1, \ldots, a_d)(b_1, \ldots, b_\delta) = K(a_1, \ldots, a_d, b_1, \ldots, b_\delta)$$

is algebraic. It follows that $M \supset K(a_1, \ldots, a_d, b_1, \ldots, b_\delta)$ is algebraic. Let $f \in K[x_1, \ldots, x_d, y_1, \ldots, y_\delta] \setminus 0$. Write

$$f = \sum_{v \in \mathbb{N}^\delta} f_v \mathbf{y}^v.$$

For some v, $f_v \neq 0$, hence since $a_1, \ldots, a_d$ are algebraically independent over K, $f_v(a_1, \ldots, a_d) \neq 0$. Then

$$g := f(a_1, \ldots, a_d, \mathbf{y}) = \sum_{v \in \mathbb{N}^\delta} f_v(a_1, \ldots, a_d)\mathbf{y}^v \in L[\mathbf{y}] \setminus 0.$$

Since $b_1, \ldots, b_\delta$ are algebraically independent over L, it follows that

$$f(a_1, \ldots, a_d, b_1, \ldots, b_\delta) = g(b_1, \ldots, b_\delta) \neq 0,$$

hence $a_1, \ldots, a_d, b_1, \ldots, b_\delta$ are algebraically independent over K. Hence $X \sqcup Y$ is a transcendence basis for M over K, hence

$$\text{trdeg}_K\ M = d + \delta = \text{trdeg}_K\ L + \text{trdeg}_L\ M.$$

□

Lemma 1.9.53. *Let $L \supset K$ be an algebraic function field in one variable with K algebraically closed. Let $a \in L \setminus K$. Then*

1. *$L \supset K(a)$ is algebraic*
2. *Suppose* char $K = 0$. *Then there is a $b \in L$ such that $L = K(a, b)$.*
3. *Consider an integral domain R with $Q(R) = L$, $K \subset R$ algebraically closed. Suppose there is a non-trivial prime ideal $I \subset R$. Then $\sigma : K \to R/I, a \mapsto a + I$ is an isomorphism.*

Proof. 1. L is algebraic over $K(t)$ for some $t \in L$. Then a is algebraic over $K(t)$, hence we may find $f \in K[x,y] \setminus 0$ such that $f(a,t) = 0$. Note that since $a \notin K$, a cannot be algebraic over K (Lemma 1.9.21). Then $\deg_y f > 0$, hence $g = f(a,y) \neq 0$ is polynomial that vanishes on t, hence $K(a,t) \supset K(a)$ is algebraic and since $L \supset K(a,t)$ is algebraic, it follows that $L \supset K(a)$ is algebraic.
2. Since $L \supset K(a)$ is algebraic it is finite, hence by the Theorem of the Primitive Element $L = K(a,b)$ for some b.
3. We prove the contrapositive: Let $I \supset R$ be any prime ideal. Suppose $\sigma : K \hookrightarrow R/I$ is not surjective. Pick $a \in R$ such that $a + I \in R/I$ is not in K and pick $b \in I$. Note that $b = 0$ or $b \in R \setminus K$, since otherwise $1 \in I$. Then since $L \supset K(a)$ is algebraic, we can find a $f = \sum_0^d g_i(a) y^i \in K[a][y] \setminus 0$ of minimal degree such that $f(b) = 0$. By Lemma 1.9.21 $g_0 = 0$ or $g_0(a) \neq 0$. In the first case we clearly have that $b = 0$, since then $f = y(\sum_1^d g_i(a) y^{i-1})$, hence $f = y$ by minimality. In the second case $f_0(a) \in I$, hence $a + I$ is integral over K which would imply that $a + I \in K$ by Lemma 1.9.21, leading to a contradiction. Since b was chosen arbitrarily it follows that $I = 0$. □

1.9.6 Graph Ideals & Algebraic Dependence of Polynomials

Definition 1.9.54. Consider a ring R. The *graph ideal* for polynomials $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ is defined to be ideal $\langle y_1 - f_m, \dots, y_m - f_m \rangle \subset R[\mathbf{x}, y_1, \dots, y_m]$

Remark 1.9.55. Note that the graph ideal of $f_1, \dots, f_m$ is just the point ideal (cf. Proposition 1.8.47) of $(x_1, \dots, x_n, f_1, \dots, f_m) \in R[\mathbf{x}, \mathbf{y}]^{n+m}$. Hence a polynomial $f \in K[\mathbf{x}, \mathbf{y}]$ is in the graph ideal of $f_1, \dots, f_m$ if and only if $f(\mathbf{x}, f_1, \dots, f_m) = 0$.

Lemma 1.9.56. *Let K be a field. Consider $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ and denote the graph ideal of $f_1, \dots, f_m$ by I. Then I is a proper ideal by Lemma 1.4.46*

Proof. Since 1 doesn't vanish on $(\mathbf{x}, f_1, \dots, f_m)$ it follows that $1 \notin I$. Hence I is proper by Lemma 1.4.46 □

Proposition 1.9.57. *Consider $f_1, \dots, f_{n+1} \in K[x_1, \dots, x_n]$ and let I denote the graph ideal in $K[\mathbf{x}, y_1, \dots, y_{n+1}]$ for these polynomials. Let $G \subset K[\mathbf{x}, \mathbf{y}]$ be a Gröbner basis for I with respect to the lexicographic term order with $x_1 > \cdots > x_n > y_1 > \cdots > y_{n+1}$. Then there is a non-zero polynomial $g \in G \cap K[\mathbf{x}]$ such that $g(f_1, \dots, f_{n+1}) = 0$.*

Proof. By Example 1.9.48 there is a polynomial $h \in K[\mathbf{y}] \setminus 0$ such that $h(f_1, \dots, f_{n+1}) = 0$. Then $h \in I$ and in particular $h \in I \cap K[\mathbf{y}]$. By Proposition 1.8.106, $G' := G \cap K[\mathbf{y}]$ is a Gröbner basis for $I \cap K[\mathbf{y}]$ with respect to the lexicographic term order with $y_1 < \cdots <$

y_{n+1}. Then $h^{G'} = 0$ by Proposition 1.8.84, meaning G' must contain a non-zero polynomial g. Since $g \in I$ we again by Lemma 1.8.106 get that $g(f_1, \ldots, f_{n+1}) = 0$. □

1.9.7 Finite Algebra Homomorphisms

Definition 1.9.58. Let S, T be R-algebras. An R-algebra homomorphism, $\sigma : S \to T$ is called *finite* if $T \supset \sigma(S)$ is module finite.

Lemma 1.9.59. *Let S, T, Q be R-algebras and $\sigma : S \to T$ and $\omega : T \to Q$ be finite R-algebra homomorphisms. Then $\omega\sigma : S \to Q$ is finite.*

Proof. For some $t_1, \ldots, t_m \in T$ and $q_1, \ldots, q_n \in Q$ we have $S = \sum_1^m \sigma(S)t_i$ and $Q = \sum_1^k \omega(T)q_i$. We Then get that

$$\sum_1^k \omega(T)q_i = \sum_1^k \omega\left(\sum_1^m \sigma(S)t_j\right) q_i = \sum_1^k \sum_1^m (\omega \circ \sigma)(S)\,\omega(t_j)q_i,$$

hence Q is a finitely generated over $(\alpha \circ \beta)(R)$ with generators

$$\omega(t_j)q_i,\ (1 \leq i \leq k, 1 \leq j \leq m).$$

Therefor, we can conclude that $\omega \circ \sigma$ is finite. □

Lemma 1.9.60. *Let S, T be R-algebras and $\sigma : S \to T$ be a surjective R-algebra homomorphism. Then σ is finite.*

Proof. Trivial since $T = \sigma(S)$. □

1.9.8 Perron's Theorem of Effective Algebraic Dependence of Polynomials

Lemma 1.9.61. *Let K be any field and $d_1, \ldots, d_n > 0$, $\mathcal{S} \subset \mathbb{N}^n$ containing $d_i e_i \in \mathcal{S}$ for each i. Set $L := K(y_v^{[i]} : v \in \mathcal{S}) = Q(K[y_v^{[i]} : v \in \mathcal{S})$. For each $i \in \{1, \ldots, n\}$, set*

$$g_i := \sum_{v \in \mathcal{S}} y_v^{[i]} \mathbf{x}^v \in L[x_1, \ldots, x_n]$$

and $d_i := \deg\ g_i$ Let $N \geq 0$ be given. Define $\Delta := \{v \in \mathbb{N}^n : |v| \leq N\}$. Then

$$B := \{g_1^{q_1} \cdots g_n^{q_n} x_1^{r_1} \cdots x_n^{r_n} : 0 \leq r_i < d_i, \sum_1^n q_i d_i + r_i \leq N\}$$

is a basis for $L[\mathbf{x}]_{\leq N}$ over K.

Proof. For each $v = (v_1, \ldots, v_n) \in \Delta$ there are unique pair of tuples

$$(q_1(v_1), \ldots, q_n(v_n)), (r_1(v_1), \ldots, r_n(v_n)) \in \mathbb{N}^n$$

such that for each $i \in \{1, \ldots, n\}$, $0 \leq r_i < d_i$ and $v = (q_1 d_1 + r_1, \ldots, q_n d_n + r_n)$. We $\nabla = \{(q_1, \ldots, q_n), (r_1, \ldots, r_n) \in \mathbb{N}^n : 0 \leq r_i < d_i, \sum_1^n (q_i d_i + r_i) \leq N\}$. We thus have that

$$(q, r) : \Delta \to \nabla$$
$$v = (v_1, \ldots, v_n) \mapsto ((q_1(v_1), \ldots, q_n(v_n)), (r_1(v_1), \ldots, r_n(v_n)))$$

defines a bijection. We define for each $v \in \Delta$,

$$\Lambda_v := \Lambda_{q(v), r(v)} := \left(\prod_1^n g_i^{r_i(v_i)} \right) \left(\prod_1^n x_i^{r_i(v_i)} \right) \in K[\mathbf{x}][y_v^{[i]} : v \in \mathcal{S}].$$

We thus have that $B = \{\Lambda_v : v \in \Delta\}$. Note that deg $\Lambda_v = |v|$ for each $v \in \Delta$, which for one means that $\Lambda_v \in L[\mathbf{x}]_{\leq N}$. Let $\sigma : K[y_v^{[i]} : v \in \mathcal{S}] \to K$ be the unique K-algebra homomorphism defined by $y_v^{[i]} \mapsto 0$ when $(v, i) \neq (d_1 e_1, 1) \ldots, (d_n e_n, n)$ and $y_{d_i e_i}^{[i]} \mapsto 1$. This map naturally extends to a $K[\mathbf{x}]$-algebra homomorphism which we also denote σ. Then $\sigma(g_i) = \mathbf{x_i}^{d_i}$ and $\sigma(\mathbf{x}^v) = \mathbf{x}^v$, hence

$$\sigma(\Lambda_v) = x_1^{q_1(v_1) d_1 + r_1(v_1)} \cdots x_n^{q_n(v_n) d_n + r_n(v_n)} = \mathbf{x}^v. \tag{5}$$

Write for each $v \in \Delta$

$$\Lambda_v = \sum_{v \in \Delta} c_{vw} \mathbf{x}^w.$$

By (5)

$$\sigma(c_{vw}) = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v \end{cases}$$

Let D denote $\#\Delta = \frac{N(N+1)}{2}$. σ naturally induces a homomorphism

$$\sigma : M_D(K[y_v : v \in \mathcal{S}]) \to M_D(K) \subset M_D(K[y_v : v \in \mathcal{S}])$$

defined by entry-wise application for which $\sigma((c_{vw})) = (\sigma(c_{vw})) = (e_{vw}) = I_D$. Set $\mathcal{V}$ to be equal to $\{\mathbf{x}^v : v \in \Delta\}$; i.e. the standard basis for $L[\mathbf{x}]_{\leq N}$ over L. Then ${}_{\mathcal{V}}T_B = (c_{vw}) \in M_D([y_v : v \bigcup \mathcal{S}_i])$. Moreover,

$$\sigma(\det\ {}_{\mathcal{V}}T_B) = \det\ \sigma({}_{\mathcal{V}}T_B) = \det\ I_D = 1 \neq 0 \implies \det\ {}_{\mathcal{V}}T_B \neq 0.$$

This means ${}_{\mathcal{V}}T_B$ is invertible in $M_D(L)$, hence B is a basis by Theorem 1.7.10. □

Remark 1.9.62. In the above setup we can therefore given any $f \in K[\mathbf{x}]$ find a family of polynomials

$$f_{r_1,\ldots,r_n} \in L[z_1,\ldots,z_n] \quad (0 \leq r_i < d_i),$$

such that

$$1.\ f = \sum_{r_1,\ldots,r_n} f_{r_1,\ldots,r_n}(g_1,\ldots,g_n)x_1^{r_1}\cdots x_n^{r_n}$$

$$2.\ \deg\ f_{r_1,\ldots,r_n}(z_1^{d_1},\ldots,z_n^{d_n}) + \sum_1^n r_i \leq \deg\ f.$$

Indeed set $N := \deg f$ and write

$$f = \sum_{v\in\Delta} a_v\Lambda_v = \sum_{(r_1,\ldots,r_n)\in\mathbb{N}^n : r_i<d_i} \left[\sum_{(q_1,\ldots,q_n):\sum_1^n (q_id_i+r_i)\leq N} a_{(q_1d_1+r_1,\ldots,q_nd_n+r_n)} g_1^{q_1}\cdots g_n^{q_n} \right] x_1^{r_1}\cdots x_n^{r_n}.$$

Setting

$$f_{r_1,\ldots,r_n} := \sum_{(q_1,\ldots,q_n)\in\mathbb{N}^n:\sum_1^n (q_id_i+r_i)\leq N} a_{(q_1d_1+r_1,\ldots,q_nd_n+r_n)} z_1^{q_1}\cdots z_n^{q_n} \quad (0 \leq r_i < d_i)$$

These polynomials will satisfy property 1. Secondly,

$$\begin{aligned}\deg\ f_{r_1,\ldots,r_n}(z_1^{d_1},\ldots,z_n^{d_n}) + \sum_1^n r_i &\leq \max_{(q_1,\ldots,q_n):\sum_1^n (q_id_i+r_i)} \deg\ z_1^{q_1d_1}\cdots z_n^{q_nd_n} + \sum_1^n r_i \\ &= \max_{(q_1,\ldots,q_n):\sum_1^n (q_id_i+r_i)} \sum_1^n (q_id_i + r_i) \leq N = \deg\ f.\end{aligned}$$

Lemma 1.9.63. *Let K be any field and $d_1,\ldots,d_n > 0$, $\mathcal{S} \subset \mathbb{N}^n$ containing $d_ie_i \in \mathcal{S}$ for each i. Set $L := K(y_v^{[i]} : v \in \mathcal{S}) = Q(K[y_v^{[i]} : v \in \mathcal{S})$. For each $i \in \{1,\ldots,n\}$, set*

$$g_i := \sum_{v\in\mathcal{S}} y_v^{[i]}\mathbf{x}^v \in L[x_1,\ldots,x_n]$$

and $d_i := \deg\ g_i$ Then for every $g_{n+1} \in L[\mathbf{x}]$ with $d_{n+1} := \deg\ g_{n+1}$ there is polynomial $P \in L[z_1,\ldots,z_{n+1}]$ that is monic in $L[z_1,\ldots,z_n][z_{n+1}]$ satisfying

$$1.\ P(g_1,\ldots,g_{n+1}) = 0$$

$$2.\ \deg\ P(z_1^{d_1},\ldots,z_{n+1}^{d_{n+1}}) \leq \prod_1^{n+1} d_i.$$

Proof. There is $d := \prod_1^n d_i$ elements in $\Omega = \{v \in \mathbb{N}^n : v_i < d_i\}$. Denote the elements in $\{\mathbf{x}^v : v \in \Omega\} = \{M_1,\ldots,M_d\}$. Then by Lemma 1.9.61 for each $i \in \{1,\ldots,d\}$ there are polynomials

$$P_{ij} \in L[\mathbf{x}] \quad (j \in \{1,\ldots d\})$$

such that

a. $M_i g_{n+1} = \sum_1^d P_{ij}(g_1, \ldots, g_n) M_i$

b. $\deg\ P_{ij}(z_1^{d_1}, \ldots, z_n^{d_n}) + \deg M_i \leq \deg\ g_{n+1} M_i = d_{n+1} + \deg\ M_i.$

Property a. shows that

$$g_{n+1} \begin{pmatrix} M_1 \\ \vdots \\ M_d \end{pmatrix} = (P_{ij}(g_1, \ldots, g_n)) \begin{pmatrix} M_1 \\ \vdots \\ M_d \end{pmatrix},$$

I.e. g_{n+1} is an eigenvalue of $(P_{ij}(g_1, \ldots, g_n))$, hence by Cramer's rule,

$$\det(P_{ij}(g_1, \ldots, g_n) - \delta_{ij} g_{n+1}) = 0.$$

Then $P := (-1)^d \det(P_{ij} - \delta_{ij} g_{n+1}) = \sum_{\pi \in S_d} \prod_1^d (P_{i\pi(i)} - \delta_{ij} g_{n+1}) \in L[\mathbf{x}] \setminus 0$ satisfies $P(g_1, \ldots, g_{n+1}) = 0$. From b. we find that

$$\deg\ P_{ij}(z_1^{d_1}, \ldots, z_n^{d_n}) - \delta_{ij} z_{n+1}^{d_{n+1}} \leq d_{n+1} + \deg\ M_i - \deg\ M_j.$$

For an arbitrary permutation $\pi \in S_d$,

$$\begin{aligned} \deg \prod_1^d (P_{i\pi(i)}(z_1^{d_1}, \ldots, z_d^{d_n}) - \delta_{i\pi(i)} z_{n+1}^{d_{n+1}}) &\leq \sum_1^d (d_{n+1} + \deg\ M_{\pi(i)} - \deg\ M_i) \\ &= \sum_1^d d_{n+1} + \sum_1^d \deg\ M_i - \sum_1^d \deg\ M_i \\ &= d d_{n+1} = \prod_1^{n+1} d_i. \end{aligned}$$

□

Lemma 1.9.64. *Let K be some field. For $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$ with $d_i := \deg\ f_i > 0$, and*

$$f_i = \sum_{v \in \mathbb{N}^n} a_v^{[i]} \mathbf{x}^v.$$

Define $L := K\left(y_v^{[i]} : v \in \mathbb{N}^n, a_v^{[i]} \neq 0 \text{ or } v = d_i e_i\right)$ and set $Y := \left\{y_v^{[i]} : v \in \mathbb{N}^n, a_v^{[i]} \neq 0 \text{ or } v = d_i e_i\right\}$. Lastly define

$$g_i := \sum_{v \in \mathbb{N}^n} y_v^{[i]} \mathbf{x}^v$$

for every i. Then there is a $K[\mathbf{x}]$-algebra homomorphism $\sigma : K[Y][\mathbf{x}] \to K[\mathbf{x}]$ such that $\sigma(g_i) = f_i$

Proof. Indeed, take σ to be the K-algebra homomorphism such that $y_v^{[i]} \mapsto a_v^{[i]}$. This trivially extends to a $K[\mathbf{x}]$-algebra homomorphism. □

Theorem 1.9.65. *(Perron's Theorem) Let K be any field and let $f_1, \ldots, f_{n+1} \in K[x_1, \ldots, x_n]$ and put $d_i :=$ deg f_i for each i. Then there is a $P \in K[z_1, \ldots, z_{n+1}] \setminus 0$ satisfying*

$$\begin{aligned} &1.\ P(f_1, \ldots, f_{n+1}) = 0, \\ &2.\ \deg\ P(z_1^{d_1}, \ldots, z_{n+1}^{d_{n+1}}) \leq \prod_1^{n+1} d_i. \end{aligned}$$

Proof. First a slight reformulation. Let M be some field and consider $H = \{h_1, \ldots, h_{n+1}\} \subset M[x_1, \ldots, x_n]$, $\delta_i :=$ deg h_i. Set

$$\Delta_{\delta_1,\ldots,\delta_{n+1}} := \left\{ v \in \mathbb{N}^{n+1} : \sum_1^{n+1} v_i \delta_i \leq \prod_1^{n+1} \delta_i \right\}.$$

Then define

$$B(H) := \left\{ h_1^{q_1} \cdots h_{n+1}^{q_{n+1}} : (q_1, \ldots, q_{n+1}) \in \Delta_{\delta_1,\ldots,\delta_{n+1}} \right\}.$$

And let $\mathcal{V}$ be the standard basis of $\{\mathbf{z}^v : v \in \Delta_{\delta_1,\ldots,\delta_n}\}$. The matrix ${}_{\mathcal{V}}T_{B(H)}$ is not invertible if and only if for suitable $a_v \in M$

$$\sum_{v \in \Delta} a_v h_1^{v_1} \cdots h_{n+1}^{v_{n+1}} = \mathrm{ev}_{h_1,\ldots,h_{n+1}} \underbrace{\left(\sum_{v \in \Delta} a_v \mathbf{z}^v \right)}_{P_H},$$

where

$$\deg\ P_H(z_1^{\delta_1}, \ldots, z_{n+1}^{\delta_{n+1}}) \leq \max_{v \in \Delta} \deg\ \mathbf{z}^{(\delta_1 v_1, \ldots, \delta_{n+1} v_{n+1})} = \max_{v \in \Delta} \sum_1^{n+1} \delta_i v_i \leq \prod_1^{n+1} \delta_i.$$

Set $F = \{f_1, \ldots, f_{n+1}\}$ and $B := \{\mathbf{z}^v : v \in \Delta_{d_1,\ldots,d_{n+1}}\}$. To prove the theorem we can equivalently prove that det ${}_BT_{B(F)} = 0$. With this in mind, we proceed with the proof of the theorem. Write $f_i = \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v$. Define L, g_i and σ as in the prior lemma. By Lemma 1.9.63, there is a $Q \in L[\mathbf{x}] \setminus 0$ such that

$$\begin{aligned} &1.\ Q(g_1, \ldots, g_n, f_{n+1}) = 0, \\ &2.\ \deg\ Q(z_1^{d_1}, \ldots, z_{n+1}^{d_{n+1}}) \leq \prod_1^{n+1} d_i. \end{aligned}$$

Set $G := \{g_1, \ldots, g_n, f_{n+1}\}$. Then $\sigma({}_BT_{B(G)}) = {}_BT_{B(F)}$. It thus follows that

$$\det\ {}_BT_{B(F)} = \det\ \sigma({}_BT_{B(G)}) = \sigma(\det\ {}_BT_{B(G)}) = \sigma(0) = 0.$$

□

1.9.9 Krull Dimension

Definition 1.9.66. Let R be a commutative ring and $\mathfrak{p}$ a prime ideal in R. We define the *height* of $\mathfrak{p}$ to be the supremum of natural numbers (including infinity) n such that there is an ascending chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

We denote the height of $\mathfrak{p}$ by

$$\text{height } \mathfrak{p}.$$

Definition 1.9.67. Given a commutative ring R. The *Krull dimension* of R, denoted dim R is the quantity

$$\sup$$

Lemma 1.9.68. *When R is an integral domain, 0 has height 0.*

Proof. Indeed, the only chain of distinct primes contained in 0 is the trivial one, hence height $0 = 0$. □

Corollary 1.9.69. *The Krull dimension of a field is 0.*

Lemma 1.9.70. *The Krull dimension of PID that is not a field is 1*

Proof. Let R be a non-field PID. Then there is a non-trivial prime ideal $\langle p \rangle$. Suppose $\langle q \rangle \subset \langle p \rangle$ is a non-zero prime. Then $q = pd$ for some d. Since q is in particular irreducible, hence d is a unit. It follows that $\langle q \rangle = \langle p \rangle$, hence the height of any non-zero prime is 1. It thus follows that dim $R = 1$. □

Corollary 1.9.71. *$\mathbb{Z}$, $K[x]$ where K is a field, and DVRs have Krull dimension 1.*

Lemma 1.9.72. *Let $R := K[x_1, \ldots, x_n]/\mathfrak{p}$ where $\mathfrak{p} \subset K[x_1, \ldots, x_n]$ is a prime ideal. Then*

$$\text{dim } R = \text{trdeg Frac}(R)$$

Proof. That's gonna take some work □

Corollary 1.9.73. *The Krull dimension of $K[x_1, \ldots, x_n]$ is n*

Lemma 1.9.74. *The Krull dimension of $R[x_1, dots, x_n]$ where R is noetherian is*

$$\text{dim } R + n$$

Proof. Is also gonna take some work to do □

1.9.10 Noether Normalizations

Lemma 1.9.75. *Let K be a field and let $f = \sum_{v\in\mathbb{N}^n} a_v \mathbf{x}^v \in K[x_1,\dots,x_n]$ be given with $d := \deg\ f > 0$. Then we have the following*

1. *There are elements $y_1,\dots,y_{n-1} \in K[x_1,\dots,x_n]$ such that $x_i = y_i + x_n^{r_i}$ for $i \in \{1,\dots,n-1\}$ for suitable $r_i > 0$ and*

$$f = ax_n^m + \sum_i^{m-1} G_i x_n^i,$$

   for some $a \in K\setminus\{0\}$, $m > 0$ and $G_i \in K[y_1,\dots,y_{n-1}]$.

2. *If $\#K = \infty$ we get the same result as in (a) with $x_i = y_i + a_i x_n$ for*

Proof. 1. Set $k = d+1$ and put $r_i = k^i$ for $i \in \{1,\dots,n-1\}$. Then for $v, w \in \mathbb{N}^n$ with $|v|, |w| \le d$ and $v \neq w$ we get

$$v_n + \sum_1^{n-1} v_i r_i = v_n + \sum_1^{n-1} v_i k^i \neq w_n + \sum_1^{k} w_i k^i = w_n + \sum_1^{n} w_i r_i,$$

by the uniqueness of k-adic expansions. This mean that we can define $m := \max_{v\in\mathbb{N}^n:|v|\le d, a_v\neq 0}\left\{v_n + \sum_1^{n-1} v_i r_i\right\}$, $v_m = \operatorname{argmax}_{v\in\mathbb{N}^n:|v|\le d, a_v\neq 0}\left\{v_n + \sum_1^{n-1} v_i r_i\right\}$. We thus pick $y_i = x_i - x_n^{r_i}$ and see that

$$f = f\left(y_1 + x_n^{r_1},\dots,y_{n-1}+x_n^{r_{n-1}},x_n\right) = \sum_{v\in\mathbb{N}^n:|v|\le d} a_v \left(\prod_1^{n-1}\left(y_i + x_n^{r_i}\right)^{v_i}\right) x_n^{v_n}$$

$$= \sum_{v\in\mathbb{N}^n:|v|\le d} a_v \left(x_n^{v_n+\sum_1^{n-1} v_i r_i} + \underbrace{\cdots}_{\text{lower degree terms}}\right) = \sum_{v\in\mathbb{N}^n:|v|\le d} a_v x_n^{v_n+\sum_1^{n-1} v_i r_i} + \underbrace{\cdots}_{\text{lower degree terms}}$$

$$= a_{v_m} x_n^m + \underbrace{\cdots}_{\text{lower degree terms}}.$$

We can write the lower order terms on the form $\sum_1^{m-1} G_i x_n^i$ for suitable $G_i \in K[y_1,\dots,y_{n-1}]$.
2. Write $f = \sum_0^d f_i$ with $f_d \neq 0$ for homogeneous $f_i \in K[x_1,\dots,x_n]$ of degree i. By Lemma 1.8.133, $f_d(x_1,\dots,x_{n-1},1) \neq 0$. Since $\#K = \infty$ we get that there are $a_1,\dots,a_{n-1} \in K$ such that $f_d(a_1,\dots,a_{n-1},1) \neq 0$. We now pick $y_i = x_i - a_i x_i$ for $i \in \{1,\dots,n-1\}$. Note that

$$\sum_{v\in\mathbb{N}^n:|v|=d} a_v (a_1 x_n)^{v_1}\cdots(a_{n-1}x_n)^{v_{n-1}} x_n^{v_n} = \left[\sum_{v\in\mathbb{N}^n:|v|=d} a_v a_1^{v_1}\cdots a_{n-1}^{v_{n-1}}\cdot 1^{v_n}\right] x_n^d = f_d(a_1,\dots,a_{n-1},1)x_n^d.$$

From this it follows that

$$\begin{aligned} f &= f(y_1 + a_1x_n, \dots, y_{n-1} + a_{n-1}x_n, x_n) \\ &= \sum_{v\in\mathbb{N}^n:|v|=d} a_v(y_1 + a_1x_n)^{v_1}\cdots(y_{n-1} + a_{n-1}x_n)^{v_{n-1}}x_n^{v_n} + \sum_1^{m-1}\sigma(F_i) \\ &\overset{(*)}{=} \left[\sum_{v\in\mathbb{N}^n:|v|=d} a_v a_1^{v_1}\cdots a_{n-1}^{v_{n-1}}\cdot 1^{v_n}\right] x_n^d + \cdots = f_d(a_1,\dots,a_{n-1},1)x_n^d + \dots. \end{aligned}$$

We then set $a = f_d(a_1, \dots, a_{n-1}, 1)$. The $\dots$ in the expressions following $(*)$ in the above signify remaining terms. One readily verifies that these have x_n-degree strictly smaller than d. Again these terms can clearly be written on the form $\sum_1^{d-1} G_i x_n^i$ for suitable $G_i \in K[y_1, \dots, y_{n-1}]$. □

Theorem 1.9.76. *(Noether Normalization Theorem) Let $A = K[x_1, \dots, x_n]/J$ for some field K and some ideal $J \subsetneq K[x_1, \dots, x_n]$. Let also $I \subsetneq A$ be an ideal.*

(a) Then there are elements $y_1, \dots, y_d \in A$, which are algebraically independent such that A is a finitely generated $K[y_1, \dots, y_d]$-module. Furthermore, for some $\delta \leq d$, $I \cap K[y_1, \dots, y_d] = \langle y_{\delta+1}, \dots, y_d\rangle$.

(b) In addition if $\#K = \infty$, we have that $y_i = \sum_{j=i}^n a_{ij}x_j$ for suitable $a_{ij} \in K$.

Proof. 1. **case 1:** We first consider the case where $A = K[\mathbf{x}]$ and $I = \langle f\rangle$ for some $f \in A$ with deg $f > 0$. We put $y_n = f$ and apply Lemma 1.9.75 1. to obtain $y_i = x_i - x_n^{r_i} \in A$ for suitable $r_i > 0$ for $i \in \{1, \dots, n-1\}$, such that

$$y_n = f = ax_n^m + \sum_1^{m-1} G_i(y_1, \dots, y_{n-1})x_n^i \iff y_n - ax_n^m + \sum_1^{m-1} G_i(y_1, \dots, y_{n-1})x_n^i = 0,$$

for some $a \in K\backslash\{0\}, m > 0, G_i \in K[y_1, \dots, y_{n-1}]$. Then x_n is integral over $K[y_1, \dots, y_n]$. Since $x_i = y_i + x_n^{r_i} \in A$ we get that $A = K[y_1, \dots, y_n][x_n]$, hence A is a finitely generated $K[y_1, \dots, y_n]$-module.

We now claim that $y_1, \dots, y_n$ are algebraically independent over K. Suppose for contradiction that this is not the case. Then $Y := \{y_1, \dots, y_n\}$ is not a transcendence basis of $K(y_1, \dots, y_n)$. However, by Lemma 1.9.44 (a) there is a subset of Y, say $y_{l_1}, \dots y_{l_k}$ for $k < n$, which constitutes a transcendence basis for $K(y_1, \dots, y_n)$. Then by Corollary 1.9.49

$$k = \text{trdeg}K(y_1, \dots, y_n) = \text{trdeg}K(x_1, \dots, x_n) = n > k,$$

leading to a contradiction.
Let $\lambda \in I \cap K[y_1, \ldots, y_n]$. Then $\lambda = gf = gy_n$ for some $g \in A$. g is integral over $K[y_1, \ldots, y_n]$, hence for suitable $h_1, \ldots, h_{k-1} \in K[y_1, \ldots, y_n]$,

$$g^k + \sum_{i=1}^{k-1} h_i g^i = 0 \implies \lambda^k = f^k g^k = -\sum_{i=1}^{k-1} h_i f^k g^i = -\sum_{i=1}^{k-1} h_i \lambda^i y_n^{k-i}.$$

This means $y_n \mid \lambda^k$, implying $y_n \mid \lambda$. From this we conclude $I \cap K[y_1, \ldots, y_n] = \langle y_n \rangle$.

Case 2: We now prove the statement for $A = K[x_1, \ldots, x_n]$ and an arbitrary ideal $I \subsetneq A$. For $I = 0$, we are done after choosing $y_i = x_i$ and $\delta = n$. We prove the statement for $I \neq 0$ by induction in $n \geq 1$. For $n = 1$, A is a PID, so I is generated by some non-zero polynomial. Then the statement follows from case 1.
Suppose now that $n > 1$ and let $f \in I \setminus \{0\}$. Again using Lemma 1.9.75 we find $y_1, \ldots, y_n \in A$ that are algebraically independent over K with $y_n = f$. Then $y_1, \ldots, y_{n-1}$ are also algebraically independent over K. By the induction hypothesis, we can find elements $t_1, \ldots, t_{d-1} \in K[y_1, \ldots, y_{n-1}]$ algebraically independent over K such that $K[y_1, \ldots, y_{n-1}]$ is a finitely generated $K[t_1, \ldots, t_{d-1}]$-module and $I \cap K[t_1, \ldots, t_{d-1}] = \langle t_{\delta+1}, \ldots, t_{d-1} \rangle$ for some $\delta < d$. We then get that $K[y_1, \ldots, y_n]$ is a finitely generated $K[t_1, \ldots, t_{d-1}, y_n]$-module, and hence A is a finitely generated $K[t_1, \ldots, t_{d-1}, y_n]$-module. Thus by a similar contradiction argument to that of case 1, we conclude that $d = n$ and $t_1, \ldots, t_{n-1}, y_n$ are algebraically independent over K.
Let $\lambda \in I \cap K[t_1, \ldots, t_{n-1}, y_n]$. Then $\lambda = g + hy_n$ for some $g \in I \cap K[t_1, \ldots, t_{n-1}] = \langle t_{\delta+1}, \ldots, t_{n-1} \rangle$ and $h \in K[t_1, \ldots, t_{n-1}, y_n]$, then $I \cap K[t_1, \ldots, t_{n-1}, y_n] = \langle t_{\delta+1}, \ldots, t_{n-1}, y_n \rangle$.

case 3: We now generalize to the case where $A = K[x_1, \ldots, x_n]/J$ and $I \subsetneq A$ for an ideal $J \subsetneq K[x_1, \ldots, x_n]$. We apply case 2 to J and find $y_1, \ldots, y_n \in A$ algebraically independent over K such that $K[x_1, \ldots, x_n]$ is a finitely generated $K[y_1, \ldots, y_n]$-module and $J \cap K[x_1, \ldots, x_n] = \langle y_{d+1}, \ldots, y_n \rangle$ for some $d \leq n$. Consider the embedding $\iota : K[y_1, \ldots, y_n] \hookrightarrow A$. By construction we have that A is a finitely generated $\iota(K[y_1, \ldots, y_n])$-module. It is easy to check that

$$\iota(K[y_1, \ldots, y_n]) \simeq \frac{K[y_1, \ldots, y_n]}{(J \cap K[y_1, \ldots, y_n])} = \frac{K[y_1, \ldots, y_n]}{\langle y_{d+1}, \ldots, y_n \rangle} \simeq K[y_1, \ldots, y_d].$$

From which it follows that A is a finitely generated $K[y_1, \ldots, y_d]$-module.
Let $I' = I \cap K[y_1, \ldots, y_d]$. Then using case 2 we find $t_1, \ldots, t_d \in K[y_1, \ldots, y_d]$ algebraically independent over K such that $K[y_1, \ldots, y_d]$ is a finitely generated $K[t_1, \ldots, t_d]$-module and $I' \cap K[t_1, \ldots, t_d] = \langle t_{\delta+1}, \ldots, t_d \rangle$ for some $\delta \leq d$. It then also follows that A

is a finitely generated $K[t_1, \ldots, t_d]$-module.

2.Suppose now that $\#K = \infty$. In case 1 the construction is also valid with $y_i = x_i - a_i x_n$ for suitable $a_i \in K$ by Lemma 1.9.75 2.
In case 2 we can choose $t_1, \ldots, t_{n-1}$ and $y_1, \ldots, y_{n-1}$ in the same way. In case 3 we can again choose $y_i = x_i - a_i x_n$ for suitable $a_i \in K$. It follows from case 2 that we can choose

$$t_j = y_i - b_j y_d = x_i - x_d - (a_i - b_j a_d) x_n + J,$$

which is of the desired form. □

Definition 1.9.77. Let A be a finitely generated K-algebra. A sequence of elements $y_1, \ldots, y_d \in A$ with the properties specified in the above theorem is called a *Noether normalization* of A.

Corollary 1.9.78. *Consider $A = K[x_1, \ldots, x_n]/I$, with $I \subset K[\mathbf{x}]$ a prime ideal.*

1. *If $y_1, \ldots, y_d$ is a Noether normalization of A, then $X = \{y_1, \ldots, y_d\}$ defines a transcendence basis of $L := Q(A) \supset K$.*

2. *If* $\operatorname{trdeg}_K Q(A) = d$*, then A has a Noether normalization $y_1, \ldots, y_d$*

Proof. 1. By assumption $y_1, \ldots, y_d$ are algebraically independent over K. Secondly A is a finitely generated $K[\mathbf{y}]$-module, hence A is integral over $K[\mathbf{y}]$. Then L is integral over $K[\mathbf{y}]$. It follows that since $L \supset K(\mathbf{y}) \supset K[\mathbf{y}]$ $K(\mathbf{y})$ must be algebraic Lemma not yet written.
2. NNT there is a Noether normalization $y_1, \ldots, y_\delta \in A$. By 1. $\delta = \operatorname{trdeg}_K Q(A) = d$. □

Corollary 1.9.79. *Let A be finitely generated K-algebra and $y_1, \ldots, y_d \in A$ be its Noether normalization. Then $\iota : K[y_1, \ldots, y_d] \hookrightarrow A$ is finite K-algebra homomorphism.*

1.9.11 Derivations and Differentials

Fix a field K.

Definition 1.9.80. Let R be a K-algebra and M and R-module. A *derivation of R into M (over K)* is a K-linear map

$$D : R \to M$$

that satisfies the Leibniz rule, i.e.

$$Dfg = fDg + (Dg)f$$

Lemma 1.9.81. *Let $D : R \to M$ be Derivation of R into M.*

1. *For each $c \in K$,*
$$Dc = 0$$
2. *Let $f \in K[x_1, \ldots, x_n]$ and $t = (t_1, \ldots, t_n) \in R^n$. Then*
$$D(f(t)) = \sum_1^n f_{x_i}(t) Dt_i$$

Proof. 1. This is obvious for $c = 0$. So suppose $c \neq 0$. Then

$$D1 = D(cc^{-1}) = cDc^{-1} + c^{-1}Dc = 2D(1) \implies D1 = 0 \implies Dc = cD1 = 0.$$

2. Since D is K-linear, it suffices to prove the statement for a monomial $\mathbf{x}^v$. Note that the statement trivially holds for $v = 0$. We prove the statement by induction in the number of variables. Let $l > 1$. By induction in the degree, we have

$$Dt^l = tDt^{l-1} + t^{l-1}Dt = (l-1)tt^{l-2}Dt + t^{l-1}Dt = lt^{l-1}Dt = x_x^l(t)Dt.$$

Consider $n > 1$. We may assume that $v \neq 0$ and $v_1 > 0$. Set $w := (v_2, \ldots, v_n)$. Then

$$\begin{aligned} D\left(t_1^{v_1} \cdots t_n^{v_n}\right) &= t_1^{v_1} D\left(t_2^{v_2} \cdots t_n^{v_n}\right) + t_2^{v_2} \cdots t_n^{v_n} Dt_1^{v_1} \\ &= \sum_2^n t^{v_1} \mathbf{x}_{x_i}^w(t_2, \ldots, t_n) Dt_i + v_1 t_1^{v_1 - 1} t_2^{v_2} \cdots t_n^{v_n} Dt_1 \\ &= \sum_2^n \mathbf{x}_{x_i}^v(t) Dt_i + v_1 t_1^{v_1 - 1} t_2^{v_2} \cdots t_n^{v_n} Dt_1 \\ &= \sum_1^n \mathbf{x}_{x_i}^v(t) Dt_i. \end{aligned}$$

□

Lemma 1.9.82. *Let R be K-algebra that is an integral domain with quotient field $L := Q(R)$ and M an L-vector space. Then any derivation $D : R \to M$ extends uniquely to a derivation $\widetilde{D} : L \to M$*

Proof. **Existence:** Define

$$\begin{aligned} \widetilde{D} &: L \to M \\ \frac{x}{y} &\mapsto \frac{1}{y}\left(Dx - \frac{x}{y}Dy\right) \end{aligned}$$

We check that this is well-defined: Suppose $\frac{x_1}{y_1} = \frac{x_2}{y_2}$. Then

$$\begin{aligned}
\frac{1}{y_1}\left(Dx_1 - \frac{x_1}{y_1}Dy_1\right) &= \frac{1}{y_1}\left(Dx_1 - \frac{x_1}{y_1}Dy_1 + \frac{1}{y_2}\left(Dx_2y_1 - Dx_1y_2\right)\right) \\
&= \frac{1}{y_1}\left(Dx_1 - \frac{x_1}{y_1}Dy_1 - \frac{x_1}{y_2}Dy_2 - Dx_1 + \frac{x_2}{y_2}Dy_1 + \frac{y_1}{y_2}Dx_2\right) \\
&= \frac{1}{y_1}\left(\frac{y_1}{y_2}Dx_2 - \frac{x_1}{y_2}Dy_2\right) = \frac{1}{y_2}\left(Dx_2 - \frac{x_1}{y_1}Dy_2\right) \\
&= \frac{1}{y_2}\left(Dx_2 - \frac{x_2}{y_2}Dy_2\right).
\end{aligned}$$

We check that it is K-linear,

$$\begin{aligned}
\widetilde{D}\left(\frac{ax_1}{y_1} + \frac{bx_2}{y_2}\right) &= \widetilde{D}\left(\frac{ax_1y_2 + bx_2y_1}{y_1y_2}\right) \\
&= \frac{1}{y_1y_2}\left(\left(Dax_1y_2 + bx_2y_1\right) - \frac{ax_1y_2 + bx_2y_1}{y_1y_2}Dy_1y_2\right) \\
&= \frac{1}{y_1y_2}\Bigg(ax_1Dy_2 + ay_2Dx_1 + bx_2Dy_1 + by_1Dx_2 \\
&\quad - \frac{ax_1y_2 + bx_2y_1}{y_1y_2}(y_1Dy_2 + y_2Dy_1)\Bigg) \\
&= a\frac{1}{y_1}\left(Dx_1 - \frac{x_1}{y_1}Dy_1\right) + b\frac{1}{y_2}\left(Dx_2 - \frac{x_2}{y_2}Dy_2\right) \\
&= a\widetilde{D}\left(\frac{x_1}{y_1}\right) + b\widetilde{D}\left(\frac{x_2}{y_2}\right).
\end{aligned}$$

We also check the Leibniz rule,

$$\begin{aligned}
\widetilde{D}\left(\frac{x_1x_2}{y_1y_2}\right) &= \frac{1}{y_1y_2}\left(\left(Dx_1x_2\right) - \frac{x_1x_2}{y_1y_2}(Dy_1y_2)\right) \\
&= \frac{1}{y_1y_2}\left(x_1Dx_2 + x_2Dx_1 - \frac{x_1x_2}{y_2}Dy_2 - \frac{x_1x_2}{y_1}Dy_1\right) \\
&= \frac{x_1}{y_1}\left(\frac{1}{y_2}\left(Dx_2 - \frac{x_2}{y_2}Dy_2\right)\right) + \frac{x_2}{y_2}\left(\frac{1}{y_1}\left(Dx_1 - \frac{x_1}{y_1}Dy_1\right)\right) \\
&= \frac{x_1}{y_1}\widetilde{D}\left(\frac{x_2}{y_2}\right) + \frac{x_2}{y_2}\widetilde{D}\left(\frac{x_1}{y_1}\right).
\end{aligned}$$

Uniqueness Let $z = \frac{x}{y} \in L$. Any derivation $\widetilde{D} : L \to M$ extending D must satisfy

$$Dx = \widetilde{D}yz = y\widetilde{D}z + zDy \implies \widetilde{D}(z) = \frac{1}{y}(Dx - zDy).$$

□

We now define differentials on a K-algebra R. Let F denote the free R-module generated by R, i.e. $F := R[R]$. For each $x \in R$, we let $[x]$ denote x seen as an element of $R[R]$. Let N be the submodule of F generated by

$$\{[x+y]-[x]-[y] : x, y \in R\} \cup \{[ax]-a[x] : x \in R, a \in K\} \cup \{[xy]-x[y]-y[x] : x, y \in R\}$$

Definition 1.9.83. Define *the R-module of differentials over K* to be

$$\Omega(R) := \Omega_K(R) := F/N.$$

We define a derivation of R into $\Omega(R)$,

$$\begin{aligned} d : R &\to \Omega(R) \\ x &\mapsto [x] + N \end{aligned}$$

Remark 1.9.84. We need to check d is indeed a derivation. It is K-linear, since for each $a, b \in K$, $x, y \in R$,

$$d(ax + by) = [ax + by] + N = a([x] + N) + b([y] + N) = adx + bdy.$$

It satisfies the Leibniz rule, since

$$d(xy) = [xy] + N = x[y] + N + y[x] + N = xdy + ydx.$$

Any derivation may be described uniquely in terms of a derivation into $\Omega(R)$.

Lemma 1.9.85. *For any R-module M and derivation $D : R \to M$, the diagram*

$$\begin{array}{ccc} R & \xrightarrow{D} & M \\ {\scriptstyle d}\downarrow & \nearrow_{\exists!\rho} & \\ \Omega(R) & & \end{array}$$

commutes

Proof. First we define

$$\begin{aligned} \rho' : F &\to M \\ \sum_{y \in R} x_y[y] &\mapsto \sum_{y \in R} x_y D \end{aligned}$$

One readily verifies that $\rho'(N) = 0$, hence there is a unique induced map $\rho : \Omega(R) \to M$ with

$$\begin{array}{ccc} F & \twoheadrightarrow & \Omega(R) \\ & {\scriptstyle \rho'}\searrow & \downarrow{\scriptstyle \rho} \\ & & M \end{array}$$

commuting. It follows that ρ is the unique map making the desired diagram commute. □

Lemma 1.9.86. *Let $(t_1, \ldots, t_n) \in R^n$. Then $\Omega(R)$ is finitely generated as a $K[t_1, \ldots, t_n]$-module by $dt_1, \ldots, dt_n$.*

Proof. This follows from Lemma 1.9.81. □

Lemma 1.9.87. *Suppose R is an integral domain with field of fractions L. Consider the derivations $d : R \to \Omega(R)$ and $d' : L \to \Omega(L)$. Then $\widetilde{d} = d'$. Moreover, if $L = K(t_1, \ldots, t_n)$, for suitable $t = (t_1, \ldots, t_n) \in L^n$, then $\Omega(L)$ is an finite dimensional L vector space.*

Proof. Indeed, consider the diagram

$$\begin{array}{ccc} R & \xrightarrow{d} & \Omega(R) \\ \downarrow & & \downarrow \iota \\ L & \xrightarrow[d']{} & \Omega(L) \end{array}$$

Since ιd is derivation, it follows from Lemma 1.9.85 that $\widetilde{d} = d'$. Let $\frac{f(t)}{g(t)} \in L = K(t_1, \ldots, t_n)$. Then

$$d' \frac{f(t)}{g(t)} = \frac{1}{g(t)} \left(d' f(t) - \frac{f(t)}{g(t)} dg(t) \right) = \sum_1^n \left[\frac{f_{x_i}(t) g(t) - f(t) g_{x_i}(t)}{g(t)^2} \right] dt_i.$$

□

2 The Real and Complex Numbers

2.1 A Topological Aside: Construction of $\mathbb{R}$ as a Completion of $\mathbb{Q}$

Definition 2.1.1. Let X be a non-empty set, K an ordered field. A map $d : X \times X \to K_{\geq 0} := \{a \in K : a \geq 0\}$ is called a *metric over* K if for every $x, y, z \in X$

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) + d(y, z) \geq d(x, z)$.

A set X with such a map d is called a *metric space over* K

Remark 2.1.2. Consider a normed vector space V over an ordered field K. Then

$$V \times V \to K_{\geq 0}, (v, w) \mapsto \|v - w\|$$

defines a metric on V.

1. $\|v - w\| = 0 \iff v - w = 0 \iff v = w$.
2. $\|v - w\| = |-1|\|v - w\| = \|-(v - w)\| = \|w - v\|$.
3. $\|v - w\| = \|(v - u) + (u - w)\| \leq \|v - u\| + \|w - u\|$.

Definition 2.1.3. Let (X, K, d) be a metric space and $\xi \in K_{>0} := \{a \in K : a > 0\}$, $x \in X$. We define *the ball of radius ξ with center x* to be

$$B_\xi(x) := \{y \in X : d(x, y) < \xi\}.$$

Lemma 2.1.4. *Let (X, K, d) be a metric space. Then*

$$\tau := \{U \subset X : \forall x(x \in U \implies \exists \xi > 0, B_\xi)\},$$

defines a topology on X

Proof. Trivially $\emptyset \in \tau$, since $x \in \emptyset$ is false. It is also obvious that $X \in \tau$, since a ball of any radius is a subset of X. Consider a family of sets in τ, $\{U_\alpha\}_{\alpha \in A}$. Then any point x in the union of these subsets is in at least one of these subsets, U_β say. Then there is an $\xi > 0$ such that $B_\xi(x) \subset U_\beta \subset \bigcup_{\alpha \in A} U_\alpha$. Consider $U_1, \ldots, U_n \in \tau$. Then for each point x in the intersection of these sets there are $\xi_1, \ldots, \xi_n > 0$, such $B_{\xi_i}(x) \subset U_i$. picking $\xi = \min_{i \in \{1,\ldots,n\}} \xi_i$, it follows that $B_\xi(x) \subset \bigcap_1^n U_i$. □

Definition 2.1.5. Let (X, K, d) be a metric space. A sequence $(q_n) \in \prod_{\mathbb{N}} X$ is said *to converge to q* if for all $\xi > 0$ there exists a $N \geq 1$ such that for every $n \geq N$, $q_n \in B_\xi(q)$. We call *q the limit of q_n as n approaches infinity* and denote it $\lim_{n\to\infty} q_n$. We shall see that such a limit is unique.

A sequence is called *Cauchy* if for every $\xi > 0$ there exists an $N \geq 0$ such that for every $n, m \geq N$, $q_n - q_m \in B_\xi(0)$. We say that two Cauchy sequences $(a_n), (b_n) \in \prod_{\mathbb{N}} X$ are equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. We write $(a_n) \equiv_d (b_n)$. We define *the completion of X with respect to d* to be the set

$$\overline{X} := \left\{ (q_n) \in \prod_{\mathbb{N}} X : (q_n) \text{ is Cauchy} \right\} / \equiv_d.$$

Remark 2.1.6. Let (x_n) be a convergent sequence with limit x. Let $y \in X \setminus \{x\}$. Set $\epsilon := d(x, y)/2$. Then for every $n \geq N$ for some $N \geq 1$, $x_n \notin B_\epsilon(y)$, hence the limit of a convergent sequence is unique. If V is a normed vector space, $\lim_{n\to\infty}$ defines a linear operator from the space of convergent sequences to K which readily follows from the homogeneity of the norm and from the triangle inequality. One also easily checks that $\lim_{n\to\infty} \circ \| \cdot \| = \| \lim_{n\to\infty} \|$. On K consider convergent sequences $(a_n), (b_n) \in \prod_{\mathbb{N}} K$. Then for any $\xi > 0$ for sufficiently large n

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| \overset{n\to\infty}{\to} 0.$$

On K consider a sequence $(a_n) \in \prod_{\mathbb{N}} K \setminus 0$ that is convergent with limit $a \in K \setminus 0$. Let $\xi > 0$. For large enough N for $n \geq N$, $|a_n||a_n - a| < \xi$. Pick $0 < \delta < |a|$.

For some M, for $n \geq M$, $|a_n| \in B_{|a|-\delta}(|a|)$, hence $|a| - |a_n| < |a| - \delta$. Then setting $\epsilon := \min\{|a_1|, \ldots, |a_{\max(N,M)}|, \delta\}$. Then for $n \geq \max(N, M)$,

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| < \frac{\xi}{|a|\epsilon}.$$

Lemma 2.1.7. *Let (X, K, d) be a metric space. Let (x_n) be a Cauchy sequence in X. Then (x_n) is bounded.*

Proof. Let $\xi > 0$ be given, then for a sufficiently large N, $d(x_n, x_N) < \xi$ for every $n \geq N$. Pick $\delta = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), \xi)$. Then $x_n \in B_\delta(x_N)$ for every $n \geq 1$. □

Definition 2.1.8. A function between metric spaces over a fixed field $f : X \to Y$ is said to be *sequentially continuous* if for every convergent sequence $(x_n) \in \prod_{\mathbb{N}} X$, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

Remark 2.1.9. Being sequentially continuous is equivalent to being continuous with respect to the topology induced by the metric on X, respectively on Y.

In general, it may be difficult to endow the completion of a metric space with a topology. If we are working over complete (i.e. every Cauchy sequence is convergent) ordered field, $([a_n], [b_n]) \mapsto \lim_{n\to\infty} d(a_n, b_n)$ defines a metric on the completion of X. Any two complete ordered fields are isomorphic as topological fields. Moreover, the completion of an ordered field can be endowed with structure of a complete ordered field.

Lemma 2.1.10. *Let K be an ordered field. We define addition on $\overline{K}$ to be*

$$[(a_n)] + [(b_n)] := [(a_n + b_n)] \quad ([(a_n)], [(b_n)] \in \overline{K}).$$

We also define multiplication by

$$[(a_n)][(b_n)] := [(a_n b_n)].$$

For $[(a_n)], [(b_n)] \in \overline{K}$ we say that $[(a_n)] \geq [(b_n)]$ if there is an $N \geq 0$ such that for $n \geq N$, $a_n \geq b_n$. With these definitions $\overline{K}$ becomes an ordered field. The subfield

$$K' := \{[(a)] \in \overline{K} : a \in K\}$$

is isomorphic to K as topological fields.

Proof. Suppose $([(a_n)], [(b_n)]) = ([(a'_n)], [(b'_n)])$. Then addition is well-defined by a single application of the triangle inequality. Multiplication is also well-defined since

$$\begin{aligned} |a_n b_n - a'_n b'_n| &= |a_n b_n - a_n b'_n + a_n b'_n - a'_n b'_n| = |a_n(b_n - b'_n) + b'_n(a_n - a'_n)| \\ &\leq |a_n||b_n - b'_n| + |b'_n||a_n - a'_n| \leq \epsilon|b_n - b'_n| + \delta|a_n - a'_n| \overset{n\to\infty}{\to} 0 \end{aligned}$$

where $\epsilon, \delta > 0$ are obtained from Lemma 2.1.7. It is fairly easy to check that upon defining $0_{\overline{K}} := 0 := [(0)]$, $1_{\overline{K}} := 1 := [(1)]$, $-[(a_n)] := [(-a_n)]$, for $[(a_n)] \in \overline{K}$, $\overline{K}$ becomes a commutative ring. Let $[(a_n)] \neq 0$. Then for n greater than some N, $a_n \neq 0$. Indeed, suppose (c_n) is Cauchy such that $c_n = 0$ for infinitely many n. Then for any $\xi > 0$ there is an $N \geq 0$ such that $|c_n - c_m| < \xi$ for $n, m \geq N$. In particular we may choose m such that $c_m = 0$, hence $|c_n - 0| < \xi$, implying $\lim_{n\to\infty} c_n = 0$, hence $[(c_n)] = 0$. We then define $b_n = 0$ for $n < N$ and $b_n = a_n^{-1}$ for $n \geq N$ and see that $[(a_n)][(b_n)] = [(1)]$, implying $\overline{K}$ is a field. It is fairly easy to check that $\leq$ defines a partial order on $\overline{K}$. We now check that $\leq$, defines a total order. Let $[(a_n)], [(b_n)] \in \overline{K}$. We prove first that the statement is true for $[(b_n)] = 0 = [(0)]$. The statement is obvious when $[(a_n)] = 0$. Note that $a_n > 0$ or $a_n < 0$ for every sufficiently large n. Suppose now (c_n) is Cauchy and $c_n > 0$ for infinitely many n and $c_n < 0$ for infinitely many n. Let (c_n') and (c_n'') the sequences entries of (c_n) being > 0 resp. < 0. Then for sufficiently large n,

$$|c_n'| + |c_n''| = c_n' - c_n'' = |c_n' - c_n''| < \xi$$

for every $\xi > 0$, using the fact that (c_n) is Cauchy, meaning $\lim_{n\to\infty} c_n = 0$. It thus follows that $a_n > 0$ for every sufficiently large n or $a_n < 0$ for every sufficiently large n, hence $[(a_n)] > 0$ or $[(a_n)] < 0$. In the general setting, we thus have that $[(a_n - b_n)] \geq 0$ or $[(a_n - b_n)] \leq 0$. This is equivalent to $a_n - b_n \geq 0$ for every large n or $a_n - b_n \leq 0$ for every large n. It follows that $[(a_n)] \geq [(b_n)]$ or $[(a_n)] \leq [(b_n)]$. It is easy to check that $\overline{K}$ becomes an ordered field with this total ordering. For instance if $[(a_n)] \leq [(b_n)]$ then $a_n + c_n \leq b_n + c_n$ for sufficiently large n, hence $[(a_n)] + [(c_n)] \leq [(b_n)] + [(c_n)]$. The map

$$K \to K'$$
$$a \mapsto [(a)]$$

is readily seen to be a ring isomorphism with mutual inverse $[(a)] \mapsto a$. Both of these maps are sequentially continuous, hence K and K' are isomorphic as topological fields. □

Remark 2.1.11. From this point on we identify K' with K.

Lemma 2.1.12. *Let $(K, |\cdot|)$ be an ordered field considered with structure of metric space in the natural way. Then K is a topological field.*

Proof. Using sequential continuity, the result follows from Remark 2.1.6. □

Lemma 2.1.13. *An ordered field is characteristic* 0.

Proof. Indeed, $0 < 1$. Suppose $0 < n$ for some $n \geq 1$. Then $1 < n + 1$, hence $0 < 1 < n + 1$. □

Remark 2.1.14. The utility of this lemma is that we may regard $\mathbb{Z}$ and $\mathbb{Q}$ as subrings in a canonical way.

Proposition 2.1.15. *For an ordered field K, we have that $\overline{K}$ is Dedekind complete, i.e. every non-empty bounded set has a least upper bound.*

Proof. Let $\emptyset \neq B \subset \overline{K}$ be bounded from above. Let $[(b_n)]$ be an upper bound. (b_n) is bounded, hence $[(b_n)] < [(u)]$ for some $u \in K$. Let $[(c_n)] \in B$ be arbitrary. Again, since $([c_n])$ is bounded, $([l]) < ([c_n])$ for some $l \in K$. Set $u_1 := u$ and $l_1 := l$. For each $n \geq 1$, if $(u_n + l_n)/2$ is an upper bound, set $u_{n+1} := (u_n + l_n)/2$ and $l_{n+1} := l_n$. Note that $|u_n - l_n| = \frac{1}{2^n}(u-l)$, which is easily verified by induction in n, hence $\lim_{n\to\infty} u_n - l_n = 0$. Then

$$|u_n - u_m| = |u_n - l_n| + |l_n - u_m| < \epsilon$$

for every $\epsilon > 0$ and sufficiently large n, m, hence (u_n) is Cauchy. A similar argument shows that (l_n) is Cauchy. We then have that $[(l_n)] = [(u_n)]$. By construction each u_n is an upper bound and each l_n is not an upper bound. We thus have that $[(u_n)]$ is an upper bound. Let $u' := [(a'_n)] \leq [(u_n)]$ be another upper bound of B. Then for every $\epsilon > 0$, for sufficiently large n

$$|a'_n - l_n| = a'_n - l_n \leq u_n - l_n = |u_n - l_n| < \epsilon,$$

hence $[(a'_n)] = [(l_n)] = [(u_n)]$, proving that $[(u_n)]$ is the least upper bound. □

Lemma 2.1.16. *Let K be a Dedekind complete ordered field. Then a monotone sequence bounded sequence is convergent.*

Proof. Let (a_n) be an increasing sequence in K. Since the image of (a_n) is bounded, it has a least upper bound a. We claim that (a_n) converges to a. Let $\epsilon > 0$. For some $N \geq 0$, $a - \epsilon < a_N \leq a < a + \epsilon$, hence for every $n \geq N$, $-\epsilon < a_n - a < \epsilon$, hence $a_n \in B_\epsilon(a)$. □

Definition 2.1.17. We define *the field of real numbers* as $\mathbb{R} := \overline{Q}$.

Definition 2.1.18. An ordered field K is *Archimedean* if $\mathbb{Q} \subset K$ is not bounded from above

Lemma 2.1.19. *Let K be an Archimedean ordered field. Then $\mathbb{Q}$ is dense in K.*

Proof. Let $0 < c \leq d$. There is a least upper bound u to the set $\{n \in \mathbb{N} : nc \leq d\}$. Since K is Archimedean we may pick a natural number $N > u$, hence $Nc > d$. The cases $a \leq 0 < b$ and $a < 0 \leq b$ are obvious. Suppose $0 < a < b$. Claim: if $c < d$ are such

that $d - c > 1$, then $\{m \in \mathbb{Z} : c < m < d\} \neq \emptyset$. Let m_0 be the greatest integer smaller than c. Then $c < m_0 + 1$ and $d - m_0 > d - c > 1$ hence $c < m_0 + 1 < d$. This proves that if $0 < a < b$ such that $a - b > 1$ then there is a rational number in between a and b. Suppose $a - b < 1$. Then there is a positive integer n such that $n(a - b) > 1$ and by the claim there is then an integer m such that $na < m < nb$ hence $a < m/n < b$. □

Definition 2.1.20. A metric space (X, K, d) is called *Cauchy complete* if every Cauchy sequence is convergent

Lemma 2.1.21. *An ordered field K is Dedekind complete if and only if it is Archimedean and Cauchy complete.*

Proof. " $\implies$ ": Suppose for a contradiction that $\mathbb{Q}$ is bounded from above. Set $s := \sup \mathbb{Q}$. We must have that s is not rational, since if it were $s+1$ would be a rational greater than s. Then $s-1 \leq q < s$ for some $q \in \mathbb{Q}$, but then $q+1$ is a rational number greater than s, leading to a contradiction. Let (a_n) be a Cauchy sequence in K. Set $(u_n) := (\sup_{k \geq n} a_n)$. This is a decreasing bounded sequence, hence u_n converges to some a. Let $\epsilon > 0$. Then there is some $N \geq 1$ such that $|a_n - a_m| < \epsilon/3$ and $|u_n - l| < \epsilon/3$ for every $n, m \geq N$. We may also find an $M \geq N$ such $u_N - \epsilon/3 < a_M$. It thus follows that

$$|a_n - a| = |a_n - a_M + a_M - u_N + u_N - a| \leq |a_n - a_M| + |a_M - u_N| + |u_N - a| < \epsilon,$$

meaning (a_n) converges to a.
" $\impliedby$ ": Let S be a non-empty, bounded subset of K. Let u be a rational upper bound of S (we may pick this as K is Archimedean) and l be a rational number smaller than an $s \in S$ (we may pick such a rational number since $\mathbb{Q}$ is dense in K which follows from K being Archimedean). Define (u_n) and (l_n) as in the prior proposition. Then (u_n) and (l_n) are Cauchy sequences, which are convergent by the assumption that K is Cauchy complete. It follows that since $|l_n - u_n| \to 0$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = u$. Since every u_n is an upper bound of S so is u. Let $v < u$. Then for large n, $u - l_n < u - v$, hence $l_n > v$. Since l_n is never an upper bound of S, we may pick a $t \in S$ such that $t \geq l_n > v$, hence u is the smallest upper bound of S. This shows that K is Dedekind complete. □

Definition 2.1.22. We say that a function $f : X \to Y$ of metric space is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$, $d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$

Lemma 2.1.23. *If $f : X \to Y$ is uniformly continuous and (x_n) is Cauchy, then $(f(x_n))$ is Cauchy.*

Proof. Let $\epsilon > 0$. We can pick $\delta > 0$ such that for $x, y \in X$, if $d(x, y) < \delta$, $d(f(x), f(y)) < \epsilon$. Pick $N \geq 1$ such that $d(x_n, x_m) < \delta$ for every $n, m \geq N$. Then $d(f(x_n), f(x_m)) < \epsilon$ for every $n, m \geq N$. □

Remark 2.1.24. Every uniformly continuous function is clearly continuous.

Lemma 2.1.25. *Let X, Y be metric spaces, where Y is complete and $f : A \to B$ be uniformly continuous where $A \subset X$ and $B \subset Y$. Then f can be uniquely extended to a uniformly continuous function $f : \text{cl}(A) \to \text{cl}(B)$.*

Proof. Let (x_n) be a convergent sequence in A. Then $(f(x_n))$ in Y is Cauchy and hence also convergent by completeness. We then define

$$f : \text{cl}(A) \to \text{cl}(B)$$
$$x \to \lim_{n\to\infty} f(x_n),$$

where (x_n) is a sequence in A converging to x. We need to check that this is well-defined. Note first that $f(A) \subset B \implies \text{cl}(f(A)) \subset \text{cl}(B)$, hence $\lim_{n\to\infty} f(x_n) \in \text{cl}(B)$. Let $(x_n), (y_n)$ be two sequences in A converging to $x \in \text{cl}(A)$. By uniform continuity (skipping a step) for large n, we get that $d(f(x_n), f(y_n)) < \epsilon$ for every $\epsilon > 0$, hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. Let $\epsilon > 0$ be given. Pick $\delta > 0$ small enough such that for every $p, q \in A$ with $d(p, q) < 2\delta \implies d(f(p), f(q)) < \epsilon/3$. Pick $x, y \in \text{cl}(A)$ such that $d(x, y) < \delta$. Pick sequences $(x_n), (y_n)$ in A converging to x respectively y. Then for large enough n, $d(x_n, y_n) \leq d(x_n, x) + d(y_n, y) < 2\delta$, hence, again for large n,

$$d(f(x), f(y)) \leq d(f(x_n), f(x)) + d(f(x_n), f(y_n)) + d(f(y_n), f(y)) < \epsilon.$$

It follows that the constructed extension of f is uniformly continuous. Let g be another such extension. Let $x \in \text{cl}(A)$ and (x_n) a sequence in A converging to x. Then since g is continuous,

$$g(x) = g(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = f(x).$$

□

Proposition 2.1.26. *Any two Dedekind complete ordered fields are isomorphic as topological fields.*

Proof. Let K, L be two such fields. There is a canonical isomorphism between the rational numbers in K and the rational numbers in L. each copy of $\mathbb{Q}$ is a dense subset of the respective fields. It is easy to show that any point in a metric space over a complete metric space can be approximated by a Cauchy sequence in a dense subset. The isomorphism $\sigma : \mathbb{Q} \subset K \to \mathbb{Q} \subset L, q \mapsto q$ is clearly uniformly continuous and can therefor by the prior lemma be extended to an isomorphism $\sigma : \text{cl}(\mathbb{Q}) = K \to \text{cl}(\mathbb{Q}) = L$. □

2.1.1 The Geometric Series, n-basis representation of Real Numbers and the Uncountability of $\mathbb{R}$

Definition 2.1.27. Let $r \in [0,1)$. We define the *geometric series with common ratio* r to be the series

$$\sum_0^\infty r^i$$

Lemma 2.1.28. *The geometric series common ratio* $r \in [0,1)$ *converges to*

$$\frac{1}{r-1}$$

Proof. It is easily seen by induction that

$$\sum_0^n r^i = \frac{1-r^{n+1}}{1-r} \to_{n\to\infty} \frac{1}{1-r}$$

□

Lemma 2.1.29. *Fix an integer* $n \geq 2$. *Given a sequence* (a_i) *in* $\{0,\ldots,n-1\}$ *the series*

$$\sum_1^\infty a_i \frac{1}{n^i}$$

converges to some real number in $[0,1]$

Proof. Indeed, the i'th term in this series is dominated by $\frac{n-1}{n^i}$. Since $n-1\sum_1^\infty \frac{1}{n^i}$ which is just the tail of a geometric series with common ratio $\frac{1}{n}$ scaled by a constant. □

Lemma 2.1.30. *Every* $x \in [0,1)$ *can be written as* $\sum_1^\infty a_i \frac{1}{n^i}$ *for some sequence* (a_i) *in* $\{0,\ldots,n-1\}$.

Proof. We define this sequence recursively. Note that $0 \leq nx < n$. Pick a_1 to be the smallest integer such that $a_1 \leq nx$ set $x_1 := nx - a_1$. Given a_i and x_i, let a_{i+1} be the small integer with $a_{i+1} \leq nx_i$ and set $x_{i+1} := nx_i - a_{i+1}$. One sees that $x_i \in [0,1)$ for each i and thus that $0 \leq a_{i+1} < n-1$. We check that $x = \sum_1^\infty a_i \frac{1}{n^i}$. Indeed, one sees readily by induction that

$$n^i x - \sum_{j=1}^{i} n^{i-j+1} a_j < 1$$

which is equivalent to

$$x - \sum_1^i a_j \frac{1}{n^j}$$

hence

$$\sum_1^\infty a_i \frac{1}{n^i} = x.$$

□

Remark 2.1.31. Such representation of a real number is called the *base n expansion*. We can extend this definition to every real number x by computing the base n expansion $n^{-k}x$ for the smallest k such that $|n^{-k}x| < 1$. Multiplying this series by n^k thus yields a sequence $(a_i)_{i=-N}^{\infty}$ in $\{0, \dots, n-1\}$ with

$$\sum_{-N}^{\infty} a_i \frac{1}{n^i}$$

for some positive integer N.

One should note that the base n expansion of a real number is not always unique. For instance, the sequence

$$1, \overline{0}$$

and

$$0, \overline{n-1}$$

both yield a base n expansion of 1. It is true however that this expansion is unique for *irrational numbers*, i.e. for $\mathbb{R} \setminus \mathbb{Q}$.

Lemma 2.1.32. *The n-ary expansion is unique for irrational numbers.*

Proof. It suffices to prove the statement for irrationals in $(0,1)$. Let $(a_i), (b_i)$ be distinct sequences in $\{0, \dots, n\}$. Let $k \geq 0$ be the smallest index with $a_k \neq b_k$. We may assume $a_k > b_k$. Then

$$n^k \left(\sum_1^{\infty} a_i n^{-i} - \sum_1^{\infty} b_i n^{-i} \right) = a_k - b_k + \sum_1^{\infty} a_i n^{-1} \underset{t}{} - \sum_1^{\infty} b_i n^{-1} > 1 + t - 1 = t > 0.$$

□

We are now ready to prove that $\mathbb{R}$ is an example of an uncountable set.

Theorem 2.1.33. $\mathbb{R}$ *is uncountable*

Proof. Let $X = \{a_1, a_2, \dots\} \subset (0,1) \setminus \mathbb{Q}$ be a countable set in $(0,1) \setminus \mathbb{Q}$ with $a_i = \sum_1^{\infty} a_{ij} 2^{-j}$. For each i set

$$b_i = \begin{cases} 1 & \text{if } a_{ii} = 0 \\ 0 & \text{otherwise} \end{cases}$$

Set $b := \sum_1^{\infty} b_i 2^{-i}$. If b is rational, $b \notin X$. Suppose b is irrational. By the prior lemma it follows that $b \neq a_i$ for any i, hence $b \notin X$. Then any countable subset of $(0,1) \setminus \mathbb{Q}$ is proper, hence $(0,1) \setminus \mathbb{Q}$ is not countable. $\mathbb{R} \simeq (0,1) \simeq (0,1) \setminus \mathbb{Q}$ is uncountable. □

2.2 From $\mathbb{R}^2$ to $\mathbb{C}$

Note that $(\mathbb{R}^2, +, |\cdot|)$ is a normed topological vector space. We seek to endow this vector space with a multiplication such that it becomes a topological field, containing $\mathbb{R}$ as a topological subfield. We will call this field *the complex numbers* and denote it $\mathbb{C}$.

Definition 2.2.1. We define

$$\cdot : \mathbb{R}^2 \to \mathbb{R}^2$$
$$((a,b),(c,d)) \mapsto (ac - bd, ad + cb)$$

Proposition 2.2.2. *With this operation $\mathbb{C} := (\mathbb{R}^2, +, \cdot)$ is a field. The multiplication is obviously continuous w.r.t. $|\cdot|$ making $\mathbb{C}$ a topological field. The set*

$$R := \{(a,0) \in \mathbb{C} : a \in \mathbb{R}\}$$

is a subfield of $\mathbb{C}$ isomorphic to $\mathbb{R}$.

Proof. Let $(a,b),(c,d),(e,f) \in \mathbb{C}$. We then have that

$$\begin{aligned}((a,b)(c,d))(e,f) &= (ac-bd, ad+cb)(e,f) = (ace - bde - adf - cbf, acf - bdf + ade + cbe)\\ &= (a(ce-df) - b(de+cf), a(cf+de) + b(ce-df)) = (a,b)(ce-df, cf+de)\\ &= (a,b)((c,d)(e,f)).\end{aligned}$$

We define $1_{\mathbb{C}} = (1,0)$. One easily checks that $(a,b)(c,d) = (c,d)(a,b)$. Then

$$1_{\mathbb{C}}(a,b) = (1,0)(a,b) = (1a - 0b, 1b + 0a) = (a,b),$$

shows that $1_{\mathbb{C}}$ is the neutral element with respect to the multiplication. We moreover have that

$$\begin{aligned}(a,b)(c+e, d+f) &= (ac + ae - bd - bf, ad + bf + bc + be)\\ &= (ac-bd, ad+bc) + (ae-bf, bf+be) = (a,b)(c,d) + (a,b)(e,f).\end{aligned}$$

Suppose $(a,b) \neq 0$. Then

$$(a,b)\left(\frac{1}{a^2+b^2}(a,-b)\right) = \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ab}{a^2+b^2}\right) = (1,0).$$

We have thus shown that $\mathbb{C}$ is a field. Note that R is subspace of $\mathbb{C}$ and that for $(a,0),(b,0) \in R$,

$$(a,0)(b,0) = (ab - 0, a0 + 0b) = (ab,0) \in R.$$

Moreover, we clearly have that $(1,0) \in R$ and that for $a \neq 0$ $(a,0)(1/a,0) = (1,0)$. We thus see that R is a subfield of $\mathbb{C}$. The map

$$\mathbb{R} \to R, a \mapsto (a,0),$$

clearly defines a bijective $\mathbb{R}$-algebra homomorphism, hence $R \simeq \mathbb{R}$. □

Remark 2.2.3. One further notices that $\mathbb{C}$ is an $\mathbb{R}$ vector space with basis $\{1, i\}$, where we define $i = (0,1)$. Note that $\pm i$ are the roots of the polynomial $x^2 + 1 \in \mathbb{C}[x]$. We therefor also write $i = \sqrt{-1}$.

3 Classical Affine Algebraic Geometry

3.1 Introducing Algebraic Sets

3.1.1 Introducing Affine Algebraic Sets & the Affine Zariski Topology

Definition 3.1.1. Let K be any field. By *the affine n-space over K*, for some positive integer n, we mean the n-fold cartesian product of K with itself. We denote affine n-space over K by $\mathbb{A}^n(K)$ or just $\mathbb{A}^n$ when this will not lead to ambiguity. Thus we have

$$\mathbb{A}^n(K) := \mathbb{A}^n := K^n.$$

We also refer to Affine 1-space as the affine line, and Affine 2-space as the affine plane.

Remark 3.1.2. Whenever we write $\mathbb{A}^n$ and no field is given prior, implicitly a field K is given and we mean $\mathbb{A}^n = \mathbb{A}^n(K)$.

Definition 3.1.3. Consider a field K. Recall the definition of the vanishing set of a subset $S \subset K[x_1, \ldots, x_n]$:

$$V^{\mathbb{A}}(S) := V(S) := \{v \in \mathbb{A}^n(K) : f(v) = 0 \text{ for every } f \in S\}.$$

When M is finite, say $M = \{f_1, \ldots, f_m\}$, we define

$$V(f_1, \ldots, f_m) := V(\{f_1, \ldots, f_m\}).$$

Definition 3.1.4. Let K be any field. A subset $X \subset \mathbb{A}^n$ is called an *affine algebraic subset*, or an *algebraic subset* when no ambiguity arises, if there is a subset $M \subset K[x_1, \ldots, x_n]$ such that $X = V(M)$

Lemma 3.1.5. *Let $M, M' \subset K[x_1, \ldots, x_n]$ with $M \supset M'$. Then $V(M) \subset V(M')$.*

Proof. Let $v \in V(M)$ and $f \in M'$. Then by assumption $f \in M$ and hence $f(v) = 0$, implying $v \in V(M')$. □

Proposition 3.1.6. *Consider a subset $M \subset K[x_1, \ldots, x_n]$ and set $I := \langle M \rangle$. Then $V(M) = V(I)$*

Proof. Let $v \in V(M)$. Let $f \in I$. Then for some $f_1, \ldots, f_m \in S$

$$f = \sum_1^m c_i f_i,$$

for suitable $c_1, \ldots, c_m \in K[x_1, \ldots, x_n]$. Then $f_i(v) = 0$ for every $i \in \{1, \ldots, m\}$ and hence

$$f(v) = \sum_1^m c_i(v) f_i(v) = \sum_1^m c_i(v) \cdot 0 = 0,$$

meaning $v \in V(I)$ and hence $V(M) \subset V(I)$.
The converse inclusion follows from Lemma 3.1.5 since $I \supset M$. □

It follows as a corollary that every affine algebraic set arises as a vanishing of some polynomial ideal.

Corollary 3.1.7. *Let $X \subset \mathbb{A}^n$ be algebraic. Then $X = V(I)$ for some ideal $I \subset K[x_1, \ldots, x_n]$.*

As another corollary to the above proposition and Hilbert's basis theorem, we have that every algebraic set arises as the vanishing set of finitely many polynomials.

Corollary 3.1.8. *Let $X \subset \mathbb{A}^n$ be an algebraic subset. Then $X = V(f_1, \ldots, f_m)$ for suitable $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$.*

Remark 3.1.9. Let an algebraic subset $X \subset \mathbb{A}^n$ be given. We can find $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ such that $X = V(f_1, \ldots, f_m)$. Consider the map

$$\varphi : \mathbb{A}^n \to \mathbb{A}^m$$
$$v \mapsto (\mathrm{ev}_v(f_1), \mathrm{ev}_v(f_2), \ldots, \mathrm{ev}_v(f_m)) = (f_1(v), f_2(v), \ldots, f_m(v)).$$

It is then clear that $V(f_1, \ldots, f_m) = \varphi^{-1}(\mathbf{0})$. Thus one concludes that every algebraic subset of $\mathbb{A}^n$ arises the zero set of a map of the same form as φ. We shall later see that φ is an element of a special class of maps, called *polynomial maps*, which are central to study of affine algebraic geometry.

Lemma 3.1.10. *We collect the following results*

(i) Let A be some indexing set. Consider a family of algebraic sets $\{X_\alpha\}_{\alpha\in A}$ in $\mathbb{A}^n$. Then $\bigcap_\alpha X_\alpha$ is an algebraic set.

(ii) Consider algebraic sets $X, Y \subset \mathbb{A}^n$. Then $X \cup Y$ is algebraic. It follows by induction that $\bigcup_1^k X_i$ is algebraic for any finite sequence of algebraic sets $X_1, \ldots, X_k$ in $\mathbb{A}^n$.

Proof. (i) By Corollary 3.1.7 for every $\alpha \in A$, we can find an ideal $I_\alpha \subset K[x_1, \ldots, x_n]$ such that $X_\alpha = V(I_\alpha)$. We claim that $\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right)$.
Let $v \in \bigcap_\alpha V(I_\alpha)$. Let $f \in \bigcup_\alpha I_\alpha$. Then $f \in I_\beta$ for some $\beta \in A$. Then since $v \in \bigcap_\alpha V(I_\alpha)$, in particular $v \in V(I_\beta)$, implying $f(v) = 0$, hence $v \in V\left(\bigcup_\alpha I_\alpha\right)$.
Let $\beta \in A$, then $I_\beta \subset \bigcup_\alpha I_\alpha$. By Lemma 3.1.5 $V\left(\bigcup_\alpha I_\alpha\right) \supset V(I_\beta)$, hence $V\left(\bigcup_\alpha I_\alpha\right) \supset \bigcap_\alpha V(I_\alpha)$.
It follows that

$$\bigcap_\alpha X_\alpha = \bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right).$$

(ii) We have that $X = V(I)$ and $Y = V(J)$ for some ideals $I, J \subset K[x_1, \ldots, x_n]$. We claim that $V(I) \cup V(J) = V(IJ)$.
$IJ \subset I$ and $IJ \subset J$, hence by Corollary 3.1.5 $V(I), V(J) \subset V(IJ)$, meaning $V(I) \cup V(J) \subset V(IJ)$.
Let $v \in V(IJ)$. If $v \in V(I)$ we find that $v \in V(I) \cup V(J)$. Suppose $v \notin V(I)$. Then for some $f \in I$ $f(v) \neq 0$. Let $g \in J$ be given. then $fg \in IJ$, hence $0 = (fg)(v) = f(v)g(v)$, since $f(v) \neq 0$ it follows that $g(v) = 0$, hence $v \in V(J) \subset V(I) \cup V(J)$.
It follows that

$$X \cup Y = V(I) \cup V(J) = V(IJ).$$

□

We collect the most fundamental examples of (affine) algebraic sets

Example 3.1.11. Fix a field K

1. Trivially the vanishing set of some subset $M \subset K[x_1, \ldots, x_n]$ is a an algebraic set.

2. Affine n-space is an algebraic set. Indeed, see that

$$\mathbb{A}^n = V(0)$$

3. The empty set $\emptyset \subset \mathbb{A}^n$ is an algebraic set. Indeed one readily verifies that

$$\emptyset = V(1).$$

4. A singleton or *point*, $\{(a_1, \ldots, a_n)\}$ for $a_1, \ldots, a_n \in K$, is an algebraic set. Indeed, one checks that

$$\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n).$$

It thus follows that from Lemma 3.1.10 (ii) that any finite subset of $\mathbb{A}^n$ is algebraic

Corollary 3.1.12. *The algebraic sets in $\mathbb{A}^n$ of the form $V(f)$, where $f \in K[\mathbf{x}]$, form a closed basis on $\mathbb{A}^n$.*

Remark 3.1.13. The topology generated by this closed basis is refered to as the *Zariski topology* on $\mathbb{A}^n$. we may opt to say that an algebraic subset $X \subset \mathbb{A}^n$ is a *Zariski closed subset*.

3.1.2 Miscellaneous Result about Algebraic sets, Examples & Non-examples

Proposition 3.1.14. *The Zariski closed subsets of $\mathbb{A}^1 = K$ are the finite subsets of $\mathbb{A}^1$ and $\mathbb{A}^1$ itself.*

Proof. Let $X \subset \mathbb{A}^1$ be Zariski closed. Suppose $X \neq V(0) = \mathbb{A}^1$, i.e. that $X = V(f_1, \ldots, f_m)$ for $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, where $f_i \neq 0$ for at least one $i \in \{1, \ldots, m\}$. Then $X \subset V(f_i)$ by Lemma 3.1.5. $V(f_i)$ is the set of roots of f_i. Since $f_i \neq 0$, $\#V(f_i) \leq \deg\ f_i < \infty$, hence

$$\#X \leq \#V(f_i) \leq \deg\ f_i < \infty.$$

□

Corollary 3.1.15. *Let K be a field with $\#K < \infty$. Let $X \subset \mathbb{A}^n$. Then X is Zariski closed.*

Proof. By assumption $\#X < \infty$, hence there are points $v_1, \ldots, v_k \in \mathbb{A}^n$ such that

$$X = \bigcup_1^k \{v_i\}.$$

By Example 3.1.11 X is Zariski closed. □

We now give an example of a countable family of algebraic subsets whose union is not algebraic.

Example 3.1.16. Let $K = \mathbb{Q}$ and let and consider the family of algebraic sets $\{\{n\}\}_{n\in\mathbb{Z}}$. The only algebraic subset of $\mathbb{A}^1(\mathbb{Q})$ are $\mathbb{A}^1(\mathbb{Q})$ itself and the finite subsets of $\mathbb{A}^1(\mathbb{Q})$ (cf. Proposition 3.1.14), but $\bigcup_{n\in\mathbb{Z}}\{n\} = \mathbb{Z}$.

Here are some examples of algebraic subsets (all from exercises in Fulton)

Example 3.1.17. 1. Let $X = \{(t, t^2, t^3) \in \mathbb{A}^3 : t \in K\}$. We prove that $X = V(f_1, f_2)$, where $f_1 = x^2 - y$, $f_2 = x^3 - z \in K[x, y, z]$. Let $v \in X$. Then $v = (t, t^2, t^3)$ for some $t \in K$. Note that

$$f_1(v) = t^2 - t^2 = 0 \text{ and } f_2(v) = t^3 - t^3 = 0,$$

implying that $v \in V(f_1, f_2)$.
Let $v = (v_1, v_2, v_3) \in V(f_1, f_2)$. Put $t = v_1$. Then $t^2 = v_1^2 = v_2$ and $t^3 = v_1^3 = v_3$, hence $v = (v_1, v_2, v_3) = (t, t^2, t^3) \in X$.

2. Let $X = \{(\cos(t), \sin(t)) \in \mathbb{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$. One sees that $X = V(x^2 + y^2 - 1)$. Indeed, for $v = (\cos(t), \sin(t)) \in X$, $\sin^2(t) + \cos^2(t) = 1$, implying that $v \in V(x^2 + y^2 - 1)$. For $v = (v_1, v_2) \in V(x^2 + y^2 - 1)$, we have that $|v|^2 = v_1^2 + v_2^2 = 1$, implying that $v \in S^1 = \text{im}(\mathbb{R} \ni t \mapsto (\cos(t), \sin(t) \in \mathbb{A}^2(\mathbb{R}))$.

3. Let $X = \left\{\left(\sin(t)\cos(t), \sin^2(t)\right) \in \mathbb{A}^2(\mathbb{R}) : t \in [0, 2\pi)\right\}$, i.e. the points in $\mathbb{A}^2(\mathbb{R})$ with polar coordinates (r, t) such that $r = \sin(t)$. Recall the trigonometric identities

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\alpha),$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

for $\alpha, \beta \in [0, 2\pi)$. For $t \in [0, 2\pi)$ this is equivalent to

$$\sin(2t) = 2\sin(t)\cos(t) \iff \sin(t)\cos(t) = \frac{1}{2}\sin(2t) \text{ and}$$
$$\cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t) \iff \sin^2(t) = \frac{1}{2}\left(1 - \cos(2t)\right),$$

hence $X = \left\{\left(\frac{1}{2}\sin(2t), \frac{1}{2}\left(1 - \cos(2t)\right)\right) : t \in [0, 2\pi)\right\}$. Let $f = x^2 + \left(\frac{1}{2} - y\right)^2 - \frac{1}{4}$. One sees that $X = V(f)$. Indeed, if $v = (\left(\frac{1}{2}\sin(2t), \frac{1}{2}\left(1 - \cos(2t)\right)\right) \in X$ we have that

$$f(v) = \frac{1}{4}\cos^2(2t) + \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{2}\sin(2t)\right)^2 - \frac{1}{4} = \frac{1}{4}\left(\cos^2(2t) + \sin^2(2t)\right) - \frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0,$$

thus we conclude $v \in V(f)$. Let $v = (v_1, v_2) \in V(f)$. We set $t = \frac{1}{2}\arcsin(2v_1)$. Then one easily checks that $\cos(2t) = v_1$. We also note that

$$v_1^2 = -\left(\frac{1}{2} - v_2\right)^2 + \frac{1}{4}.$$

Therefor we get that

$$\begin{aligned}\frac{1}{2}(1-\cos(2t)) &= \frac{1}{2}\left(1-\cos\left(2\frac{1}{2}\arcsin(2v_1)\right)\right) = \frac{1}{2}\left(1-\sqrt{1-4v_1^2}\right)\\ &= \frac{1}{2}\left(1-\sqrt{1+4\left(\left(\frac{1}{2}-v_2\right)^2-\frac{1}{4}\right)}\right) = \frac{1}{2}\left(1-\sqrt{1-1+4\left(\frac{1}{2}-v_2\right)^2}\right)\\ &= \frac{1}{2}\left(1-2\left(\frac{1}{2}-v_2\right)\right) = \frac{1}{2}\cdot 2v_2 = v_2.\end{aligned}$$

Hence $v = (v_1, v_2) = \left(\frac{1}{2}\sin(2t), \frac{1}{2}(1-\cos(2t))\right) \in X$.

Definition 3.1.18. An algebraic subset of an $\mathbb{A}^2$ is called an *algebraic curve*.

Definition 3.1.19. The vanishing set of a non-zero polynomial $f \in K[x_1, \ldots, x_n]$ is called a *hypersurface*. If deg $f = 1$, then its vanishing set is called a *hyperplane*.

Definition 3.1.20. A hypersurface in $\mathbb{A}^2$ is called an *affine plane curve*. A hyperplane in $\mathbb{A}^2$ is called a *line*.

Remark 3.1.21. Consider a line $V(f) \subset \mathbb{A}^2$, where $f = ax + by + c$, for $a, b, c \in K$ where a or b are non-zero. Without loss of generality we may assume that $b \neq 0$. Thus

$$V(f) = V(f) \cup \emptyset = V(f) \cup V\left(b^{-1}\right) = V\left(b^{-1}(ax+by+c)\right) = V\left(y + ab^{-1}x + cb^{-1}\right).$$

This means that any line can be expressed as the vanishing set of some polynomial of the form

$$y - Ax + B,$$

for some $A, B \in K$.

Proposition 3.1.22. *Consider $l = y - (ax + b) \in K[x, y]$ and a line $L = V(l)$ and an affine plane curve $C = V(f)$, where $f \in K[x, y]$ with $n =$ deg f such that $L \not\subset C$. Then $C \cap L$ is a finite set containing at most n points.*

Proof. The case where $\#K < \infty$ is trivial. Suppose then that $\#K = \infty$. Set

$$X := \{(t, at + b) \in \mathbb{A}^2 : f(t, at + b) = 0\}.$$

Note that

$$t \in V(f(x, ax + b)) \iff (t, at + b) \in X$$

and thus that there is a $t \in \mathbb{A}^1 \setminus V(f(x, ax + b)$, since $L \not\subset C$. This then implies that $\#V(f(x, ax + b)) \leq n$ (cf. Example 3.1.14), and hence $\#X \leq n$. It is then sufficient to prove that $C \cap L = X$. Let $v = (v_1, v_2) \in C \cap L$. Since $v \in L$ we have that

$$v_2 = av_1 + b,$$

meaning $v = (v_1, v_2) = (v_1, av_1 + b) \in X$. Conversely, let $v = (t, at + b) \in X$. Then

$$l(v) = at + b - (at + b) = 0,$$

hence $v \in L$. Clearly we also have that $v \in C$, meaning $v \in C \cap L$. □

Corollary 3.1.23. *An algebraic curve $X \subset \mathbb{A}^2$ intersects a line L not contained in X in only finitely many points.*

Proof. $X = V(f_1, \ldots, f_m)$, for some $f_1, \ldots, f_m \in K[x, y]$, where without loss of generality $f_1 \neq 0$. Then $X \subset V(f_1)$ and $L \not\subset V(f_1)$, which by Proposition 3.1.22 implies that $\#X \leq \#V(f_1) \leq \deg\ f_1 < \infty$. □

We use this result to give some non-examples of algebraic subsets

Example 3.1.24. 1. Consider the graph of the sine function,

$$G := \{(t, \sin(t)) \in \mathbb{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$$

Consider also the line $L := V(y)$. One easily verifies that $L = \{(t, 0) \in \mathbb{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$. Thus $G \cap L = \{(n\pi, 0) \in \mathbb{A}^2(\mathbb{R}) : n \in \mathbb{Z}\} \neq L$, hence $\#(G \cap L) = \infty$, and G is not algebraic by Corollary 3.1.23.

2. Consider the sphere

$$S := \left\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\right\}.$$

Consider again the line $L := V(y)$. Note that $S \cap L = \{(z, 0) \in \mathbb{A}^2(\mathbb{C}) : |z| = 1\} \neq L$ and that this set is in bijection with S^1. But clearly $\#S^1 = \infty$, hence S cannot be algebraic by Corollary 3.1.23.

Definition 3.1.25. Given an algebraic set $X \subset \mathbb{A}^n$ and $(b_1, \ldots, b_n) \in \mathbb{A}^n$, a *translation* is a map

$$\varphi : X \to \mathbb{A}^n$$
$$(v_1, \ldots, v_n) \mapsto (v_1 + b_1, \ldots, v_n + b_n)$$

Lemma 3.1.26. *Algebraic sets are closed under translation. In other words, let $X := V(f_1, \ldots, f_m) \subset \mathbb{A}^n$ be an algebraic set. Let $b_1, \ldots, b_n \in K$. Then the image of the map*

$$\varphi : X \to \mathbb{A}^n$$
$$(v_1, \ldots, v_n) \mapsto (v_1 + b_1, \ldots, v_n + b_n)$$

is an algebraic set

Proof. One checks that im $\varphi = V(f_1(x_1 - b_1, \ldots, x_n - b_n), \ldots, f_m(x_1 - b_1, \ldots, x_n - b_n))$. Indeed, let $(v_1 + b_1, \ldots, v_n + b_n) \in$ im φ. Then

$$f_i(v_1 + b_1 - b_1, \ldots, v_n + b_n - b_n) = f_i(v_1, \ldots, v_n) = 0.$$

On the other hand, let

$$(v_1, \ldots, v_n) \in V(f_1(x_1 - b_1, \ldots, x_n - b_n), \ldots, f_m(x_1 - b_1, \ldots, x_n - b_n)),$$

Then $(v_1 - b_1, \ldots, v_n - b_n) \in V(f_1, \ldots, f_n)$ and

$$(v_1, \ldots, v_n) = \varphi(v_1 - b_1, \ldots, v_n - b_n) \implies (v_1, \ldots, v_n) \in \text{im } \varphi.$$

□

Lemma 3.1.27. *Let $X \subset \mathbb{A}^n$ be algebraic and $\omega \in \mathcal{S}_n$. Then the image of*

$$\varphi : X \to \mathbb{A}^n$$
$$(v_1, \ldots, v_n) \mapsto (v_{\omega(1)}, \ldots, v_{\omega(n)})$$

is algebraic.

Proof. For suitable $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, $X = V(f_1, \ldots, f_m)$. Clearly,

$$\text{im } \varphi = V\left(f_1\left(x_{\omega^{-1}(1)}, \ldots, x_{\omega^{-1}(n)}\right), \ldots, f_m\left(x_{\omega^{-1}(1)}, \ldots, x_{\omega^{-1}(n)}\right)\right).$$

Indeed, if $w = (v_{\omega(1)}, \ldots, v_{\omega(n)}) \in$ im φ. Then

$$f_i\left(w_{\omega^{-1}(1)}, \ldots, w_{\omega^{-1}(n)}\right) = f_i\left(v_{\omega^{-1}(\omega(1))}, \ldots, v_{\omega^{-1}(\omega(n))}\right) = f_i(v_1, \ldots, v_n) = 0.$$

Conversely, if $v = (v_1, \ldots, v_n)$ is an element of the right-hand side, then picking $w = \left((v_{\omega^{-1}(1)}, \ldots, v_{\omega^{-1}(n)}\right)$, then $f_i(w) = 0$ and $\varphi(w) = v$. □

Definition 3.1.28. We define a *line* in $\mathbb{A}^n$ given by $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$, where $a_j \neq 0$ for some j to be the set

$$\left\{ \begin{pmatrix} a_1 t + b_1 \\ \vdots \\ a_n t + b_n \end{pmatrix} \in \mathbb{A}^n : t \in K \right\}$$

Lemma 3.1.29. *A line in $\mathbb{A}^n$ is Zariski closed.*

Proof. We consider first the case where $a_1 \neq 0$. It is easy to check that

$$V(\{a_1x_i - a_ix_1 - a_1b_i + a_ib_1 : i \neq 1\}) = \left\{ \begin{pmatrix} a_1t + b_1 \\ a_2t + b_2 \\ \vdots \\ a_nt + b_n \end{pmatrix} \in \mathbb{A}^n : t \in K \right\}.$$

Any other line can be obtained as the image translation composed with a map induced by a permutation of a line of the above form, hence by Lemmas 3.1.26 and 3.1.27 any line is Zariski closed. □

The intersection of a proper configuration of $n-1$ hyperplanes is a line for $n \geq 2$.

Proposition 3.1.30. *Consider hyperplanes $V(p_1 + b_i), \ldots, V(p_{n-1} + c_{n-1})$, with $p_i = \sum_1^n a_{ij}x_j \in K[x_1, \ldots, x_n]$, $b_i \in K$ such that the vectors*

$$a_i := \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}, \quad (i \in \{1, \ldots, n-1\})$$

are linearly independent. The intersection of these hyperplanes, L say, is a line.

Proof. Our assumptions lets us show that WLOG

$$L = \left\{ v \in \mathbb{A}^n : \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)n} \\ 0 & \cdots & 0 \end{pmatrix} v = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ 0 \end{pmatrix} \right\} = \left\{ v \in \mathbb{A}^n : \begin{pmatrix} c_{ij} \end{pmatrix} v = \begin{pmatrix} b'_1 \\ \vdots \\ b'_{n-1} \\ 0 \end{pmatrix} \right\}$$

where $c_{ii} = 1$ for $i \in \{1, \ldots, n-1\}$, $c_{nn} = 0$ and $c_{ij} = 0$ for $i, j \in \{1, \ldots, n\}$ where $i > j$ and $n > j > i$. In other words

$$L = \bigcap_1^{n-1} V(x_i - c_ix_n - b'_i),$$

for suitable $c'_ib'_i \in K$. This means that

$$L = \{(c_1t + b'_1, \ldots, c_{n-1}t + b'_{n-1}, t) \in \mathbb{A}^n : t \in K\}.$$

□

We can extend the result in Proposition 3.1.22 in the following way

Proposition 3.1.31. *Consider also a hypersurface $H := V(f) \subset \mathbb{A}^n$ with $d := \deg f > 0$ and a line $L \subset \mathbb{A}^n$ with $n \geq 2$ such that $L \not\subset H$. Then $\#(L \cap H) \leq d$*

Proof. For suitable $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ with $a_j \neq 0$ for some j we have that

$$L = \left\{ \begin{pmatrix} a_1 t + b_1 \\ \vdots \\ a_n t + b_n \end{pmatrix} \in \mathbb{A}^n : t \in K \right\}$$

We may then easily prove that

$$L \cap H = \{(a_1 t + b_1, \ldots, a_n t + b_n) \in \mathbb{A}^n : f(a_1 t + b_1, \ldots, a_n t + b_n) = 0\}$$

and that $(a_1 t + b_1, \ldots, a_n t + b_n) \in L \cap H$ if and only if

$$t \in V(f(a_1 x + b_1, \ldots, a_n x + b_n)).$$

Since $L \cap H \neq L$, there is a $t \in \mathbb{A}^1$ such that $t \notin V(f(a_1 x + b_1, \ldots, a_n x + b_n))$, hence

$$\#(L \cap H) = \#V(f(a_1 x + b_1, \ldots, a_n x + b_n)) \leq d$$

□

Corollary 3.1.32. *Consider a line $L \in \mathbb{A}^n$. Then L intersects any algebraic set not containing L only finitely many times.*

Example 3.1.33. Consider the helix

$$H := \left\{ (\cos(t), \sin(t), t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R} \right\}.$$

Consider also the line $L := \{(0, 1, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ and note that $L \cap H = \{(0, 1, 2n\pi + \pi/2) : n \in \mathbb{Z}\} \neq \mathbb{A}^1$, but we also have that $\#L \cap H = \infty$, hence H cannot be algebraic by 3.1.32.

A later strengthening of above results (cf. Theorem 4.3.40), generalises this way of constructing non-examples in affine case by stating that the number of points counted with multiplicity in the (in a certain sense) extented intersection of two plane curves is equal to the product of their degrees.

Proposition 3.1.34. *Suppose K is an infinite field. Let $f \in K[x_1, \ldots, x_n]$ with* $\deg\ f > 0$

1. *Suppose $n \geq 1$. Then $\#\left(\mathbb{A}^n \setminus V(f)\right) = \infty$.*
2. *Suppose that K is algebraically closed and that $n \geq 2$. Then $\#V(f) = \infty$*

Proof. 1. In the one variable case every algebraic subset that is not affine n-space is finite, hence $\#(\mathbb{A}^1 \setminus V(f)) = \infty$. Suppose then that $n \geq 2$. Since f is non-constant, we can write

$$f = \sum_0^d f_i x_n^i,$$

for a $d \geq 1$ and for suitable $f_i \in K[x_1, \ldots, x_{n-1}]$ with $f_j \neq 0$ for some $j \in \{1, \ldots, d\}$. By Proposition 1.8.56 there is a point $v \in \mathbb{A}^{n-1}$ such that $f_j(v) \neq 0$. Hence

$$f' := \sum_0^d f_i(v) x_n^i \in K[x_n]$$

is a non-zero polynomial in one variable, implying there are infinitely many $v_n \in K$ such that $f'(v_n) = f(v_1, \ldots, v_n) \neq 0$, hence $\mathbb{A}^n \setminus V(f)$ is infinite.
2. We again write $f = \sum_0^d f_i x_n^i$, for suitable $f_i \in K[x_1, \ldots, x_{n-1}]$, with $f_j \neq 0$ for some $j \in \{1, \ldots, d\}$. If every non-zero f_i is constant, then for any choice of $v \in \mathbb{A}^n$, there exists $a_v \in K$ such that for $f_v = \sum_0^d f_i(v) x_n^i$,

$$0 = f_v(a_v) = f(v_1, \ldots, v_{n-1}, a_v).$$

If deg $f_j > 0$ for some j. Then there are infinitely many $v \in \mathbb{A}^{n-1}$ such $f_j(v) \neq 0$. For such a v there is at least one $a_v \in K$ that is a root in $f(v_1, \ldots, v_{n-1}, x_n)$. Thus in any case f has infinitely many zeroes. □

We present a way of constructing algebraic sets via Cartesian product

Proposition 3.1.35. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be algebraic then. $X \times Y \subset \mathbb{A}^{n+m}$ is algebraic.*

Proof. For suitable $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$ and $g_1, \ldots, g_l \in K[y_1, \ldots, y_m]$, we have $X = V(f_1, \ldots, f_m)$ and $Y = V(g_1, \ldots, g_l)$. We prove that $X \times Y = V(f_1, \ldots, f_k, g_1, \ldots, g_l)$, where we consider $f_1, \ldots, f_k, g_1, \ldots, g_l$ as elements of $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let $(v, w) \in X \times Y$. then clearly $f_i(v, w) = 0$ and $g_j(v, w) = 0$ for every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, l\}$. Let $(v, w) \in V(f_1, \ldots, f_k, g_1, \ldots, g_l)$. Considering $f_1, \ldots, f_k$ and $g_1, \ldots, g_l$ as elements of the subsrings $K[x_1, \ldots, x_n]$ and $K[y_1, \ldots, y_m]$ respectively. We easily see that $f_i(v) = 0$ for every i and $g_j(w) = 0$ for every j, hence $v \in X$ and $w \in Y$, meaning $(v, w) \in X \times Y$. □

3.1.3 A Correspondence between Algebraic sets and Polynomial Ideals

Definition 3.1.36. Let $X \subset \mathbb{A}^n$ be any subset. We define the *ideal of X* to be the set

$$I^{\mathbb{A}}(X) := I(X) := \{f \in K[x_1, \ldots, x_n] : f(v) = 0 \text{ for every } v \in \mathbb{A}^n\}$$

Lemma 3.1.37. *The ideal of any subset $X \subset \mathbb{A}^n$ is an ideal in $K[x_1, \ldots, x_n]$.*

Proof. Let $f, g \in I(X)$ and $h \in K[x_1, \ldots, x_n]$. Let $v \in X$. Note first that $0 \in K[x_1, \ldots, x_n]$ trivially vanishes on v, hence $0 \in I(X)$. Furthermore we have that

$$(hf + g))(v) = h(v)f(v) + g(v) = 0 \implies hf + g \in I(X)$$

□

Example 3.1.38. We consider some initial examples of ideals of subsets of $\mathbb{A}^n$.

1. $I(\emptyset) = K[x_1, \ldots, x_n]$. For $f \in K[x_1, \ldots, x_n]$, the statement, f vanishes on every $v \in \emptyset$ is vacuously true, hence $1 \in I(\emptyset)$.

2. Suppose $\#K = \infty$. Then $I(\mathbb{A}^n) = 0$. Since K is infinite, if $f \in I(\mathbb{A}^n)$, then $f(v) = 0$ for every $v \in \mathbb{A}^n$, hence $f = 0$ by Proposition 1.8.56

3. Consider a point $v \in \mathbb{A}^n$. Then $I(\{v\}) = \langle x_1 - v_1, \ldots, x_n - v_n \rangle$. This follows from Proposition 1.8.47.

Lemma 3.1.39. *Let $X, Y \subset \mathbb{A}^n$ with $X \subset Y$. Then $I(X) \supset I(Y)$.*

Proof. Let $f \in I(Y)$, and let $v \in X$. Then $v \in Y$, hence $f(v) = 0$, which implies $f \in I(X)$. □

Lemma 3.1.40. *Let $M \subset K[x_1, \ldots, x_n]$ and $X \subset \mathbb{A}^n$. Then we have the following*

1. *$I(V(M)) \supset M$.*

2. *$V(I(X)) \supset X$.*

3. *$V(I(V(M))) = V(M)$. Hence if X is algebraic $X = V(I(X))$.*

4. *$I(V(I(X))) = I(X)$. Hence if M is an ideal of some algebraic set, $M = I(V(M))$*

Proof. 1. Let $f \in M$. Let $v \in V(M)$. Then $f(v) = 0$, hence $f \in I(V(M))$.
2. Let $v \in X$. Let $f \in I(X)$. Then $f(v) = 0$, hence $v \in V(I(X))$.
3. By 1. $I(V(M)) \supset M$, hence $V(M) \supset V(I(V(M)))$ by Lemma 3.1.5. By 2. $V(I(V(M))) \supset V(M)$.
4. Since $V(I(X)) \supset X$ by 2, it follows that $I(V(I(X))) \subset I(X)$ by Lemma 3.1.39. By 1. $I(V(I(X))) \supset I(X)$. □

Lemma 3.1.41. *Let $X \subset \mathbb{A}^n$. Then $I(X)$ is radical.*

Proof. Let $F \in \mathrm{rad}(I(X))$, then $F^n \in I(X)$ for some $n > 0$. Let $v \in X$. Then

$$F(v)^n = (F^n)(v) = 0.$$

Since K is an integral domain, this implies that $F(v) = 0$, hence $F \in I(X)$. □

Lemma 3.1.42. *Let $X, Y \subset \mathbb{A}^n$ be algebraic subsets. Then*

$$X = Y \iff I(X) = I(Y).$$

Proof. " $\Longrightarrow$ ": Follows from Lemma 3.1.39.
" $\Longleftarrow$ ": By Lemma 3.1.40 3.

$$X = V(I(X)) = V(I(Y)) = Y.$$

□

Corollary 3.1.43. *Let $X \subsetneq \mathbb{A}^n$ be an algebraic subset.*

1. *Consider $p \in \mathbb{A}^n \setminus X$. Then there is some polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(q) = 0$ for every $q \in X$ and $f(p) = 1$.*
2. *Consider distinct points $p_1, \ldots, p_k \in \mathbb{A}^n \setminus X$. Then there are polynomials $f_1, \ldots, f_k \in I(X)$ such that $f_i(p_j) = 0$ whenever $i \neq j$ and $f_i(p_i) = 1$.*
3. *Let $p_1, \ldots, p_k \in \mathbb{A}^n \setminus X$ and $a_{ij} \in K$ for $i, j \in \{1, \ldots, k\}$. Then there are $G_i \in I(X)$ with $G_i(p_j) = a_{ij}$ for every i and j.*

Proof. 1.Note that $X \subsetneq X \cup \{p\}$. By Lemma 3.1.39 and Lemma 3.1.42, this implies $I(X) \supsetneq I(X \cup \{p\})$. Hence there is some $g \in I(X) \setminus I(X \cup \{p\})$. Clearly $g(q) = 0$ for every $q \in X$, while $g(q) \neq 0$, for otherwise $g \in I(X \cup \{p\})$. Upon taking

$$f = g(p)^{-1} g \in K[x_1, \ldots, x_n],$$

we are done.
2. Let $i \in \{1, \ldots, k\}$ and put

$$Y = X \cup \bigcup_{j \in \{1,\ldots,k\} : j \neq i} \{p_j\}.$$

Using 1. we can then find $f_i \in K[x_1, \ldots, x_n]$ such that $f_i(v) = 0$ for every $v \in Y$ and $f_i(p_i) = 1$. In particular, $f_i(p_j) = 0$ whenever $i \neq j$.
3. We construct $f_1, \ldots, f_k \in I(X)$ as in 2. Set

$$G_i = \sum_{h=1}^{k} a_{ij} f_j \in I(X).$$

Then

$$G_i(p_j) = \sum_{h=1}^{k} a_{ih} f_h(p_j) = a_{ij} f_j(p_j) = a_{ij}.$$

□

From Lemma 3.1.42 we get that

$$I(\bullet) : \tau_{\mathcal{Z}} \ni X \mapsto I(X) \in \{I \subset K[x_1, \ldots, x_n] : I \text{ is a radical ideal}\},$$

is an injective well defined function. A simple counter example shows that it is not surjective in general.

Example 3.1.44. Consider $I := \langle x^2 + 1 \rangle \subset \mathbb{R}[x]$. We prove that I is prime and therefor radical by Lemma 1.6.36. Put $f = x^2 + 1$ and let $f_1, f_2 \in \mathbb{R}[x]$ such that $f_1 f_2 \in I$, i.e. such that $f \mid f_1 f_2$. Note that f has no roots in $\mathbb{R}$ and therefor is irreducible in $\mathbb{R}[x]$. Since $\mathbb{R}[x]$ is a UFD it follows that $f \mid f_1$ or $f \mid f_2$, hence $f_1 \in I$ or $f_2 \in I$. From the fact that $x^2 + 1$ vanishes on no points in $\mathbb{R}$, it follows that $x^2 + 1 \notin I(X)$ for any non-empty $X \subset \mathbb{R}$, hence $I(X) \neq I$. Since deg $1 = 0$ it follows that $x \notin I$, hence $I \subsetneq I(\emptyset) = \langle 1 \rangle = \mathbb{R}[x]$. In conclusion, $I \neq I(X)$ for any set $X \subset \mathbb{R}$.

We shall later ammend this by only considering algebraically closed fields as our base field.

Lemma 3.1.45. *Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Then $V(I) = V(\text{rad}(I))$.*

Proof. Since $I \subset \text{rad}(I)$, $V(I) \supset V(\text{rad}(I))$ by Lemma 3.1.5. Let $v \in V(I)$ and let $f \in \text{rad}(I)$. Then for some $n > 0$, $f^n \in I$. Then $0 = f(v)^n$, and since K is an integral domain, this implies $f(v) = 0$, hence $v \in V(\text{rad}(I))$. In conclusion $V(I) = V(\text{rad}(I))$. □

Lemma 3.1.46. *Since $I \subset K[x_1, \ldots, x_n]$ be an ideal. Then* $\text{rad}(I) \subset I(V(I))$.

Proof. Let $f \in \text{rad}(I)$. Then for some $n > 0$, $f^n \in I$. Let $v \in V(I)$. Then $f^n(v) = 0$ hence $f(v) = 0$, implying that $f \in I(V(I))$. □

Lemma 3.1.47. *Let $a_1, \ldots, a_n \in K$ and $I := \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Then I is maximal ideal and $K \simeq K[x_1, \ldots, x_n]/I$ via the canonical embedding of K in $K[x_1, \ldots, x_n]/I$*

Proof. Proving that I is maximal is equivalent to proving that $K[x_1, \ldots, x_n]/I$ is a field. Hence if we prove that $K \simeq K[x_1, \ldots, x_n]/I$ the first claim follows. Consider the canonical embedding of K in $K[x_1, \ldots, x_n]/I$

$$\iota : K \to K[x_1, \ldots, x_n]/I$$
$$a \to a + I$$

Note since I is proper, this is a non-trivial morphism of fields and thus an injective map. Let $f + I \in K[\mathbf{x}]/I$ be given. Then

$$f + I = f(a_1, \ldots, a_n) + I = \iota(f(a_1, \ldots, a_n)),$$

hence ι is surjective. □

3.2 Affine Varieties

Definition 3.2.1. An algebraic subset $X \subset \mathbb{A}^n$ is said to be *reducible* if there are algebraic subsets $Y, Z \subsetneq X$ such that $X = Y \cup Z$. An algebraic subset $V \subset \mathbb{A}^n$ that is not reducible is said to be *irreducible*. An irreducible algebraic subset is also called an *(affine) variety*.

Remark 3.2.2. If $f \in K[x_1, \ldots, x_n]$ is a reducible polynomial, then $V(f)$ is reducible. Indeed, $f = gh$ for some non-constant $g, h \in K[\mathbf{x}]$, hence $V(f) = V(gh) = V(g) \cup V(h)$. We also have that $\langle f \rangle \subsetneq \langle g \rangle$ and $\langle f \rangle \subsetneq \langle h \rangle$, since $g, h \mid f$ and $\deg\ g, \deg\ h < \deg\ f$. Later we will see an example of a variety $V(f)$ where f is not irreducible.

Example 3.2.3. For an infinite field K, $\mathbb{A}^n(K)$ is irreducible. Indeed, $I(\mathbb{A}^n(K)) = 0$, which is trivially prime, hence $\mathbb{A}^n(K)$ is irreducible by Theorem 3.2.6. One thus concludes that

$$\mathbb{A}^n(K) \text{ is irreducible } \iff \#K = \infty$$

Proposition 3.2.4. *Let $V \subset \mathbb{A}^n$ be a finite non-empty algebraic set. Then V is a variety if and only if V is a point*

Proof. " $\implies$ ":Suppose V is not a point. Then $V = \{p_1, \ldots, p_m\}$ for $m > 1$, hence $V = \{p_1\} \cup \bigcup_2^m \{p_i\}$. Noting that $\{p_1\}$ and $\bigcup_2^m \{p_i\}$ are disjoint algebraic sets contained in V, it follows that V is reducible.
" $\impliedby$ ": Suppose V is a point. Then if $V = X \cup Y$ for algebraic sets $X, Y \subset \mathbb{A}^n$, then either $X = V$ or $Y = V$, hence V is a variety. □

Definition 3.2.5. Let $V \subset \mathbb{A}^n$ be a variety. We define *the coordinate ring of V* to be the ring $\Gamma(V) := K[x_1, \ldots, x_n]/I(V)$.

Proposition 3.2.6. *Let $V \subset \mathbb{A}^n$ be an algebraic set. The following are equivalent:*

1. *V is a variety.*
2. *$I(V)$ is a prime ideal.*
3. *$\Gamma(V)$ is an integral domain.*

Proof. "1. $\implies$ 2.": Suppose $I(V)$ is not prime. Then there are $f_1, f_2 \in K[x_1, \ldots, x_n] \setminus I(V)$ such that $f_1 f_2 \in I(V)$ then $V \subset V(f_1 f_2)$, hence

$$V = V \cap V(f_1 f_2) = V \cap (V(f_1) \cup V(f_2)) = (V \cap V(f_1)) \cup (V \cap V(f_2)).$$

Since $f_1, f_2 \notin I(V)$, $I(V) \subsetneq I(V) + \langle f_i \rangle$ for $i = 1, 2$, hence

$$V \subsetneq V(I(V) + \langle f_i \rangle) = V(I(V)) \cap V(f_i) = V \cap V(f_i),$$

meaning V is reducible.
"2. $\implies$ 1.": Suppose V is reducible. Then $V = V_1 \cup V_2$ for algebraic subsets $V_1, V_2 \subsetneq V$. Then $I(V_1), I(V_2) \supsetneq I(V)$, meaning that there exists $f_1 \in I(V_1) \setminus I(V)$ and $f_2 \in I(V_2) \setminus I(V)$. Furthermore, $f_1 f_2$ vanish on ever point in $V_1 \cup V_2 = V$, hence $f_1 f_2 \in I(V)$, meaning $I(V)$ is not prime.
"2. $\iff$ 3.": This follows from Lemma 1.6.27. □

We demonstrate the strength of the above theorem in the following example

Example 3.2.7. 1. Let $f = y - x^2 \in \mathbb{C}[x, y]$. Since $\deg_y f = 1$ and f is monic with respect to x, f is irreducible in $(\mathbb{C}[y])[x] \simeq \mathbb{C}[x, y]$, hence since $\mathbb{C}[x, y]$ is a UFD (cf. Corollary 1.8.82), f is prime, and thus $\langle f \rangle$ is prime. Suppose $a \in I(V(f))$. We aim to prove that $a(x, x^2) = 0$, which would imply

$$a(x, y) + \langle f \rangle = a(x, x^2) + \langle f \rangle = 0 + \langle f \rangle \implies a \in \langle f \rangle.$$

Indeed every point in $V(f)$ is of the form $(\alpha, \alpha^2) \in \mathbb{A}^2(\mathbb{C})$, hence upon putting $b = a(x, x^2) \in \mathbb{C}[x]$, we see that $b(\alpha) = 0$ for every $\alpha \in \mathbb{C}$, implying $a(x, x^2) = b = 0$. We thus conclude $I(V(f)) = \langle f \rangle$, which means $V(f) \subset \mathbb{A}^2(\mathbb{C})$ is irreducible by the above theorem.

2. Let $g = y^4 - x^2 \in \mathbb{C}[x, y]$ and $h = y^4 - x^2 y^2 + xy^2 - x^3 \in \mathbb{C}[x, y]$. Setting $g_1 = y^2 - x$ and $g_2 = y^2 + x$, then $g = g_1 g_2$. We can thus write

$$V(g, h) = V(g) \cap V(h) = V(g_1 g_2) \cap V(h) = (V(g_1) \cup V(g_2)) \cap V(h) = V(g_1, h) \cup V(g_2, h).$$

Note also that

$$h = y^4 - x^2 y^2 + xy^2 - x^3 = y^2(y^2 + x) - x^2(y^2 - x) = -x^2 g_1 + y^2 g_2 \in \langle g_1, g_2 \rangle.$$

We aim to prove that $V(g_i, h) = \{0\}$. Let $v \in V(g_1, h)$. Then $v_1 = v_2^2$. Then $h(v) = v_2^2 g_2(v)$, hence $0 = v_2^2 = v_1$, in which case $v = 0$, or $g_2(v) = 0$. Since $V(g_1) \cap V(g_2) = \{0\}$, it follows that in the second case $v = 0$. One proves $V(g_2, h) = \{0\}$ similarly. Hence $V(g, h) = \{0\}$ is irreducible.

3. One should note that a polynomial being irreducible, does not imply that its vanishing set is irreducible. Take for instance, $f = y^2 + x^2(x-1)^2 \in \mathbb{R}[x,y]$. Consider f as a polynomial in $\mathbb{C}[x,y]$ we have that

$$f = (y - ix(x-1))(y + ix(x-1)),$$

hence since the prime factorization of f is unique (since $\mathbb{C}[x,y]$ is a UFD) and $y - ix(x-1), y + ix(x-1) \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$, there is no non-trivial factors of f in $\mathbb{R}[x,y]$, which means f is irreducible in $\mathbb{R}[x,y]$. Note that

$$V(f) \supset \{(0,1)\} \cup \{(0,0)\} = V(x-1,y) \cup V(x,y).$$

Let $(v_1, v_2) \in V(f)$. Then $v_2^2 = -v_1^2(v_1-1)^2$. Note that $v_2^2 \geq 0$, and $v_1^2(v_1-1)^2 = (v_1(v_1-1))^2 \geq 0$, hence $v_2^2 = -v_1^2(v_1-1)^2 \leq 0$. This implies $v_2 = 0$, hence $v_1^2 = 0$ or $(v_1-1)^2 = 0$, implying $v_1 = 0$ or $v_1 - 1 = 0$. In conclusion $(v_1, v_2) = (0,0)$ or $(v_1, v_2) = (1,0)$, meaning $(v_1, v_2) \in \{(0,1)\} \cup \{(0,0)\}$. We thus see that

$$V(f) = V(x-1,y) \cup V(x,y).$$

Lemma 3.2.8. *A non-empty family of algebraic sets in $\mathbb{A}^n$ has a minimal element.*

Proof. Indeed, consider

$$\{V(I) : I \in \mathcal{I}\}$$

where $\mathcal{I}$ is a non-empty family of ideals in $K[x_1, \ldots, x_n]$. Then $\mathcal{I}$ has a maximal element J, since $K[x_1, \ldots, x_n]$ is Noetherian. Then $V(I) \subset V(J)$ where $I \in \mathcal{I}$ implies $I \supset J$, hence $I = J$, meaning $V(I) = V(J)$. We thus conclude that $V(J)$ is a minimal element of $\{V(I) : I \in \mathcal{I}\}$. □

Theorem 3.2.9. *Let $X \subset \mathbb{A}^n$ be an algebraic set. Then there is a finite sequence of varieties $V_1, \ldots, V_n \subset \mathbb{A}^n$ unique (up to re-ordering) such that*

$$X = \bigcup_1^n V_i$$

and $V_i \not\subset V_j$ for $i \neq j$.

Proof. **Existence:** Consider the set

$$\mathcal{A} := \{X \subset \mathbb{A}^n : X \text{ is algebraic and not a finite union of varieties}\}.$$

Suppose for a contradiction that $\mathcal{A} = \emptyset$. Then by Lemma 3.2.8, there is a minimal element $X_0 \in \mathcal{A}$. On the one hand X_0 is not irreducible. Hence There are $Y, Z \subsetneq X_0$ such that

$X_0 = Y \cup Z$. However, by minimality $Y, Z \notin \mathcal{A}$, hence $Y = \bigcup_1^k V_i$ and $Z = \bigcup_1^l W_i$ for varieties $V_1, \ldots, V_k, W_1, \ldots, W_l \subset \mathbb{A}^n$. But then

$$X_0 = Y \cup Z = \bigcup_1^k V_i \cup \bigcup_1^l W_i,$$

leading to a contradiction. This means $\mathcal{A} = \emptyset$. Hence every algebraic set can be written as a union of varieties.

Let $X \subset \mathbb{A}^n$ be an algebraic set, and pick varieties $V_1, \ldots, V_m \subset \mathbb{A}^n$ whose union is equal to X. Let $\{V_{k_1}, \ldots, V_{k_l}\}$ be the maximal elements of $\{V_1, \ldots, V_m\}$. Let $j \in \{1, \ldots, m\}$. The set $S := \{V_i : V_j \subset V_i\}$ has a maximal element V_k. Let $h \in \{1, \ldots, m\}$ and suppose $V_k \subset V_h$. Then in particular $V_j \subset V_h$, hence $V_h \in S$, implying $V_h = V_k$. This means V_k is a maximal element of $\{V_1, \ldots, V_m\}$. In other words $k = k_i$ for some i, hence $V_j \subset V_{k_i}$, implying

$$X = \bigcup_1^m V_i = \bigcup_1^l V_{k_i}$$

Uniqueness: Suppose $X \subset \mathbb{A}^n$ be algebraic sets. Suppose there are varieties $V_1, \ldots, V_l, W_1, \ldots, W_m \subset \mathbb{A}^n$ such that

$$\bigcup_1^l V_i = X = \bigcup_1^m W_i,$$

and $V_i \not\subset V_j$ for $j \neq i$ and $W_i \not\subset W_j$ for $i \neq j$. For each $i \in \{1, \ldots, l\}$ we have that

$$V_i = V_i \cap V = V_i \cap \bigcup_1^m W_j = \bigcup_1^m (V_i \cap W_j).$$

Then there is a $j(i)$ such that $V_i \subset W_{j(i)}$. By similar argument, $W_{i(j)} \subset V_k$ for some k hence $i = k$, implying $V_i = W_{j(i)}$. similarly $W_j = V_{i(j)}$ for every j some $i(j)$. Hence there is a bijection of $\{V_1, \ldots, V_l\}$ with $\{W_1, \ldots, W_m\}$. □

Definition 3.2.10. The varieties $V_1, \ldots, V_m \subset \mathbb{A}^n$ involved in the *decomposition*, described in the above theorem, of an algebraic set $X \subset \mathbb{A}^n$ are called the *(irreducible) components* of X.

Proposition 3.2.11. *Let $X, Y \subset \mathbb{A}^n$ be algebraic subsets such that $X \subset Y$ with irreducible components $V_1, \ldots, V_m$ respectively $W_1, \ldots, W_l$. Then for each $i \in \{1, \ldots, m\}$, V_i is contained in W_j for some $j \in \{1, \ldots, l\}$.*

Proof. Let $\{X_{ij}^{k_{ij}}\}$ be the components of $V_i \cap W_j$. Then

$$\bigcup_1^m V_i = \bigcup_1^m \bigcup_1^l V_i \cap W_j = \bigcup_1^m \bigcup_1^l \bigcup X_{ij}^{(k_{ij})} = \bigcup_1^m X_{i_h j_h}^{(k_{i_h j_h})}$$

for suitable (i_h, j_h) such that $X_{i_h j_h} \not\subset X_{i_g j_g}$ for $h \neq g$. Given $i \in \{1, \ldots, m\}$ for some $k_{i_h j_h}$,

$$V_i = X_{i_h j_h}^{(k_{i_h j_h})} \subset V_{i_h} \cap W_{j_h}.$$

Then $V_i \subset W_{j_h}$.

□

Proposition 3.2.12. *Let $X \subset \mathbb{A}^n$ be an algebraic subset with irreducible components $V_1, \ldots, V_m$. Then for each $i \in \{1, \ldots, m\}$,*

$$V_i \not\subset \bigcup_{j \in \{1,\ldots,m\} \setminus \{i\}} V_j.$$

Proof. Indeed, if a variety V is a subset of $\bigcup_1^k W_i$, then $V \subset W_h$ for some h by the prior proposition. Since $V_i \not\subset V_j$ for $i \neq j$, the result follows. □

3.2.1 The Three Types of Algebraic Subsets of the Plane

Theorem 3.2.13. *Let $f, g \in K[x, y]$ such that* $\gcd(f, g) = 1$. *Then* $\#V(f, g) < \infty$.

Proof. By Add result 1 is also greatest common divisor of f and g in $K(x)[y]$. since $K(x)[y]$ is a PID it follows that $\langle f, g \rangle = \langle 1 \rangle \subset K(x)[y]$ Result. Then there are $\mu, \lambda \in K(x)[y]$ such that $\mu f + \lambda g = 1$. For some $\delta \in K[x]$, we have that $\alpha f + \beta g = \delta$. Hence if $(v_1, v_2) \in V(f, g)$, then $v_1 \in V(\delta)$. Since $\#V(\delta) < \infty$, we have that

$$V_1 := \{v_1 \in K : (v_1, v_2) \in V(f, g) \text{ for some } v_2 \in K\}$$

is finite. We can similarly show by a symmetric argument that

$$V_2 := \{v_2 \in K : (v_1, v_2) \in V(f, g) \text{ for some } v_1 \in K\}$$

is finite. One easily sees that $V(f, g) \subset V_1 \times V_2$, hence $\#V(f, g) < \infty$. □

Corollary 3.2.14. *Let $f \in K[x, y]$ be irreducible such that $\#V(f) = \infty$. Then $I(V(f)) = \langle f \rangle$ and $V(f)$ is irreducible.*

Proof. If $g \in I(V(f))$ then $V(f, g) \supset V(f)$, hence $\#V(f, g) = \infty$. Then f and g has a non-trivial common factor by the above theorem. Then since f is assumed irreducible this means $f \mid g$ or equivalently that $g \in \langle f \rangle$, hence $I(V(f)) \subset \langle f \rangle$, implying $I(V(f)) = \langle f \rangle$ by Lemma 3.1.40. This implies that $I(V(f))$ is prime by Lemma 1.6.32 and the fact that $K[x, y]$ is a UFD. This implies that $V(f)$ is irreducible by Proposition 3.2.6. □

Corollary 3.2.15. *Suppose K is infinite. The varieties in $\mathbb{A}^2$ are $\mathbb{A}^2$, $\emptyset$, points and plane curves $V(f)$ where $f \in K[x,y]$ is irreducible and $\#V(f) = \infty$.*

Proof. Suppose $V \subset \mathbb{A}^2$ is a non-empty variety. By Proposition 3.2.4, V is finite if and only if V is a point. Suppose V is infinite. If $I(V) = 0$ then $V = \mathbb{A}^2$. So suppose $I(V)$ contains a non-constant f. Since $K[x,y]$ is a UFD, we can write f as a product irreducible factors $f_1, \cdots, f_m \in K[x,y]$, since $I(V)$ is prime by Proposition 3.2.6 it follows that $f_i \in I(V)$ for some i. Suppose for a contradiction that there is a $g \in I(V) \setminus \langle f_i \rangle$, then $\langle g, f_i \rangle \subset I(V)$ and $f_i \nmid g$, hence $\gcd(g, f_i) = 1$ and $V(g, f_i) \supset V(I(V)) = V$, implying that V would be finite by the above theorem. It thus follows that $I(V) = \langle f_i \rangle$, hence $V = V(I(V)) = V(f_i)$. □

Corollary 3.2.16. *Let K be an algebraically closed field, $f \in K[x,y]$, $\deg f > 0$. Let distinct irreducible polynomials $f_1, \ldots, f_l \in K[x,y]$ and positive integers $r_1, \ldots, r_l$ be given such that $f = \prod_1^r f_i^{r_i}$. Then $V_i := V(f_i)$ for $i = 1, \ldots, l$ are the irreducible components of $V := V(f)$ and $V(f) = \left\langle \prod_1^l f_i \right\rangle$.*

Proof. Proposition 3.1.34 2. shows that $V(f_i)$ is an infinite set for each i. Hence by Corollary 3.2.14 V_i is irreducible and $I(V_i) = \langle f_i \rangle$ for each i. Furthermore, for each i, $V\left(f_i^{r_i}\right) = V\left(\operatorname{rad}\left(f_i^{r_i}\right)\right) = V(f_i)$, hence $V = \bigcup_1^l V_i$. Since $f_i \nmid f_j$ for $i \neq j$, $\langle f_j \rangle \not\subset \langle f_i \rangle$, hence $V_i \not\subset V_j$. It follows that

$$I\left(\bigcup_1^l V_i\right) = \bigcap_1^l I(V_i) = \bigcap_1^l \langle f_i \rangle = \left\langle \prod_1^l f_i \right\rangle.$$

□

Example 3.2.17. We give some examples of why we will want to assume that we work over non-algebraically closed field. The main reason though, is that Hilbert's Nullstellensatz holds for algebraically cloesed fields.

1. Let $f = x^2+y^2+1 \in \mathbb{R}[x,y]$, $I = \langle f \rangle$. Note that for any $(v_1, v_2) \in \mathbb{A}^2(\mathbb{R})$, $v_1^2+v_2^2 \geq 0$, hence $v_1^2 + v_2^2 + 1 \geq 1 > 0$, hence $V(I) = \emptyset$. Hence $I(V(I)) = I(\emptyset) = \langle 1 \rangle$ Let $f_1 = a_1x + b_1y + c_1$, $f_2 = a_2x + b_2y + c_2 \in \mathbb{R}[x,y]$. Then

   $$f_1f_2 = a_1a_2x^2 + b_1b_2y^2 + (a_1b_2 + a_2b_1)xy + (a_1c_2 + a_2c_1)x + (b_1c_2 + b_2c_1)y + c_1c_2.$$

   Consider the following system of equations

   $$\begin{cases} a_1a_2 = \alpha & b_1b_2 = \beta \\ (a_1b_2 + a_2b_1) = 0 \end{cases}$$

where $\alpha, \beta > 0$ This system of equations clearly cannot have any real solutions since using $a_1a_2 = 1$ and $b_1b_2 = 1$, we get that

$$(a_1b_2 + a_2b_1) = 0 \iff a_1b_2 = -a_2b_1 \iff a_1^2b_2^2 = -a_2a_1b_1b_2 = -1.$$

It thus follows that any polynomial in $\mathbb{R}[x, y]$ of the form $\alpha x^2 + \beta y^2 + 0xy + \ldots \in \mathbb{R}[x, y]$ where α, β cannot be a product of two degree 1 polynomials in $\mathbb{R}[x, y]$ and therefor is irreducible in $\mathbb{R}[x]$. Alternatively one could also see this by applying the rational root theorem and Proposition 1.8.114 to $x^2 + y^2 + 1 \in \mathbb{R}(x)[y]$.
The point in any case is that $I(V(I)) \neq I$.

2. Every algebraic subset of $\mathbb{A}^2(\mathbb{R})$ can be written as $V(f)$ for some $f \in \mathbb{R}[x, y]$: If $P = \{(v_1, v_2)\}$, then $P = V(g)$, where $g = (x - v_2)^2 + (y - v_1)^2$. Indeed, $g(v_1, v_2) = 0$ and if $(w_1, w_2) \in V(g)$, then since $(w_1 - v_1)^2, (w_2 - v_2)^2 \geq 0$ and $(w_1 - v_1)^2 + (w_2 - v_2)^2 = 0$, we have $w_1 - v_1 = 0$ and $w_2 - v_2 = 0$. Now if V is an arbitrary algebraic subset of $\mathbb{A}^2(\mathbb{R})$, it has a decomposition into irreducible components $V_1, \ldots, V_l$. By Corollary 3.2.15 and the remark we made about points at the beginning, it follows that $V_i = V(f_i)$ for some $f_i \in \mathbb{R}[x, y]$, hence

$$V = \bigcup_1^l V(f_i) = V\left(\prod_1^l f_i\right).$$

Example 3.2.18. We have now classified the varieties of the plane over an infinite field K. Let's determine the irreducible decompositions of some algebraic subsets

1. Let $f = y^2 - xy - x^2y + x^3$. Set $f_1 = x - y$ and $f_2 = y - x^2$. Then one sees that $f = f_1f_2$ and hence $V(f) = V(f_1) \cup V(f_2)$. Eisenstein's criterion shows that $f_1 \in K(x)[y]$, $f_2 \in K(y)[x]$ are irreducible, since $x \nmid 1, x \mid x, x^2 \nmid x$ and $y \nmid 1, y \mid y, y^2 \nmid y$. We then have that (since f_1, f_2 are primitive) that $f_1, f_2 \in K[x, y]$ are irreducible (This is for any field K). Note that $V_{\mathbb{R}}(f_1) = \{(\lambda, \lambda) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$, which is infinite. Note also that $V_{\mathbb{R}}(f_2) = \{(\lambda, \lambda^2) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$, which is also infinite since it is the graph of the function $\lambda \mapsto \lambda^2$. We thus see that $V_{\mathbb{R}}(f_1), V_{\mathbb{R}}(f_2)$ and $V_{\mathbb{C}}(f_1), V_{\mathbb{C}}(f_2)$ are the irreducible components of $V_{\mathbb{R}}(f)$ resp. $V_{\mathbb{C}}(f)$ by Corollary 3.2.15, since $V_{\mathbb{C}}(f_1) \cap V_{\mathbb{C}}(f_2) = \{(0, 0), (1, 1)\}$ implying that neither set is contained in the other.

2. Consider $f = y^2 - x(x^2 - 1)$. The set $V(f)$ is an example of an elliptic curve (these are in general of the form $V(y^2 - (ax^3 + bx^2 + cx + d))$). Using Eisenstein's criterion for f in $K[x][y]$ testing the criterion against $x \in K[x]$, it follows that

$f \in K[x,y]$ is irreducible. We aim to show that $V_{\mathbb{R}}(f)$ is infinite. Let $\lambda \in \mathbb{R}_{\geq 1}$. Then $\lambda^2 - 1 \geq 0$, hence $\lambda(\lambda^2 - 1) \geq 0$. It thus follows that

$$\left\{\left(\lambda, \pm\sqrt{\lambda(\lambda^2-1)}\right) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\right\} \subset V_{\mathbb{R}}(f) \subset V_{\mathbb{C}}(f).$$

Since this subset is infinite, it follows that both $V_{\mathbb{R}}(f), V_{\mathbb{C}}(f)$ are irreducible by Corollary 3.2.15.

3. Consider $f = x^3 + x - x^2y - y$. Put $f_1 = x - y$, $f_2 = x^2 + 1$. Then $f = f_1 f_2$. We have already seen that f_1 is irreducible in $K[x,y]$ and that $V(f_1)$ is infinite. We also know that $V_{\mathbb{R}}(f_2) = \emptyset$. It thus follows that $V_{\mathbb{R}}(f) = V_{\mathbb{R}}(f_1) \cup V_{\mathbb{R}}(f_2) = V_{\mathbb{R}}(f_1)$, hence $V_{\mathbb{R}}(f_1)$ is the only irreducible component over $\mathbb{R}$. Note that $f_2 = (x+i)(x-i)$ in $\mathbb{C}[x,y]$, and $f_2' = x + i$, and f_2'' are irreducible in $\mathbb{C}[x,y]$ being of degree 1. Since $V_{\mathbb{C}}(f_2') = \{(-i,\lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}$ and $V_{\mathbb{C}}(f_2'') = \{(i,\lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}$ are infinite sets, it thus follows that $V_{\mathbb{C}}(f_2'), V_{\mathbb{C}}(f_2'')$ are also irreducible sets in $\mathbb{A}^2(\mathbb{C})$. Clearly $V_{\mathbb{C}}(f_2')$ and $V_{\mathbb{C}}(f_2'')$ are disjoint. Furthermore, $V_{\mathbb{C}}(f_1) \cap V_{\mathbb{C}}(f_2') = \{(-i,-i)\}$ and $V_{\mathbb{C}}(f_1) \cap V_{\mathbb{C}}(f_2'') = \{(i,i)\}$, hence there is no containment of these sets in each other. It thus follows that the irreducible components of $V_{\mathbb{C}}(f)$ are $V_{\mathbb{C}}(f_1), V_{\mathbb{C}}(f_2'), V_{\mathbb{C}}(f_2'')$.

3.2.2 Hilbert's Nullstellensatz

In the discussion involving Hilbert's Nullstellensatz, we shall assume that K is algebraically closed. Before presenting the full Hilbert Nullstellensatz we prove a weak version, which we shall see implies Hilbert's Nullstellensatz.

Theorem 3.2.19. *(The Weak Nullstellensatz/WNS)*
Let K be an algebraically closed field and consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then $V(f_1, \ldots, f_m) = \emptyset$ if and only if there are polynomials $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that

$$\sum_1^m \lambda_i f_i = 1.$$

In other words

$$V(f_1, \ldots, f_m) = \emptyset \iff \langle f_1, \ldots, f_m \rangle = \langle 1 \rangle = K[\mathbf{x}].$$

Proof. Set $I := \langle f_1, \ldots, f_m \rangle$.
" $\Longleftarrow$ ": If $I = \langle 1 \rangle$, then $V(I) = V(1) = \emptyset$.
" $\Longrightarrow$ ": Suppose $I \neq \langle 1 \rangle$, i.e. I is a proper ideal. We aim to show $V(I) \neq \emptyset$. $I \subset I'$ for some maximal ideal (cf. Proposition 1.6.17). Therefore $V(I) \supset V(I')$. This means that if $V(I') \neq \emptyset$, then $V(I) \neq \emptyset$. It is then sufficient to prove the statement in the case where I

is maximal. It follows from Corollary 1.9.27 that $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for suitable $a_i \in K$. Hence $V(I) = \{(a_1, \ldots, a_n)\} \neq \emptyset$. □

Remark 3.2.20. The equation

$$\sum_1^m f_i y_i = 1 \tag{6}$$

is called *the Hilbert equation associated with* $f_1, \ldots, f_m$. Thus WNS says that $f_1, \ldots, f_m$ has no common zeros if and only if the Hilbert equation associated with $f_1, \ldots, f_m$ is soluble over $K[\mathbf{x}]$

Theorem 3.2.21. *(Hilbert's Nullstellensatz/HNS)*
Let K be an algebraically closed field and consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Put $I := \langle f_1, \ldots, f_m \rangle$. Then for every $f \in I(V(I))$ there are $\lambda_1, \ldots, \lambda_n \in K[\mathbf{x}]$ and an integer $k \geq 0$ such that

$$\sum_1^m \lambda_i f_i = f^k.$$

In other words $I(V(I)) = \text{rad}(I)$.

Proof. $\text{rad}(I) \subset I(V(I))$ by Lemma 3.1.46. Let $g \in I(V(I))$. Put $J := \langle f_1, \ldots, f_m, x_{n+1}g - 1 \rangle \subset K[x_1, \ldots, x_{n+1}]$. Suppose $v \in \mathbb{A}^{n+1}$ vanishes on $f_1, \ldots, f_m$. Then $v_{n+1}g(v) - 1 = -1 \neq 0$. If $v \in \mathbb{A}^{n+1}$ vanishes on $x_{n+1}g - 1$. Then $v_{n+1}g(v) = 1 \neq 0$, hence v doesn't vanish on $f_1, \ldots, f_m$. This means $V(J) = \emptyset$. By the weak Nullstellensatz we can then find $\lambda_1, \ldots, \lambda_{m+1} \in K[x_1, \ldots, x_{n+1}]$ such that

$$1 = \left[\sum_1^m \lambda_i f_i\right] + \lambda_{m+1}(x_{n+1}G - 1),$$

Let $y = \frac{1}{g} \in K(x_1, \ldots, x_{n+1})$. Then

$$1 = \text{ev}_{x_1,\ldots,x_n,y}(1) = \text{ev}_{x_1,\ldots,x_n,y}\left(\left[\sum_1^m \lambda_i f_i\right] + \lambda_{m+1}(x_{n+1}G - 1)\right)$$
$$= \left[\sum_1^m \lambda_i(x_1, \ldots, x_n, y) f_i\right] + \lambda_{m+1}(x_1, \ldots, x_n, y)(yG - 1) = \sum_1^m \lambda_i(x_1, \ldots, x_n, y) f_i.$$

Then taking $N \geq \max_{i \in \{1,\ldots,m\}}(\deg_{x_{n+1}} \lambda_i f_i)$,

$$g^N = g^N \sum_1^m \lambda_i(x_1, \ldots, x_n, y) f_i = \sum_1^m \lambda_i' f_i,$$

for suitable $\lambda_i' \in K[x_1, \ldots, x_n]$. This implies $g \in \text{rad}(I)$ □

We give a few corollaries the first of which is apparent.

Corollary 3.2.22. *The mapping taking a radical ideal $I \subset K[x_1, \ldots, x_n]$ to $V(I) \subset \mathbb{A}^n$ and the mapping taking an algebraic set $V = V(I) \subset \mathbb{A}^n$ to $I(V) = \text{rad}(I)$ establishes a one-to-one correspondence between algebraic sets and polynomial radical ideals for each $n \geq 1$.*

Corollary 3.2.23. *The mapping taking a prime ideal $I \subset K[x_1, \ldots, x_n]$ to $V(I) \subset \mathbb{A}^n$ and the mapping taking a variety $V = V(I) \subset \mathbb{A}^n$ to $I(V) = \text{rad}(I)$ establishes a one-to-one correspondence between irreducible algebraic sets and polynomial prime ideals for each $n \geq 1$.*

Proof. Let $I \subset K[\mathbf{x}]$ be a prime ideal. By Lemma 1.6.36 and HNS,

$$I(V(I)) = \text{rad}(I) = I, \tag{7}$$

hence by Proposition 3.2.6, $V(I)$ is indeed irreducible. Given a variety $V \subset \mathbb{A}^n$, also by Proposition 3.2.6, $I(V)$ is prime. Furthermore, $V(I(V)) = V$ by Lemma 3.1.40. This with (7) shows the mappings are mutual inverses. □

Corollary 3.2.24. *The mapping taking a maximal ideal $I \subset K[x_1, \ldots, x_n]$ to $V(I) \subset \mathbb{A}^n$ and the mapping taking a point $P = \{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n) \subset \mathbb{A}^n$ to $I(P) = \text{rad}(x_1 - a_1, \ldots, x_n - a_n)$ establishes a one-to-one correspondence between and polynomial maximal ideals in $K[x_1, \ldots, x_n]$ and points in $\mathbb{A}^n$ for each $n \geq 1$.*

Proof. Let $I \subset K[\mathbf{x}]$ be a maximal ideal. Then $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, hence $V(I) = \{(a_1, \ldots, a_n)\}$. Since I is maximal, it is prime, hence radical, and thus $I(V(I)) = I$ by HNS. Let $P = \{(a_1, \ldots, a_n)\} \subset \mathbb{A}^n$. Then $I(P) = I(V(\langle x_1 - a_1, \ldots, x_n - a_n \rangle)) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, which is maximal, and of course we know $V(I(P)) = P$. □

HNS establishes a hands-on way of decomposing ANY hypersurface into its irreducible components, which generalizes the result already obtained in the two variable case.

Corollary 3.2.25. *Consider a hypersurface $X := V(f) \subset \mathbb{A}^n$. We can write*

$$f = \prod_1^r f_i^{v_i},$$

for suitable distinct irreducible polynomials $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$ and $v_1, \ldots, v_r \geq 1$. The irreducible components of X are $V_1, \ldots, V_r$, where $V_i := V(f_i)$. We thus establish a one-to-one correspondence between irreducible hypersurfaces and irreducible polynomials up to scalar multiplication from K. Lastly $I(X) = \langle \prod_1^r f_i \rangle$

Proof. Since the ideal $\langle f_i \rangle$ are prime, it follows from HNS that $V_i = V(f_i)$ is irreducible. Since $f_i \nmid f_j$ for $i \neq j$, it follows that $V_i \not\subset V_j$, hence $V_1, \ldots, V_r$ are the irreducible components of V. To prove the last statement, it is sufficient, by HNS, to prove that $\text{rad}(\langle f \rangle) = \langle \prod_1^r f_i \rangle$. This follows from Lemma 1.6.78. □

Corollary 3.2.26. *Consider $I := \langle f_1, \ldots, f_m \rangle \subset K[x_1, \ldots, x_n]$. $V(I)$ is finite if and only if $K[\mathbf{x}]/I$ is a finite dimensional vector space over K. If this occurs $\#V(f) \leq \dim_K K[\mathbf{x}/I]$.*

Proof. " $\Longrightarrow$ ": Suppose $V(I) = \{p_1, \ldots, p_l\}$, where $p_i = (a_{i1}, \ldots, a_{il}) \in \mathbb{A}^n$. Define for $j \in \{1, \ldots, n\}$, $g_j := \prod_{i=1}^l (x_j - a_{ij})$. Then $g_j \in I(V(I)) = \text{rad}(I)$, hence for some $N_j \geq 1$, $g_j^{N_j} \in I$. Take $N = \max\ N_j$. Then for suitable $\lambda_{ij} \in K$,

$$0 + I = F_j^N + I = \left[X_j^{lN} + \sum_0^{lN-1} \lambda_{ij} X_j^i \right] + I \implies X_j^{lN} + I = \left[-\sum_0^{lN-1} \lambda_{ij} X_j^i \right] + I.$$

For each $s \geq 0$, we thus find that $X_j^s + I$ is a linear combination of $1+I, X_j+I, \ldots, X_j^{rN-1}+I$. This means

$$K[\mathbf{x}]/I = \text{Span}_K \left(\left\{ x_j^k + I : j \in \{1, \ldots, n\}, k \in \{0, \ldots, lN\} \right\} \right),$$

hence $K[\mathbf{x}]/I$ is finite dimensional.
" $\Longleftarrow$ ": Set $d := \dim_K\ K[\mathbf{x}]/I$ and let distinct points $p_1, \ldots, p_l \in V(I)$ be given. There are polynomials $g_1, \ldots, g_l \in K[\mathbf{x}]$ such that $g_i(p_i) = 1$ and $g_i(p_j) = 0$ for $i \neq j$ by an earlier result. Let $\lambda_1, \ldots, \lambda_l \in K$ be given such that

$$\left[\sum_1^l \lambda_i g_i \right] + I = 0 + I.$$

Then $\left[\sum_1^l \lambda_i g_i \right] \in I$, so for each i,

$$\lambda_i = \lambda_i g_i(p_i) = \sum_1^m \lambda_j f_j(p_i) = 0,$$

hence $\{g_1+I, \ldots, g_l+I\} \subset K[\mathbf{x}]/I$ are linearly independent over K. Then $\sum_1^l K(g_i+I) \subset K[\mathbf{x}]/I$ has dimension l, and $l \leq d$. This means there can be at most d distinct points in $V(I)$. □

Example 3.2.27. 1. In general the Nullstellensatz provides a way to find irreducible components. Consider for example $f := x^2 + y^2 - 1, g := x^2 - z^2 - 1 \in \mathbb{C}[x, y, z]$ and set $X := V(f, g)$. The fact that f, g are both irreducible will make the task of

providing the irreducible components for X difficult. It is therefor useful to find an alternative generating set for $I := \langle f, g\rangle$. Note that $h := y^2 + z^2 = f - g \in I$ and $f = (f-g)+g = h+g \in J := \langle h, g\rangle$. It thus follows that $I = J$. Setting $h_1 := y - iz$ and $h_2 := y + iz$, one sees that $h = h_1 h_2$. This means

$$X = V(I) = V(J) = (V(h_1) \cup V(h_2)) \cap V(g) = \underbrace{(V(h_1) \cap V(g))}_{V_1} \cup \underbrace{(V(h_2) \cap V(g))}_{V_2}.$$

We now claim that V_1, V_2 are the irreducible components of X. Consider the ring homomorphism

$$\begin{aligned} \sigma : \mathbb{C}[x, y, z] &\to \mathbb{C}[x, z]/\langle g\rangle \\ f &\mapsto f(x, iz, z) \end{aligned}$$

This is clearly a surjective ring homomorphism. Note that $\sigma(h_1) = 0$ and $\sigma(g) = 0$, hence $\langle h_1, f\rangle \subset \ker\ \sigma$. Let $f \in \ker\ \sigma$. Then $f(x, iz, z) + \langle g\rangle = 0$. This means

$$f + \langle h_1, g\rangle = f(x, iz, z) + \langle h_1, g\rangle = 0 + \langle h_1, g\rangle,$$

hence $\ker\ \sigma = \langle h_1, g\rangle$, which shows that

$$\mathbb{C}[x, y, z]/\langle h_1, g\rangle \simeq \mathbb{C}[x, z]/\langle g\rangle.$$

Since g is irreducible, it is prime and thus $\mathbb{C}[x, z]/\langle g\rangle$ is an integral domain hence $\mathbb{C}[x, y, z]/\langle h_1, g\rangle$ is an integral, meaning $\langle h_1, g\rangle$ is a prime ideal. By HNS, it follows that V_1 is irreducible. Similarly one can show that $\mathbb{C}[x, y, z]/\langle h_2, g\rangle$ is an integral domain, and hence $\langle h_2, g\rangle$ is prime, meaning V_2 is irreducible. Any point vanishing on h_1 clearly does not vanish on h_2, hence $V_i \not\subset V_j$ for $i \neq j$. We thus conclude that V_1, V_2 are the irreducible components of X.

2. Set $X := \{(v, v^2, v^3) : v \in \mathbb{C}^3\}$. Note that by Example 3.1.17, $X = V(f, g)$, where $f_1 = x^2 - y$ and $f_2 = x^3 - z$. How does one determine $I(X)$?. Consider the surjective ring homomorphism

$$\begin{aligned} \sigma : \mathbb{C}[x, y, z] &\to \mathbb{C}[t] \\ f &\mapsto f\left(t, t^2, t^3\right) \end{aligned}$$

Note that $I(V) = \ker\ \sigma$. Indeed, $f \in \mathbb{C}[x, y, z]$ vanishes on (v, v^2, v^3), then $f\left(t, t^2, t^3\right) = 0$, since $\mathbb{C}$ is infinite. Conversely if $f\left(t, t^2, t^3\right) = 0$, then for any $v \in \mathbb{C}$, $f\left(v, v^2, v^3\right) = 0$. Since $V(I(V)) = V$, it would be nice to compute the generators of $I(V)$, as this perhaps leads to a nicer sets of generators, that allows

us to say something about irreducible components. Note that ker $\sigma = \mathbb{C}[x,y,z] \cap \langle x-t, y-t^2, z-t^3\rangle$. This is indeed just a reformulation of Lemma 1.8.47. In general, how do we compute the generators of $\mathbb{C}[x,y,z] \cap \langle x-t, y-t^2, z-t^3\rangle$? First let $\leq$ be the lexicographic term order where $x < y < z < t$ or $x < y < z$. The answer is that we turn Gröbner basis theory to compute a Gröbner basis for the ideal $I := \langle x-t, y-t^2, z-t^3\rangle \subset \mathbb{C}[x,y,z,t]$ with respect to $\leq$ using Buchberger's algorithm and then determine $G \cap K[x,y,z]$ which will be a Gröbner basis for $K[x,y,z] \cap I$ with respect to $\leq$ by Theorem 1.8.106. Set $g_1 := -t + x$, $g_2 := -t^2 + y$, $g_3 := -t^3 + z$. We follow Buchberger's algorithm

$$S(g_1, g_2) = tx - y,$$

which has residue $f_1 = -y + x$ when divided by $\{g_1, g_2, g_3\}$. For step 2 we find that

$$S(g_1, g_3) = t^2 x - z,$$

which has residue $f_2 = -z + x^3$ when divided by $\{g_1, g_2, g_3, f_1\}$. For step 3 we find that

$$S(g_2, g_3) = ty - z,$$

which one verifies to have residue 0 when divided by $\{g_1, g_2, g_3, f_1, f_2\}$. Note now that $S(g_i, g_j) \to_{\{g_1,g_2,g_3,f_1,f_2\}} 0$ and the initial terms of the remaining pairs have pairwise greatest common divisor 1, hence the S-polynomial of these pairs also reduce to 0 modulo $\{g_1, g_2, g_3, f_1, f_2\}$ by Proposition 1.8.100. Thus $G := \{g_1, g_2, g_3, f_1, f_2\}$ is a Gröbner basis for I, hence $G' := G \cap \mathbb{C}[x,y,z] = \{f_1, f_2\}$ is a Gröbner basis for ker $\sigma = I(V)$. From this one concludes that $\langle x - y^2, x - z^3\rangle$ is radical. However, we can say even more! We actually find that

$$K[x,y,z]/I(V) = K[x,y,z]/\langle f_1, f_2\rangle \simeq K[t].$$

This means $K[x,y,z]/I(V)$ is an integral domain, implying V is irreducible by Proposition 3.2.6.

Example 3.2.28. Consider $f = y^2 - x(x-1)(x-a) \in K[x,y]$ where K is an algebraically closed field and $a \in K$. $V(f)$ is irreducible. Indeed, when $a \neq 1$, $x - 1 \nmid y^2$, $x \mid f$ and $(x-1)^2 \nmid f$ thus by Eisenstein's criterion f is irreducible/prime. When $a = 1$, $a \neq 0$, hence $x \nmid y^2, x \mid f$ and $x^2 \nmid f$ implies again by Eisenstein's criterion that f is irreducible/prime. The ideal $\langle f\rangle$ is thus prime, hence by HNS $V(f)$ is irreducible.

Example 3.2.29. Consider $f_1 = x^2 - y^2 = (x + y)(x - y) \in \mathbb{C}[x, y]$ and $f_2 = x^2 + y^2 = (x + iy)(x - iy) \in \mathbb{C}[x, y]$. Set $V := V(f_1, f_2)$. Let's find the irreducible components. There is a naive approach: Note that

$$\begin{aligned} V &= (V(x + y) \cup V(x - y)) \cap (V(x + iy) \cup V(x - iy)) \\ &= V(x + y, x + iy) \cup V(x + y, x - iy) \cup V(x - y, x + iy) \cup V(x - y, x - iy). \end{aligned}$$

We can thus decompose V into 4 linear equations all of which has $\{(0, 0)\}$ as there only solution, hence $V = \{(0, 0)\}$.
One can be smarter: Note that $1/2(f_1 + f_2) = x^2$ and $1/2(f_2 - f_1) = y^2$, hence $I = J := \langle x^2, y^2 \rangle$, hence clearly if $v \in V(I)$, $v = (0, 0)$. Note now that

$$\mathbb{C}[x, y]/I = \mathbb{C}[x, y]/J = \{ax + by + cxy + d + J : a, b, c, d \in \mathbb{C}\},$$

and hence is isomorphic to the 4 dimensional vector space $\text{Span}_{\mathbb{C}}(1, x, y, xy) \subset \mathbb{C}[x, y]$. So the inequality from Corollary 3.2.26 doesn't always hold with equality. One notes, however that when decomposing V into a system of linear equations, we get 4 "copies" of the point $(0, 0)$, hence it seems that the dimension of the vector space $\mathbb{C}[x, y]/I$ matches the number of points in $V(I)$ counted with some sort of multiplicity.

Corollary 3.2.30. *If a variety $V \subset \mathbb{A}^n$ is a hypersurface. It is either $\mathbb{A}^n$, $\emptyset$ or $V(f)$ where f is irreducible. As a consequence, such a hypersurface is of the third type, it contains no variety W.*

Proof. This follows from Lemma 1.6.79 and HNS. □

Definition 3.2.31. Let $V \subset \mathbb{A}^n$ be a variety. A variety $W \subset \mathbb{A}^n$ is a *subvariety of* V if it is also a subset of V.

Proposition 3.2.32. *Let $V \subset \mathbb{A}^n$ be a variety. We have one-to-one correspondences between*

1. *Algebraic subsets contained in V and radical ideals of $\Gamma(V)$.*
2. *Subvarieties of V and prime ideals of $\Gamma(V)$.*
3. *Points of V and maximal ideals of $\Gamma(V)$.*

Proof. This follows from HNS (or more precisely the one-to-one correspondences obtained from it) and Proposition 1.6.41. □

Definition 3.2.33. Let $V \subset \mathbb{A}^n$ be a variety and $W \subset V$ a subvariety. We define $I_V(W)$ to be the ideal $I(W)/I(V) \subset \Gamma(V)$.

Lemma 3.2.34. *Let $V \subset \mathbb{A}^n$ be a variety and $W \subset V$ a subvariety. Then $\Gamma(W) \simeq \Gamma(V)/I_V(W)$*

Proof. This follows from an isomorphism theorem:

$$\Gamma(V)/I_V(W) = \frac{K[\mathbf{x}]/I(V)}{I(W)/I(V)} \simeq K[\mathbf{x}]/I(W) = \Gamma(W).$$

□

Proposition 3.2.35. *Let $V \subset \mathbb{A}^n$ be a non-empty variety. The following are equivalent:*

1. *V is a point.*
2. *$\Gamma(V) = K$.*
3. *$\dim_K \Gamma(V) < \infty$.*

Proof. "1. $\implies$ 2.": By HNS, $\Gamma(V)$ is a field, and $K[x] \to \Gamma(V), f \mapsto f + I(V)$ is surjective K-algebra homomorphism hence by Corollary 1.9.26, $\Gamma(V) = K$.
"2. $\implies$ 3.": In this case $\dim_K \Gamma(V) = \dim_K K = 1 < \infty$.
"3. $\implies$ 1.": Since $\dim_K \Gamma(V) < \infty$, V is finite by Corollary 3.2.26, and since V is irreducible it cannot consist of multiple points. □

Example 3.2.36. Set $f := x^2 - y^3$,$g := y^2 - z^3$ and $I := \langle f, g \rangle$. Consider the K-algebra homomorphism

$$\alpha : K[x, y, z] \to K[t]$$
$$x \mapsto t^9, y \mapsto t^6, z \mapsto t^4$$

Let $h = \sum_{v=(a,b,c)\in\mathbb{N}^3} \alpha_v x^a y^b z^c \in K[x, y, z]$. Note that for $a \geq 2$, $x^a + I = x^{2q_a+r_a} + I = x^{r_a} y^{3q_a} + I$ for some $q_a, r_a \geq 0$ with $r_a \in \{0, 1\}$. For $b \geq 2$, $y^b = y^{2q_b+r_b} + I = y^{r_b} z^{3q_b} + I$ for some $q_b, r_b \geq 0$, $r_b \in \{0, 1\}$. So for a monomial $x^a y^b z^c$ we get that

$$x^a y^b z^c + I = x^{r_a} y^{r_b} z^{q_a+q_b+c} + I = \begin{cases} z^{q_a+q_b+c} + I \\ x z^{q_a+q_b+c} + I \\ y z^{q_a+q_b+c} + I \\ xy z^{q_a+q_b+c} + I \end{cases}$$

. This means that

$$h + I = Ax + By + Cz + Dxy + I,$$

for some $A, B, C, D \in K[z]$. We aim to prove that $I = \ker\ \alpha$. Suppose h is such that $h(t^9, t^6, t^4) = 0$. Then $\alpha(Ax + By + Cy + Dxy) = 0$, hence $A(t^4)t^9 + B(t^4)t^6 + C(t^4)t^4 + D(t^4)t^{15} = 0$. One notes that terms in $A(t^4)t^9$ have degree $4d_A + 1$; in $B(t^4)t^6$, $4d_B + 2$; in $C(t^4)t^4$, $4d_C$; in $D(t^4)t^{15}$, $4d_D + 3$, so $A(t^4) = B(t^4) = C(t^4) = D(t^4) = 0$, hence $A = B = C = D = 0$. Then $h + I = 0$, meaning $h \in I$. Then I is prime hence $V(f, g)$ is irreducible

3.2.3 Introduction to Effective Nullstellensätze: Degree Bounds and a Gröbner Basis Method

We fix an algebraically closed field K. An *effective Nullstellensatz (ENS)* is a theorem that proves the weak Nullstellensatz in a way that gives rise to an algorithm for computing a solution to Hilbert Equations. The proof of the weak Nullstellensatz provided in these notes thus far does not give a way for constructing an explicit solution to the Hilbert Equation associated with polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, which vanish nowhere simultaneously and is therefor not an ENS. We are thus presented with the question of how to construct a solution to this Hilbert equation explicitly.

One way to accomplish this, is to prove that there is an upper bound $B \geq 0$ (which will typically dependent on deg $f_1, \ldots,$ deg f_m) on $\{\deg\ \lambda_i\}_1^m$ or equivalently on $\{\deg\ \lambda_i f_i\}_1^m$. To see that this is indeed the case, write

$$f_i = \sum_{v \in \mathbb{N}^n} a_v^{(i)} \mathbf{x}^v \in K[\mathbf{x}],$$

for suitable $a_v^{(i)} \in K$, $i \in \{1, \ldots, m\}$. Define for $u \in \mathbb{N}^n$ with $|u| \leq B$,

$$c_u = \sum_{\substack{i \in \{1,\ldots,m\}, \\ v,w \in \mathbb{N}^n: \\ v+w=u}} a_v^{(i)} y_w^{(i)},$$

where $y_w^{(i)}$ (say for $|w| \leq B$) are unknowns. Consider the equation:

$$1 = \sum_1^m \sum_{v \in \mathbb{N}^n} \sum_{w \in \mathbb{N}^n} a_v^{(i)} y_w^{(i)} \mathbf{x}^{v+w} = \sum_{u \in \mathbb{N}^n : |u| \leq B} c_u \mathbf{x}^u. \tag{8}$$

This equation has a solution if and only if the following linear equation has a solution,

$$\begin{cases} c_u = 0 & \text{for } 0 < |u| \leq B, \\ c_{(0,\ldots,0)} = 1. \end{cases} \tag{9}$$

The upshot is that the existence of a solution $b_w^{(i)} \in K$ to (8) is seen to be equivalent the existence of a solution to the Hilbert equation bounded by B when putting

$$\lambda_i = \sum_{w \in \mathbb{N}^n} b_w^{(i)} \mathbf{x}^v.$$

Indeed we get that

$$\sum_1^m \lambda_i f_i = \sum_{u \in \mathbb{N}^n : |u| \leq B} c_u \mathbf{x}^u = 1$$

and by construction deg $\lambda_i f_i \leq B$. Conversely if a solution to the Hilbert equation $\lambda_i \in K[x_1, \ldots, x_n]$ with deg $\lambda_i f_i \leq B$ exists then the coefficients of λ_i constitutes a solution to (8). Then we can produce an explicit solution to the Hilbert equation by solving the system of linear equations (9) and we know that a solution to this system of equations exists exactly when a degree bounded solution to the Hilbert equation exists.

Another approach to providing an ENS relies exclusively on Gröbner basis theory. We present this approach here. We first need to reformulate WNS such that the statement can be observed by a Gröbner basis. To do this we give the following definition

Definition 3.2.37. Consider polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. A polynomial $p(\mathbf{x}, y_1 \ldots, y_m) \in K[\mathbf{x}, y_1, \ldots, y_m]$ is called a *final polynomial for* $f_1, \ldots, f_m$ if

1. $p(\mathbf{x}, f_1, \ldots, f_m) = 0$
2. $p(\mathbf{x}, \mathbf{0}) \in K \setminus \{0\}$.

Furthermore, if $p \in K[\mathbf{y}] \subset K[\mathbf{x}, \mathbf{y}]$ is a final polynomial for $f_1, \ldots, f_m$, it is called a *final syzygy* (for $f_1, \ldots, f_m$).

Lemma 3.2.38. *Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then there is a final polynomial for $f_1, \ldots, f_m$ if and only if there is a solution to the Hilbert equation for $f_1, \ldots, f_m$.*

Proof. " $\Longrightarrow$ ": Suppose $p \in K[\mathbf{x}, y_1, \ldots, y_m]$ is a final polynomial for $f_1, \ldots, f_m$. We can then write

$$p = -c + \sum_1^m G_i y_i,$$

for suitable $G_i \in K[\mathbf{x}, \mathbf{y}]$ and $c \in K$. The second defining property of final polynomials shows that $c \neq 0$. Put $\lambda_i = G_i(\mathbf{x}, f_1, \ldots, f_m) \in K[\mathbf{x}]$ for each i. From the first defining property for final polynomials it follows that

$$c = p(\mathbf{x}, f_1, \ldots, f_m) = \sum_1^m G_i(\mathbf{x}, f_1, \ldots, f_m) f_i = \sum_1^m \lambda_i f_i,$$

hence (after scaling by c^{-1}), $\{\lambda_i\}_1^m$ is a solution to the Hilbert equation for $f_1, \ldots, f_m$.
" $\Longleftarrow$ ": Let $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ be a solution to the Hilbert equation for $f_1, \ldots, f_m$. Put $p = 1 - \sum_1^m \lambda_i y_i \in K[\mathbf{x}, \mathbf{y}]$. Then

$$p(\mathbf{x}, f_1, \ldots, f_m) = 1 - \sum_1^m \lambda_i f_i = 1 - 1 = 0,$$

hence p satisfies the first defining property for final polynomials. Secondly, we have that

$$p(\mathbf{x}, \mathbf{0}) = 1 - \sum_1^m \lambda_i \cdot 0 = 1 \in K \setminus 0,$$

hence p is a final polynomial for $f_1, \ldots, f_m$. □

This gives rise to the following reformulation of the weak Nullstellensatz

Theorem 3.2.39. *Let K be algebraically closed and consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then $V(f_1, \ldots, f_m) = \emptyset$ if and only if there exists a final polynomial for $f_1, \ldots, f_m$.*

Proposition 3.2.40. *Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n$, denote the graph ideal of these polynomials by I and let G be a Gröbner basis for I with respect $\leq_{lex}$ with $x_1 > \cdots > x_n > y_1 > \cdots > y_m$. Then $f_1, \ldots, f_m$ has a final syzygy if and only if G contains a final syzygy for $f_1, \ldots, f_m$.*

Proof. Note by Theorem 1.8.106 that $G' = G \cap K[\mathbf{y}] \subset G$ is a Gröbner basis for the ideal $I \cap K[\mathbf{y}]$ with respect to the $\leq_{\text{lex}}$ with $y_1 < \ldots < y_m$. Suppose $p \in K[\mathbf{y}]$ is a final syzygy for $f_1, \ldots, f_m$. We see that $p \in I \cap K[y]$ by Proposition 1.8.47, which implies $\widehat{G} \cap K[\mathbf{y}]$ contains a non-zero polynomial. The final syzygy p is on the form $-[\sum_1^m h_i y_i] + c$ for suitable $h_i \in K[\mathbf{y}]$, $c \in K \setminus 0$ and $p^{G'} = 0$ by Proposition 1.8.84. For this to be true there must be a polynomial $p' \in G'$ with non-zero constant term, implying $p'(0) \in K \setminus 0$. Since $p' \in I$, it satisfies $p'(f_1, \ldots, f_m) = 0$. One thus concludes that p' is a final syzygy for $f_1, \ldots, f_m$. □

To extend this approach to the case where only a final polynomial to exist we need to modify the definition of final polynomial slightly.

Definition 3.2.41. Consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. A polynomial $p(z, \mathbf{x}, \mathbf{y}) \in K[z, \mathbf{x}, y_1, \ldots, y_n]$ is called an *extended final polynomial* if

1. $p(z, \mathbf{x}, zf_1(\mathbf{x}), \ldots, zf_m(\mathbf{x})) = 0$
2. $p(z, \mathbf{x}, 0) = cz \quad (c \in K \setminus \{0\})$.

Remark 3.2.42. It is easy to see that an extended final polynomial $p(z, \mathbf{x}, \mathbf{y})$ for $f_1, \ldots, f_m \in K[\mathbf{x}]$ gives rise to a final polynomial for $f_1, \ldots, f_m$ given by $p(1, \mathbf{x}, \mathbf{y})$.
Conversely, if $f_1, \ldots, f_m$ admits a final polynomial $p(\mathbf{x}, \mathbf{y}) = 1 - \sum_1^m \lambda_i y_i \in K[\mathbf{x}, \mathbf{y}]$ where $\lambda_i \in K[\mathbf{x}]$, we can construct an extended final polynomial for $f_1, \ldots, f_m$ given by $z - \sum_1^m \lambda_i y_i$.

Theorem 3.2.43. *Let $f_1, \ldots, f_m \in K[\mathbf{x}]$ and let $I = \langle zf_1 - y_1, \ldots, zf_m - y_m \rangle \subset K[z, \mathbf{x}, y_1, \ldots, y_m]$. Let $\leq$ be a term order on $K[z, \mathbf{x}, \mathbf{y}]$, where z^k is greater than any monomial in $K[\mathbf{x}, \mathbf{y}]$ for $k \geq 1$ (e.g. $\leq_{lex}$ with $z > x_1 > \cdots > x_n > y_1 > \cdots > y_m$). Consider a Gröbner basis $G \subset K[z, \mathbf{x}, \mathbf{y}]$ of I with respect to $\leq$. Then $V(\mathcal{F}) = \emptyset$ if and only if G contains an extended final polynomial for $f_1, \ldots, f_m$.*

Proof. The first direction is trivial. For the converse statement, suppose $V(\mathcal{F}) = \emptyset$. Then by the Nullstellensatz there exist $\lambda_i \in K[\mathbf{x}]$ such that

$$1 = \sum_1^m \lambda_i f_i,$$

hence

$$p(z, \mathbf{x}, \mathbf{y}) = z - \sum_1^m \lambda_i y_i,$$

defines an extended final polynomial for $f_1, \ldots, f_m$. By Lemma 1.9.56, $p \in I$. Then, since G is a Gröbner basis, there is a $g \in G$ such that $\text{in}_{\leq} g \mid \text{in}_{\leq} p = z$. Again by Lemma 1.9.56, $1 \notin I$, hence we can assume that $\text{in}_{\leq} g = cz$ for some $c \in K \setminus \{0\}$. Since $g \in I$, we have that

$$g = \sum_1^m \mu_i(z, \mathbf{x}, \mathbf{y})(zf_i - y_i),$$

for suitable $\mu_i \in K[z, \mathbf{x}, \mathbf{y}]$. This, for one, implies that $g(z, \mathbf{x}, zf_1, \ldots, zf_m) = 0$. Furthermore, we can rearrange terms in the above expression to write

$$g = \underbrace{z \sum_1^m \mu_i' f_i}_{s_1} - \underbrace{\sum_1^m \mu_i'' y_i}_{s_2},$$

for suitable $\mu_i' \in K[z, \mathbf{x}]$ and $\mu_i'' \in K[z, \mathbf{x}, \mathbf{y}]$. Since $s_2 \in \langle y_1, \ldots, y_m \rangle$ and $s_1 \in K[z, \mathbf{x}, \mathbf{y}] \setminus \langle y_1, \ldots, y_m \rangle$ there is no cancellation between terms of s_1 and s_2. This means $\sum_1^m \mu_i' f_i = c$ and hence $g(z, \mathbf{x}, 0) = cz$, which implies g is an extended final polynomial for $f_1, \ldots, f_m$. □

3.3 A Theory of Affine Varieties

In this section we will assume that any field is algebraically closed.

3.3.1 Morphisms of Affine Varieties: Polynomial maps

Definition 3.3.1. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be affine varieties. A map $\varphi : V \to W$ is called a *polynomial map* if there are polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ such that for $v \in V$,

$$\varphi(v) = (f_1(v), \ldots, f_m(v)).$$

When $W = K$, we call such a polynomial map a *polynomial function*. We denote the set of polynomial maps from V to W by $\mathrm{Pol}(V, W)$

Remark 3.3.2.

1. Note that $\mathrm{Pol}(V, K)$ admits a structure of ring where $0, 1 \in \mathrm{Pol}(V, K)$ are the constant functions mapping every point to respectively 0 and 1 and where for $f, g \in \mathrm{Pol}(V, K)$, $f + g \in \mathrm{Pol}(V, K)$ is defined by

$$(f + g)(v) := f(v) + g(v) \quad (v \in V)$$

and $fg \in \mathrm{Pol}(V, K)$ is defined by

$$(fg)(v) = f(v)g(v) \quad (v \in V).$$

2. Suppose we have polynomials $f_1, \ldots, f_m, g_1, \ldots, g_m \in K[x_1, \ldots, x_n]$ such that for $v \in V$

$$(f_1(v), \ldots, f_m(v)) = \varphi(v) = (g_1(v), \ldots, g_m(v)),$$

then $(f_i - g_i)(v) = f_i(v) - g_i(v) = 0$ for every $v \in V$ and $i \in \{1, \ldots, m\}$, i.e. $f_i - g_i \in I(V)$. Hence a polynomial map is uniquely defined up to $I(V)$-residues.

Example 3.3.3. Here are a few examples of polynomial maps.

1. Consider for $1 \leq i \leq j \leq n$ the map

$$\pi_{i,j} : \mathbb{A}^n \to \mathbb{A}^m$$
$$(v_1, \ldots, v_n) \mapsto (v_i, \ldots, v_j)$$

Note that setting $f_i = x_i \in K[x_1, \ldots, x_n]$ we get that

$$(f_i(v), \ldots, f_j(v)) = (v_i, \ldots, v_j) = \pi_{i,j}(v),$$

for every $v \in \mathbb{A}^n$, hence $\pi_{i,j}$ is polynomial. In particular the projection map $\pi_i = \pi_{i,i}$ is polynomial.

2. Let $\omega \in \mathcal{S}_n$, i.e. a permutation of n elements. Then

$$\varphi : \mathbb{A}^n \to \mathbb{A}^n$$
$$(v_1, \ldots, v_n) \mapsto (v_{\omega(1)}, \ldots, v_{\omega(n)})$$

Setting $f_i = x_{\omega(i)}$ for each i, we get that

$$(f_1(v), \ldots, f_n(v)) = (v_{\omega(1)}, \ldots, v_{\omega(n)}) = \varphi(v),$$

for every $v \in \mathbb{A}^n$.

3. Given a variety $V \subset \mathbb{A}^n$. The identity map is a polynomial map, given by $x_1, \ldots, x_n \in K[x_1, \ldots, x_n]$.

4. A simple example is given varieties $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ and $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, where $f_1(v), \ldots, f_m(v) \in W$. $V \ni v \mapsto (f_1(v), \ldots, f_m(v)) \in W$ is a polynomial map.

Lemma 3.3.4. *Let $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ be varieties and $\varphi : V \to W$ a polynomial map defined by $f_1 \ldots, f_m$. Let $X = V(g_1, \ldots, g_l) \subset \mathbb{A}^m$ be an algebraic set, where $g_1, \ldots, g_l \in K[y_1, \ldots, y_m]$. We then have that*

1. *$\varphi^{-1}(X) \subset V$ is an algebraic set.*

2. *If $\varphi^{-1}(X)$ is a variety and $X \subset$ Im φ, then X is irreducible.*

Proof. 1. We prove that $\varphi^{-1}(X) = V(g_1(f_1, \ldots, f_m), \ldots, g_l(f_1, \ldots, f_m))$. Indeed,

$$\begin{aligned} v \in \varphi^{-1}(X) &\iff (f_1(v), \ldots, f_m(v)) = \varphi(v) \in X \in X = V(g_1, \ldots, g_l) \\ &\iff g_i(f_1, \ldots, f_m)(v) = g_i(f_1(v), \ldots, f_m(v)) = 0 \ \forall i \\ &\iff v \in V(g_1(f_1, \ldots, f_m), \ldots, g_l(f_1, \ldots, f_m)) \end{aligned}$$

2. Suppose $X = Y \cup Z$ for some algebraic sets $Y, Z \subset X$. Then $\varphi^{-1}(Y), \varphi^{-1}(Z)$ are algebraic sets contained in V such that

$$\varphi^{-1}(X) = \varphi^{-1}(Y \cup Z) = \varphi^{-1}(Y) \cup \varphi^{-1}(Z),$$

hence WLOG $\varphi^{-1}(Y) = \varphi^{-1}(X)$. Then since $X \subset \varphi(V)$,

$$Y = \varphi(\varphi^{-1}(Y)) = \varphi(\varphi^{-1}(X)) = X.$$

□

The above result can be used to determine wether an algebraic set is a variety.

Example 3.3.5. 1. Consider the algebraic set $V = \{(t, t^2, t^3) : t \in \mathbb{C}\} \subset \mathbb{A}^3(\mathbb{C})$ from Example 3.2.27. There is an easier way of showing that it is a variety. Indeed consider the surjective polynomial map

$$\varphi : \mathbb{A}^1(\mathbb{C}) \to V$$
$$t \mapsto \left(t, t^2, t^3\right)$$

This has an inverse given by projection onto the first coordinate, i.e. the inverse is the map

$$\varphi^{-1} : V \to \mathbb{A}^1(\mathbb{C})$$
$$\left(t, t^2, t^3\right) \mapsto t$$

This mean $\varphi^{-1}(V) = \mathbb{A}^1(\mathbb{C})$ which is a variety, hence by part 2. of the above proposition V is a variety.

2. Consider $I = \langle f_1, f_2, f_3\rangle \subset \mathbb{C}[x, y, z]$, where $f_1 = xz - y^2$, $f_2 = yz - x^3$, $f_3 = z^2 - x^2y$ and set $V = V(I)$. Consider

$$\varphi : \mathbb{A}(\mathbb{C})^1 \to V$$
$$t \mapsto \left(t^3, t^4, t^5\right)$$

One easily sees that $f_i\left(t^3, t^4, t^5\right) = 0$ for $i = 1, 2, 3$. Let $v \in V$. And let t be any solution to the equation $X^3 - v_1 = 0$. If $v_1 = 0$, then $0 = v_1v_3 - v_2^2 = -v_2^2$, hence $v_2 = 0$. Furthermore $v_3 = v_1^2v_2 = 0$. In this case, we then have $\varphi(0) = v$. Suppose $v_1 \neq 0$. Then $v_1v_3 = v_2^2$, hence $v_3 = v_2^2/v_1$, hence $v_2v_2^2/v_1 = v_2v_3 = v_1^3$, implying $v_2^3 = v_1^4 = t^{12}$, hence t is a solution to $X^4 - v_2 = 0$. Lastly $v_3^2 = v_1^2v_2 = t^6t^4 = t^10$, hence t is a solution to $X^5 - v_3 = 0$. It thus follows that $\varphi(t) = v$. Note that $\varphi^{-1}(V) = V(f_1\left(t^3, t^4, t^5\right), f_2\left(t^3, t^4, t^5\right), f_3\left(t^3, t^4, t^5\right) = V(0) = \mathbb{A}^1(\mathbb{C})$, which is a variety. Again by part 2. of the above theorem V is a variety.

Proposition 3.3.6. *Affine varieties with polynomial maps chosen as morphisms define a category.*

Proof. It is clear that composition of polynomial maps is associative and the identity function $\mathrm{id}_V : \mathbb{A}^n \supset V \ni v \mapsto v \in V$ is the identity morphism, since this map is polynomial (defined by $x_1, \ldots, x_n \in K[x_1, \ldots, x_n]$). □

Proposition 3.3.7. *A polynomial map is continuous in the Zariski topology.*

Proof. Let varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be given. Consider a polynomial map $\varphi : V \to W$ defined by $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Consider a Zariski closed subset $U \subset W$. Then by Proposition 3.3.4 1. $\varphi^{-1}(U)$ is Zariski closed, hence it is known from topology that φ is continuous. □

Definition 3.3.8. A polynomial map $\varphi : V \to W$, where $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are affine varieties is called an *isomorphism (of affine varieties)* if it is bijective and $\varphi^{-1} : W \to V$ is a polynomial map. In other words an isomorphism of affine varieties is a *bi-polynomial map*.

Proposition 3.3.9. *Let $V \subset \mathbb{A}^n$ be an affine variety. Then*

$$\Gamma(V) \simeq \text{Pol}(V, K).$$

Proof. Consider the ring homomorphism

$$\sigma : K[x_1, \ldots, x_n] \to \text{Pol}(V, K)$$
$$f \mapsto (V \ni v \mapsto f(v) \in K),$$

This is obviously surjective. Indeed, a polynomial function $\text{Pol}(V, K)$ is given by $v \mapsto f(v)$ for some $f \in K[x_1, \ldots, x_n]$ by definition. If $f \in K[x_1, \ldots, x_n]$, is given such that $f(v) = \sigma(f)(v) = 0$ for all $v \in V$, then $f \in I(V)$. Hence one sees that $I(V) = \ker \sigma$. Then by Theorem **??**,

$$\Gamma(V) = K[\mathbf{x}]/I(V) = K[\mathbf{x}]/\ker \sigma \simeq \text{Pol}(V, K).$$

□

Definition 3.3.10. Let $V \subset \mathbb{A}^n$,$W \subset \mathbb{A}^n$ be varieties and $\varphi : V \to W$ be a polynomial map defined by polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. The *$K$-algebra homomorphism induced by* φ is the map

$$\widetilde{\varphi} : \Gamma(W) \to \Gamma(V)$$
$$f + I(W) \mapsto f \circ \varphi + I(V)$$

where $f \circ \varphi + I(V) = f(f_1, \ldots, f_m) + I(V)$

Remark 3.3.11. This map is well-defined. Indeed the map described above is due to Proposition 3.3.9 given by the composition

$$\Gamma(W) \xrightarrow{\sim} \text{Pol}(W, K) \to \text{Pol}(V, K) \xrightarrow{\sim} \Gamma(V)$$
$$f + I(W) \mapsto \text{ev}_\bullet(f) \mapsto \text{ev}_\bullet(f) \circ \varphi \mapsto f(f_1, \ldots, f_m) + I(V).$$

Proposition 3.3.12. *Let $V \subset \mathbb{A}^n$,$W \subset \mathbb{A}^n$ be varieties and $\varphi : V \to W$ be a polynomial map defined by $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. $\widetilde{\varphi} \in \mathrm{Hom}^{K-Alg}(\Gamma(W), \Gamma(V))$.*

Proof. The map

$$\sigma : K[y_1, \ldots, y_m] \to \Gamma(V)$$
$$f \mapsto f \circ \varphi + I(V)$$

is clearly a ring homomorphism. Suppose $f \in I(W)$. Then for every $v \in V$, $\varphi(v) \in W$. Hence $(f \circ \varphi)(v) = f(\varphi(v)) = 0$, hence $\sigma(f) = f \circ \varphi \in I(V)$, hence by Corollary **??**, $\widetilde{\varphi}$ is a well-defined ring homomorphism. Let $k \in K$. Then

$$\widetilde{\varphi}(k + I(W)) = k \circ \varphi + I(V) = k + I(V) = k(1 + I(V)) = k\widetilde{\varphi}(1 + I(W))$$

□

Lemma 3.3.13. *The mapping taking a variety V to $\Gamma(V)$ and a polynomial map $\varphi \in \mathrm{Pol}(V, W)$ to $\widetilde{\varphi} \in \mathrm{Hom}^{K-Alg}(\Gamma(W), \Gamma(V))$ is a contravariant functor.*

Proof. For varieties $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$, $U \subset \mathbb{A}^l$, let $\varphi \in \mathrm{Pol}(V, W)$ and $\psi \in \mathrm{Pol}(W, U)$ be defined by polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ and $g_1, \ldots, g_l \in K[y_1, \ldots, y_m]$. Then viewing an arbitrary $f + I(W) \in \Gamma(W)$ as an element $f \in \Gamma(W, K)$, we have that

$$\widetilde{\psi\varphi}(f) = f(\psi\varphi) = (f\psi)\varphi = \widetilde{\varphi}(f\psi) = (\widetilde{\varphi}\widetilde{\psi})(f) \implies \widetilde{\psi\varphi} = \widetilde{\varphi}\widetilde{\psi}.$$

Furthermore we have that

$$\widetilde{\mathrm{id}_V}(f) = f\mathrm{id}_V = f = \mathrm{id}_{\Gamma(V)}f \implies \widetilde{\mathrm{id}_V} = \mathrm{id}_{\Gamma(V)}.$$

□

Theorem 3.3.14. *For varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, the map*

$$\widetilde{\bullet} : Pol(V, W) \ni \varphi \mapsto \widetilde{\varphi} \in Hom^{K-Alg}(\Gamma(W), \Gamma(V)),$$

is bijective.

Proof. Let $\sigma \in \mathrm{Hom}(\Gamma(W), \Gamma(V))$. For each $i \in \{1, \ldots, m\}$ there is some $f_i \in K[x_1, \ldots, x_n]$ such that $\sigma(y_i + I(W)) = f_i + I(V)$ for each $i \in \{1, \ldots, m\}$. Define

$$\psi_\sigma : \mathbb{A}^n \to \mathbb{A}^m$$
$$v \mapsto (f_1(v), \ldots, f_m(v))$$

which is a polynomial map. Note that for $f \in I(W)$

$$\begin{aligned}\widetilde{\psi_\sigma}(f) + I(V) &= f(f_1, \ldots, f_m) + I(V) = f(f_1 + I(V), \ldots, f_m + I(V)) \\ &= f(\sigma(y_1 + I(W)), \ldots, \sigma(y_m + I(W))) = \sigma\left(f\left(y_1 + I\left(W\right), \ldots, y_m + I\left(W\right)\right)\right) \\ &= \sigma(f(y_1, \ldots, y_m) + I(W)) = \sigma(f + I(W)) = \sigma(0 + I(W)) = 0 + I(V).\end{aligned}$$

From this we see that $\widetilde{\psi_\sigma}(I(W)) \subset I(V)$. Note that by HNS $V = V(I(V)$ and $W = V(I(W))$. So given $\psi_\sigma(v) \in \psi_\sigma(V)$, if $f \in I(W)$, then $f(\psi_\sigma(v)) = \psi_\sigma(f)(v) = 0$, hence $\psi_\sigma(V) \subset W$. This implies that

$$\begin{aligned}\varphi_\sigma : V &\to W \\ v &\mapsto \widetilde{\psi_\sigma}(v)\end{aligned}$$

is a well-defined polynomial map. It remains to check that $\sigma \mapsto \varphi_\sigma$ is the mutual inverse of $\varphi \mapsto \widetilde{\varphi}$. Indeed, for $\varphi \in \mathrm{Pol}(V, W)$ defined by $f_1, \ldots, f_m \in K[\mathbf{y}]$, since $\widetilde{\varphi}(y_i + I(W)) = f_i + I(V)$, we have for any $v \in V$ that

$$\varphi_{\widetilde{\varphi}}(v) = (f_1(v), \ldots, f_m(v)) = \varphi(v) \implies \varphi_{\widetilde{\varphi}} = \varphi.$$

Conversely for $\sigma \in \mathrm{Hom}(\Gamma(W), \Gamma(V))$ for any $f + I(W) \in \Gamma(W)$

$$\begin{aligned}\widetilde{\varphi_\sigma}(f + I(W)) &= f \circ \varphi_\sigma + I(W) = f(f_1, \ldots, f_m) + I(W) = f(f_1 + I(W), \ldots, f_m + I(W)) \\ &= f(\sigma(y_1 + I(V)), \ldots, \sigma(y_m + I(V))) = \sigma(f(y_1 + I(V), \ldots, y_m + I(V)) \\ &= \sigma(f(y_1, \ldots, y_m) + I(V)) = \sigma(f + I(V)),\end{aligned}$$

implying $\widetilde{\varphi_\sigma} = \sigma$. □

Remark 3.3.15. We thus have a contravariant functor $(\Gamma(\bullet), \widetilde{\bullet})$ between the category of affine varieties over K with polynomial maps and integral finitely generated K-algebras mapping a variety V to $\Gamma(V)$ and a polynomial map $\varphi \in \mathrm{Pol}(V, W)$ to $\widetilde{\varphi} \in \mathrm{Hom}(\Gamma(W), \Gamma(V))$. The above theorem shows that this functor is fully faithful. The functor is also essentially surjective on objects. Indeed, if A is a finitely generated K-algebra that is also an integral domain, then $A \simeq K[x_1, \ldots, x_n]/I$ for some prime ideal $I \subset K[x_1, \ldots, x_n]$, hence $A \simeq K[x_1, \ldots, x_n]/I(V(I)) = \Gamma(V(I))$. We can thus conclude that $\Gamma(\bullet)$ defines an equivalence of categories.

Remark 3.3.16. With the above in mind, we can reformulate Proposition 3.3.9 in the following way: There is a natural isomorphism

$$\Gamma \equiv \mathrm{Pol}(_, K)$$

whose components are

$$\Gamma(V) = K[\mathrm{x}]/I(V) \to \mathrm{Pol}(V,K)$$
$$f + I(v) \mapsto (v \mapsto f(v))$$

Corollary 3.3.17. *Let $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ be varieties. Then*

$$V \simeq W \iff \Gamma(V) \simeq \Gamma(W),$$

where $\simeq$ refers to an isomorphism of varieties, resp. K-algebras.

Proposition 3.3.18. *Let $V \subset \mathbb{A}^n$ be a variety and $W \subset V$ a subvariety. The ideal $I_V(W) \subset \Gamma(V)$ corresponds to the ideal of polynomial functions in $\mathrm{Pol}(V,K)$ vanishing on W, which we therefor also denote by $I_V(W)$. Thus the map $\mathrm{Pol}(V,K) \to \mathrm{Pol}(W,K), f \mapsto f\mid_W$ has kernel $I_V(W)$.*

Proof. This follows from the above proposition and Proposition 3.2.33. □

Proposition 3.3.19. *Let $\varphi : V \to W$ be a polynomial map and $V' \subset V$, $W' \subset W$ subvarieties. Suppose $\varphi(V') \subset W'$. Thus $\varphi\mid_{V'}: V' \to W'$ is a polynomial map. Furthermore $\widetilde{\varphi}(I_W(W')) \subset I_V(V')$.*

Proof. Let $f + I(W) \in I(W')/I(W)$ and $v \in V'$. Then $\varphi(v) \in W'$, hence $f \circ \varphi(v) = 0$, meaning $f \circ \varphi \in I(W')$, hence $\widetilde{\varphi}(f + I(W)) = f \circ \varphi + I(V) \in I_V(V')$. □

Lemma 3.3.20. *Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic set and $\varphi : V \to W$ be a map such that $\varphi(v) = (f_1(v), \ldots, f_m(v))$ for $v \in V$ where $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. We then have*

1. *The graph of φ,*

$$G_\varphi := \{(v_1, \ldots, v_n, \varphi(v)) \in \mathbb{A}^{n+m} : v \in V\} \subset V \times W,$$

   is an algebraic set.

2. *If V, W are varieties, φ is a polynomial map. In this case G_φ is a variety isomorphic to V*

Proof. For suitable $I = \langle g_1, \ldots, g_l\rangle \subset K[x_1, \ldots, x_n]$, $V = V(I)$.
1. One easily verifies that $G_\varphi = V(g_1, \ldots, g_l, y_1, \ldots, y_m)$. Indeed, if $(v, w) \in G_\varphi$, $v \in V$, and $w_i = f_i(v)$, hence $g_i(v, w) = 0$ (here we think of g_i in the canonical way as an element of $K[\mathbf{x}, \mathbf{y}]$) and $\mathrm{ev}_{(v,w)}(y_i - f_i) = f_i(v) - f_i(v) = 0$. Conversely, if (v, w) is in the right-hand side. Then $g_i(v) = 0$ for every i, hence $v \in V$. Furthermore $w_i - f_i(v) = 0$, hence

$w_i = f_i(v)$, hence $w = \varphi(v)$.
2. By HNS we can choose I to be a prime ideal. Let $J = \langle g_1, \ldots, g_l, y_1 - f_1, \ldots, y_m - f_m \rangle \subset K[\mathbf{x}, \mathbf{y}]$. By Corollary 1.8.50

$$K[\mathbf{x}]/I \simeq K[\mathbf{x}, \mathbf{y}]/J,$$

hence J is prime. It follows that $G_\varphi = V(J) \subset \mathbb{A}^{n+m}$ is a variety. We can see more from this isomorphism. By HNS $I = I(V)$ and $J = I(G_\varphi)$, thus $\Gamma(V) \simeq \Gamma(G_\varphi)$, hence by Corollary 3.3.17 $V \simeq G_\varphi$. Another way to explicitly construct an isomorphism. Is to consider $\phi : V \to G_\varphi, v \mapsto (v, \varphi(v))$ and $\pi : G_\varphi \to V, (v, \varphi(v)) \mapsto v$. These are clearly mutually inverse polynomial maps. □

Example 3.3.21. Let us consider a couple of examples of a bijective polynomial map that is NOT isomorphisms of varieties and a central example of an elliptic curve "almost" being in bijection with the affine line.

1. Consider the map

$$\varphi : \mathbb{A}^1 \to V := V(y^2 - x^3)$$
$$t \mapsto \left(t^2, t^3\right)$$

This is clearly a well-defined polynomial map. Let $(a, b) \in V$. Let t be a solution to the equation $X^2 - a = 0$. Then $t^2 = a$. Moreover, $b^2 = a^3 = t^6$, hence $\pm t$ is a solution to $X^3 - b = 0$, since then $(b + t^3)(b - t^3) = 0$. Suppose $(t^2, t^3) = (s^2, s^3)$. Then $0 = t^2 - s^2 = (t + s)(t - s)$. Then either $t = -s$ or $t = s$. In the first case, we get $s^3 = t^3 = -s^3$, hence $2s^3 = 0$, implying $s = 0$, and hence $t = 0$. It follows that φ is a bijection. Note that $y^2 - x^3$ is irreducible hence $\Gamma(V) = K[x, y]/I$. The induced K-algebra homomorphism is given by

$$\widetilde{\varphi} : \Gamma(V) \to K[z]$$
$$f + I \mapsto f(z^2, z^3)$$

One notes that $\widetilde{\varphi}(\Gamma(V)) = K[z^2, z^3] \subsetneq K[z]$, meaning that $\widetilde{\varphi}$ is not an isomorphism. By Corollary 3.3.17 it follows that φ cannot be an isomorphism.

2. This is a continuation of Example 3.2.36. In this example we considered $f = x^2 - y^3$, $g = y^2 - z^3$,$I = \langle f, g \rangle$ and saw that $V := V(I)$ was irreducible by showing that I is the kernel of $\alpha := \mathrm{ev}_{T^9, T^6, T^4} : K[x, y, z] \to K[T]$. One recovers a polynomial map $\varphi : \mathbb{A}^1 \to V$ by applying the inverse functor to $\overline{\alpha} : \Gamma(V) = K[x, y, z]/I \to K[T] = \Gamma(\mathbb{A}^1)$. Then $\varphi(s) := (s^9, s^6, s^4)$, since $\widetilde{\varphi}(f + I) = f(T^9, T^6, T^4) = \overline{\alpha}$. Suppose

$(\mu^9, \mu^6, \mu^4) = (\nu^9, \nu^6, \nu^4)$. $\mu^4 = \nu^4$ implies $\mu^2 = \pm\nu^2$. $(\pm\nu^2)^3 = (\mu^2)^3 = \mu^6 = \nu^6 = (\nu^2)^3$ implies $\mu^2 = \nu^2$, hence $\mu = \pm\nu$. Then $(\pm\nu)^9 = \mu^9 = \nu^9$, hence $\mu = \nu$. Let $(\alpha, \beta, \gamma) \in V(I)$. Note that

$$\alpha = \pm\beta^{3/2} = \pm(\pm\gamma^{3/2})^{3/2} = \pm\beta^{3/2} = (\pm(\pm\gamma^{1/2})^{1/2})^9.$$

Note also that

$$\beta = \pm\gamma^{3/2} = (\pm\gamma^{1/2})^3 = (\pm\gamma^{1/2})^{6/2} = (\pm(\pm\gamma^{1/2})^{1/2})^6.$$

So picking

$$t = \begin{cases} (\gamma^{1/2})^{1/2} & \text{if } \beta = \gamma^{3/2}, \alpha = \beta^{3/2} \\ -(-\gamma^{1/2})^{1/2} & \text{if } \beta = -\gamma^{3/2}, \alpha = -\beta^{3/2} \\ -(\gamma^{1/2})^{1/2} & \text{if } \beta = \gamma^{3/2}, \alpha = -\beta^{3/2} \\ (-\gamma^{1/2})^{1/2} & \text{if } \beta = -\gamma^{3/2}, \alpha = \beta^{3/2} \end{cases},$$

we get that $\varphi(t) = (\alpha, \beta, \gamma)$. Then φ is a bijection. However the image of $\widetilde{\varphi} = \overline{\alpha}$ is $K[T^9, T^6, T^4] \subsetneq K[T]$, hence φ is not an isomorphism by the full faithfulness of $(\Gamma(\bullet), \widetilde{\bullet})$.

3. Consider the polynomial map

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to V := V(y^2 - x^2(x+1)) \\ t &\mapsto (t^2 - 1, t(t^2 - 1)) \end{aligned}$$

One readily verifies that this is well-defined. Let $(a, b) \in V$. Let t be a solution to $X^2 - a + 1$. Then

$$b^2 = (t^2 - 1)^2 t^2 \implies b = \pm t(t^2 - 1).$$

It thus follows that

$$\varphi(\pm t) = (t^2 - 1, \pm t(t^2 - 1)) = (a, b).$$

Suppose $(t^2 - 1, t(t^2 - 1)) = (s^2 - 1, s(s^2 - 1))$ for $t \neq \pm 1$. Then $t^2 = s^2$, hence $s = \pm t$. If $s = -t$, then

$$0 = t(t^2 - 1) - s(s^2 - 1) = -2t(t^2 - 1) \implies t = 0 \text{ or } t^2 - 1 = 0.$$

by the assumption $t \neq \pm 1$, one concludes $s = t = 0$. Note that $\varphi(\pm 1) = (0, 0)$. One thus concludes that φ is onto and injective but for a double point $(0, 0)$.

3.3.2 (Affine) Coordinate Changes

Definition 3.3.22. Let $I = \langle g_1, \ldots, g_l \rangle \subset K[y_1, \ldots, y_m]$ and $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ a polynomial map given by polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_m]$. We define

$$I(\varphi) = \langle \{ f(f_1, \ldots, f_m) \in K[x_1, \ldots, x_n] : f \in I \} \rangle$$

Remark 3.3.23. One easily checks that

$$\langle \{ f(f_1, \ldots, f_m) \in K[x_1, \ldots, x_n] : f \in I \} \rangle = \langle g_1(f_1, \ldots, f_m), \ldots, g_m(f_1, \ldots, f_m) \rangle.$$

Note that the set $I(\varphi)$ is independent of choice of representative of φ, since φ is uniquely determined by it's defining polynomials. Let $W \subset \mathbb{A}^m$ be a variety. Note that $V^{\varphi} := \varphi^{-1}(W) = V(I(\varphi))$, setting $I := I(W)$.

Definition 3.3.24. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ given by polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. φ is called *an affine change of coordinates*, if $\deg\ f_i = 1$ and φ is bijective for each i.

Remark 3.3.25. For a moment let us remove the condition that φ is bijective. We may write $f_i = b_i + \sum_{j=1}^n a_{ij} x_i$, hence for each $v \in \mathbb{A}^n$ we get

$$\varphi(v) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} v + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

In other words φ is a given by a composition $t \circ l$, where $t : \mathbb{A}^n \to \mathbb{A}^n$ is a translation and $l : \mathbb{A}^n \to \mathbb{A}^n$ is a linear map. It follows that φ is an affine change of coordinates if and only if l is bijective, as t is always bijective. Note that the inverse of a linear map and a translation is also respectively a linear map and a translation, hence in particular these are polynomial maps. An affine change of coordinates is there automatically an isomorphism of affine varieties.

Definition 3.3.26. A variety $V \subset \mathbb{A}^n$ is called a *linear subvariety* if $V = V(f_1, \ldots, f_m)$ for degree 1 polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$

Lemma 3.3.27. *Let R be a commutative ring. Let $f_1, \ldots, f_n, g \in R[x_1, \ldots, x_n]$ with $f_i = a_i + \sum_{j=1}^n a_{ij} x_j$ such that $(a_{ij}) \in M_n(R)$ is invertible and $g = c + \sum_{i=1}^n b_i x_i \in R[x]$. Then*

$$g(f_1, \ldots, f_n) = 0 \iff g = 0.$$

If $\deg\ g = 1$, *then* $\deg\ g(f_1, \ldots, f_n) = 1$

Proof. " $\Longleftarrow$ ": This is obvious.

" $\Longrightarrow$ ": Note that

$$\begin{aligned} g(f_1,\dots,f_n) &= c + \sum_{i=1}^{n} b_i \left(a_i + \sum_{j=1}^{n} a_{ij} x_j \right) = c + \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_{ij} x_j \\ &= c + \sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{n} \left[\sum_{i=1}^{n} b_i a_{ij} \right] x_j. \end{aligned}$$

Thus

$$g(f_1,\dots,f_n) = 0 \iff \begin{cases} c + \sum_{i=1}^{n} a_i b_i = 0, \\ \sum_{i=1}^{n} b_i a_{ij} = 0 \end{cases}$$

Note that

$$\begin{pmatrix} \sum_{i=1}^{n} b_i a_{i1} \\ \vdots \\ \sum_{i=1}^{n} b_i a_{in} \end{pmatrix} = \mathbf{0} \iff (a_{ij})^T \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{0} \iff \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{0}. \tag{10}$$

We thus also have that $c = 0$, hence $g = 0$. If $\deg\ g = 1$, then $b_i \neq 0$ for some i. Then $\sum_1^n b_i a_{ij} \neq 0$ for some j by (10). □

Lemma 3.3.28. *1. Let $\varphi, \psi : \mathbb{A}^n \to \mathbb{A}^n$ be polynomial maps and $V \subset \mathbb{A}^n$ a variety such that V^φ is a variety. Then*

$$(V^\varphi)^\psi = V^{\varphi\circ\psi}.$$

2. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n, v \mapsto (f_1(v),\dots,f_n(v))$ be an isomorphism. Let $V = V(g_1,\dots,g_m) \subset \mathbb{A}^n$ be a variety. Then

$$(V^\varphi)^{\varphi^{-1}} = V.$$

3. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map. Then

$$\emptyset^\varphi = \emptyset.$$

Proof. Let $V = V(f_1,\dots,f_m)$.

1. Then

$$\begin{aligned} (V^\varphi)^\psi &= V(f_1\circ\varphi,\dots,f_m\circ\varphi)^\psi = V((f_1\circ\varphi)\circ\psi,\dots,(f_m\circ\varphi)\circ\psi) \\ &= V(f_1\circ(\varphi\circ\psi),\dots,f_m\circ(\varphi\circ\psi)) = V^{\varphi\circ\psi}. \end{aligned}$$

2. From 1. we find

$$(V^\varphi)^{\varphi^{-1}} = V^{\varphi\circ\varphi^{-1}} = V^{\mathrm{id}} = V(f_1\circ\mathrm{id},\dots,f_m\circ\mathrm{id}) = V(f_1,\dots,f_m) = V.$$

□

Proposition 3.3.29. *Let $V \subset \mathbb{A}^n$ be a linear subvariety.*

1. *If $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ given by $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$ is an affine change of coordinates, then $V^\varphi \subset \mathbb{A}^n$ is a linear subvariety.*

2. *If $V \neq \emptyset$, then there is an affine change of coordinates $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ such that $V^\varphi = V(x_{d+1}, \ldots, x_n)$ for some $1 \leq d \leq n$. Moreover, it thus follows that a linear subvariety is a variety.*

3. *The integer d above is unique, i.e. independent of choice of affine change of coordinates.*

Proof. Let degree 1 polynomials $g_1, \ldots, g_m \in K[x_1, \ldots, x_n]$ be given such that $V = V(g_1, \ldots, g_m)$.
1. Since deg $g_i = 1$ for each i, deg $f_j(g_1, \ldots, g_n) = 1$ by Lemma 3.3.27, hence $V^\varphi = V(g_1(f_1, \ldots, f_n), \ldots, g_m(f_1, \ldots, f_n))$ is a linear subvariety.
2. For the case $n = 1$, note $g_i = g_j$ for every i and j, hence $V = V(b + ax)$ where $a \neq 0$. Let $\varphi : v \mapsto a^{-1}v - a^{-1}b$. Then $g \circ \varphi = aa^{-1}x - a^{-1}b + b = x$, hence $V^\varphi = V(x_n)$.
We prove the result for $n \geq 2$ by induction in m.
Consider first the case $m = 1$. Let $V = V(g)$, where $g = b + \sum_1^n a_i x_i \in K[\mathbf{x}]$ is of degree 1. WLOG $a_n \neq 0$. Put $f_i := a_n x_i$ for $i \in \{1, \ldots, n-1\}$ and $f_n := \left[-\sum_1^{n-1} a_i x_i\right] + a_n^{-1}x_n - a_n^{-1}b$. Then

$$g(f_1, \ldots, f_n) = b + \sum_1^n a_i f_i = b + \sum_1^{n-1} a_i a_n x_i - \left[\sum_1^{n-1} a_i a_n x_i\right] + a_n a_n^{-1} x_n - a_n a_n^{-1} b = x_n.$$

Choosing $\varphi : \mathbb{A}^n \to \mathbb{A}^n, v \mapsto (f_1(v), \ldots, f_n(v))$, one finds that $V^\varphi = V(g(f_1, \ldots, f_n)) = V(x_n) = V(x_{n-1+1})$. It remains to check that φ is invertible. Note that for each $v \in \mathbb{A}^n$, that $\varphi = (a_{ij})v + be_n$, where $a_{ii} = a_n$, $a_{ni} = -a_i$ for $i \in \{1, \ldots, n-1\}$, $a_{nn} = a_n^{-1}$. A is a lower triangular matrix, hence det $A = \prod_1^n a_{ii} = a_n^{n-2} \neq 0$, meaning A is invertible.
Suppose the statement is true for some $m \geq 1$. Consider $V = V(g_1, \ldots, g_{m+1})$ for degree 1 polynomials $g_1, \ldots, g_{m+1} \in K[\mathbf{x}] \setminus 0$. Consider $W = V(g_1, \ldots, g_m)$. There is an affine change of coordinates $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ such that $W^\varphi = V(x_{d+1}, \ldots, x_n)$ for some $1 \leq d \leq n$. Then

$$V^\varphi = V(x_{d+1}, \ldots, x_n) \cap V(g_{m+1} \circ \varphi)) = V(x_{d+1}, \ldots, x_n) \cap V(\underbrace{(g_{m+1} \circ \varphi)(x_1, \ldots, x_d, \mathbf{0})}_{h}).$$

If $h = 0$, $V^\varphi = V(x_1, \ldots, x_d)$. Otherwise deg $h = 1$, since if not $V(h) = \emptyset$, hence $V^\varphi = \emptyset$, hence $V = \emptyset$leading to a contradiction (here we only rely on the fact that φ is bijective). In

this case there following the same procedure as for $m = 1$ is a polynomial map $\psi : \mathbb{A}^d \to \mathbb{A}^d$, such that $V(h) = V(x_d)$. Setting $\phi : \mathbb{A}^n \to \mathbb{A}^n, v \mapsto (\psi(v_1, \ldots, v_d), v_{d+1}, \ldots, v_n)$, it follows that

$$V^{\varphi\circ\phi} = V(x_{d+1}, \ldots, x_n)^{\phi} \cap V(h \circ \phi) = V(x_d, \ldots, x_n),$$

hence $d - 1$ works.

3. Suppose there are affine change of coordinates φ, ϕ such that $V^{\varphi} = V(x_{d+1}, \ldots, x_n)$ and $V^{\phi} = V(x_{\delta+1}, \ldots, x_n)$. Hence setting $\psi := \varphi^{-1} \circ \phi$, one gets $(V^{\varphi})^{\psi} = V^{\phi}$.

One observation that could make now is that since ψ is an isomorphism,

$V^{\varphi} \simeq V^{\phi}$. Then

$$K[x_1, \ldots, x_d] \simeq \Gamma\left(V^{\varphi}\right) \simeq \Gamma\left(V^{\phi}\right) \simeq K[x_1, \ldots, x_{\delta}].$$

This then implies that $\delta = d$ using that transcendence degrees must agree.

Here we are invoking a lot of theory. One can take a more elementary approach to arguing $d = \delta$ only using linear algebra. WLOG $d \leq \delta$. Pick $f_1, \ldots, f_n \in K[\mathbf{x}]$ such that $\psi(v) = (f_1(v), \ldots, f_n(v))$. Write $f_i = b_i + \sum_{j=1}^{n} a_{ij} x_j$. Then

$$\psi(v) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} v + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \left(v \in \mathbb{A}^n\right).$$

Note that (a_{ij}) is invertible, hence it's rows are linearly independent. Consider the polynomial map,

$$\kappa : \mathbb{A}^n \to \mathbb{A}^{n-d}$$
$$v \mapsto (f_{d+1}(v), \ldots, f_n(v)).$$

Note that

$$\kappa(v) = \underbrace{\begin{pmatrix} a_{d+1,1} & \cdots & a_{d+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{A} v + \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{\mathbf{b}} \quad \left(v \in \mathbb{A}^{n-d}\right).$$

Then $A = \kappa - \mathbf{b}$ is a linear map whose kernel is 0. We also have that

$$\{v \in \mathbb{A}^n : v = (v_1, \ldots, v_{\delta}, \mathbf{0})\} = V(x_{\delta+1}, \ldots, x_n) = V(x_{d+1}, \ldots, x_n)^{\psi} = \{v \in \mathbb{A}^n : Av = -\mathbf{b}\}.$$

If $d < \delta$, we get that $e_d, e_{d+1} \in V(x_{d+1}, \ldots, x_n)^{\psi}$. Here is the catch

$$0 = \psi(-e_j) = -\begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{pmatrix} + \mathbf{b} \iff \begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{pmatrix} = \mathbf{b}.$$

This implies that $A(e_d - e_{d+1}) = \mathbf{b} - \mathbf{b} = 0$, which contradicts the fact that ker $A = 0$, since $e_d \neq e_{d+1}$. We conclude that $d = \delta$. □

Remark 3.3.30. A further note on V^{φ}, is that since the pre-image is equal to the image of the inverse of φ, we have that if $V^{\varphi} = W$ for some variety W, then $V = \varphi(W) = W^{\varphi^{-1}}$.

Definition 3.3.31. The integer k above is called the *dimension* of V, denoted dim V.

Remark 3.3.32. One easily sees that the concept of dimension defined above generalizes the concept of dimension of finite dimensional vector spaces. We shall see later that the above notion of dimension can be generalized to any non-empty variety, through the concept of transcendence degree or equivalently Krull-dimension.

Lemma 3.3.33. *Consider linear subvarieties $V, W \subset \mathbb{A}^n$ such that there exists an affine change of coordinates $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ with $\varphi(V) = W$. Then $V \overset{\varphi}{\simeq} W$ and* dim V = dim W.

Proof. □

Definition 3.3.34. Let $v, w \in \mathbb{A}^n$ be distinct points. The *line through v and w* is the set

$$L(v, w) := \{t(w - v) + v : t \in K\}$$

Remark 3.3.35. $L(v, w)$ is clearly a line in the sense of Definition 3.1.28

Lemma 3.3.36. *Let $v, w \in \mathbb{A}^n$ be distinct points and $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ an affine change of coordinates. Then*

$$\varphi(L(v, w)) = L(\varphi(v), \varphi(w)).$$

Proof. Pick $A \in M_n(K)$ and $u \in \mathbb{A}^n$ such that $\varphi(x) = Ax + u$ for $x \in \mathbb{A}^n$. Let $t \in K$. The statement follows directly from the following computation

$$t(\varphi(w) - \varphi(v)) + \varphi(v) = t(Aw + u - Av - u) + Av + u = A(t(w - v) + v) + u = \varphi(t(w - v) + v).$$

□

Lemma 3.3.37. *A line $L \subset \mathbb{A}^n$ is a linear subvariety of dimension* 1.

Proof. Lemma 3.1.29 shows that L. WLOG

$$L = \left\{ \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \\ \vdots \\ a_n t + b_n \end{pmatrix} \in \mathbb{A}^n : t \in K \right\},$$

where $a_1 \neq 0$. From the proof of Lemma 3.1.29 we see that

$$L = V(\{\underbrace{a_1x_i - a_ix_1 - a_1b_i + a_ib_1}_{g_i} : i \in \{2, \dots, n\}).$$

Let $A = (a_{ij}) \in M_n(K)$ be the matrix with $a_{11} = 1$, $a_{ii} = a_1^{-1}$ for $i > 1$, $a_{i1} = a_1^{-1}a_i$, $a_{ij} = 0$ for $j > 1$ and $i \neq j$. Since (a_{ij}) is lower triangular, $\det(a_{ij}) = \prod_1^n a_{ii} = a_1^{-(n-1)} \neq 0$, meaning (a_{ij}) is invertible. Set $w = (b_1, \dots, b_n)^T$. Then $\varphi : \mathbb{A}^n \to \mathbb{A}^n, v \mapsto Av + w$ is an affine change of coordinates, given by $f_1 = x_1 + b_1$ and $f_i = a_1^{-1}x_i + a_1^{-1}a_ix_1 + b_i$. We thus find that

$$g_i \circ \varphi = a_1a_1^{-1}x_i + a_1a_1^{-1}a_ix_1 + a_1b_i - a_ix_1 - a_ib_1 - a_1b_i + a_ib_1 = x_i,$$

for $i > 1$. We thus find that

$$L^\varphi = V(g_2 \circ \varphi, \dots, g_n \circ \varphi) = V(x_2, \dots, x_n),$$

hence dim $L = 1$. □

Proposition 3.3.38. *Let $v, v' \in \mathbb{A}^2$, L_1, L_2 distinct lines through v and L_1', L_2' distinct lines through v'. There is an affine change of coordinates $\mathbb{A}^2$ such that $\varphi(v) = v'$ and $\varphi(L_i) = L_i'$ for both i.*

Proof. There are distinct $w_1, w_2, w_1', w_2' \in \mathbb{A}^2$ such that $L_i = L(v, w_i)$ for $i = 1, 2$ and $L_i' = L(v, w_i')$ for $i = 1, 2$. We first perform a translation $T : \mathbb{A}^2 \to \mathbb{A}^2, x \mapsto Ix - v$. Then by Lemma 3.3.36,

$$\Lambda_i := T(L_i) = L(0, \underbrace{w_i - v}_{u_i}),$$

and since T is bijective $\Lambda_1 \neq \Lambda_2$. Note also that $u_1 \neq u_2$. Since Λ_1 and Λ_2 are distinct and intersect at 0, $tu_1 \neq su_2$ for each $s, t \in K \setminus 0$ by Proposition 3.1.22, hence u_1 and u_2 are algebraically independent. This means that the matrix $u = (u_{ij}) \in M_2(K)$ is invertible. Then

$$\Lambda_i' := u^{-1}\Lambda_i = L(0, e_i).$$

One notes that $w_1' - v', w_2' - v'$ are linearly independent again since $L(0, w_1' - v')$ intersect only in 0. Thus taking $\psi : \mathbb{A}^n \to \mathbb{A}^n, x \mapsto (w_{ij}' - v_j')x + v'$. Then

$$\psi(0) = v', \psi(e_i) = \begin{pmatrix} w_{i1}' - v_1 \\ w_{i2}' - v_2 \end{pmatrix} + v' = w_i' - v' + v' = w_i',$$

hence putting $\varphi = \psi \circ u \circ T$,

$$\varphi(v) = v', \varphi(w_i) = w_i', \varphi(L_i) = L_i'.$$

□

Lemma 3.3.39. *Let $\mathbb{A}^n(\mathbb{C})$ be equipped with the usual metric induced topology. For any countable set $S \subset \mathbb{A}^n(\mathbb{C})$, $\mathbb{A}^n(\mathbb{C}) \setminus S$ is path-connected.*

Proof. Let $S = \{v_1, \ldots, v_m\}$ we prove the statement by induction in m. for $m = 1$, let $p, q \in \mathbb{A}^n(\mathbb{C}) \setminus S$ be given suppose L is the line parametrized by $[0,1] \to \mathbb{A}^n(\mathbb{C}), t \mapsto (1-t)p + tq$. If $v_1 \notin L$ we are done. If $v_1 \in L$, pick a point $w \in \mathbb{C} \setminus L$. Let $\psi : [0,1] \to \mathbb{C}$ be the composition of the line from p to w with the line from w to q. The line segments parametrized by ψ_i have each exactly one intersection with L at p respectively q. One sees this from the fact that these three line segments are each respectively subsets of $L(p,q), L(p,w)$ and $L(w,q)$ (as lines in $\mathbb{A}^n(\mathbb{C})$ viewed as $\mathbb{A}^{2n}(\mathbb{R})$) and it follows from Proposition 3.1.31 that $\#\left(L(p,q) \cap L(p,w)\right) = 1$ and $\#\left(L(p,q) \cap L(w,q)\right) = 1$. It thus follows that v is not an element in the line path ψ, hence $\mathbb{A}^n(\mathbb{C})$ is path-connected in the case $m = 1$.
Suppose for each set $S = \{v_1, \ldots, v_m\}$, $p, q \in \mathbb{A}^n(\mathbb{C}) \setminus S$ there is a finite composition of line segments in $\mathbb{A}^n(\mathbb{C})$ connecting p and q.
Let $S = \{v_1, \ldots, v_{m+1}$ and $p, q \in \mathbb{A}^n(\mathbb{C}) \setminus S$, let ψ be the composition of line segments in $A^n(\mathbb{C}) \setminus \{v_1, \ldots, v_m\}$. If $v_{m+1} \notin \psi$ we are done. If $v_{m+1} \in \psi$, let x, y be the respective start and end point of the line segment containing v_{m+1} denoted ρ. We can write this line segment as the composition of a line ρ_1 from x to v_{m+1} and ρ_2 from v_{m+1} to y to Let $0 < \epsilon < \min_{i \in \{1,\ldots,m\}} |v_i - v_{m+1}|$. By continuity of ρ_1 and ρ_2 there are points $x' \in B_\epsilon(v_{m+1}) \cap \rho_1$ and $y' \in B_\epsilon(v_{m+1}) \cap \rho_2$ distinct from v_{m+1}. Let ρ_1' be the line segment from x to x' and ρ_2' be the line segment from y' to y. Consider a third point $z \in B_\epsilon(v_{m+1})$ distinct from x', y', v. Let ξ_1 and ξ_2 be the line segment from x' to z respectively z to y'. By convexity of $B_\epsilon(v)$, $\xi_i \subset B_\epsilon(v)$ for $i = 1, 2$. We thus get a path $\rho' := \rho_1' \cup \xi_1 \cup \xi_2 \cup \rho_2'$ from x to y not containing v_{m+1} or indeed any $v_1, \ldots, v_m$ by the choice of ϵ. Replacing ρ by ρ' we get a composition of line segments in $\mathbb{A}^n(\mathbb{C}) \setminus S$ from p to q. Thus $\mathbb{A}^n(\mathbb{C}) \setminus S$ is path-connected. □

Proposition 3.3.40. *Let $V \subset \mathbb{A}^n(\mathbb{C})$ be Zariski closed. $\mathbb{A}^n(\mathbb{C}) \setminus V$ is path-connected with respect to the metric induced topology.*

Proof. Let $p, q \in \mathbb{A}^n(\mathbb{C}) \setminus V$. $L(p,q) \cap V = \{w_1, \ldots, w_m\}$ by Corollary 3.1.32. By Lemma 3.3.37, $L(p,q) \simeq V(x_2, \ldots, x_n) \simeq \mathbb{A}^1(\mathbb{C})$. Note that a polynomial map $\varphi : \mathbb{A}^k(\mathbb{C}) \to \mathbb{A}^l(\mathbb{C})$ is continuous with respect to the usual metric topologies on $\mathbb{A}^k(\mathbb{C})$ and $\mathbb{A}^l(\mathbb{C})$. Hence $L(p,q)$ is homeomorphic to $\mathbb{A}^1(\mathbb{C})$ via a map, ψ, say. Let $S := \psi(L(p,q) \cap V) = \{v_1, \ldots, v_m\}$. Then $\mathbb{A}^1(\mathbb{C}) \setminus S$ is path-connected by Lemma 3.3.39, hence $L(p,q) \setminus V = L(p,q) \setminus (L(p,q) \cap V)$ is path-connected, meaning $\mathbb{A}^n(\mathbb{C}) \setminus V \supset L(p,q) \setminus V$ is path-connected. □

3.3.3 The Field of Rational Functions on a Variety & the Local Ring of Rational Functions Defined at a Point

Definition 3.3.41. Given a non-empty variety $V \subset \mathbb{A}^n$, we define the *the field of rational functions on* V to be the field

$$K(V) := Q(\Gamma(V)).$$

The elements of $K(V)$ are called *rational functions on* V.

Remark 3.3.42. This is well-defined by Proposition 3.2.6. The term rational functions is well chosen as $\Gamma(V) \simeq \text{Pol}(V, K)$ (cf. Proposition 3.3.9), hence indeed

$$K(V) = Q(\text{Pol}(V, K)) = \left\{ \frac{f}{g} : f, g \in \text{Pol}(V, K), g \neq 0 \right\}.$$

Definition 3.3.43. For a variety $V \subset \mathbb{A}^n$ and a point $v \in V$, a rational function $f \in K(V)$ is *defined at* v if there are $a, b \in \Gamma(V)$ with $f = \frac{a}{b}$ such that $b(v) \neq 0$.

Remark 3.3.44. If $\Gamma(V)$ is a UFD there is are unique (up to scalar multiplication over K) polynomial functions $a, b \in \Gamma(V)$ with $\gcd(a, b) = 1$ such that $f = \frac{a}{b}$. Hence f is defined at v if and only if $b(v) \neq 0$.

Definition 3.3.45. Let $V \subset \mathbb{A}^n$ be a variety and $P \in \mathbb{A}^n$. *The local ring of* V *at* P is the ring

$$\mathcal{O}_P(V) := \{q \in K(V) : q \text{ defined at } P\}$$

Remark 3.3.46. One readily verifies that $\mathcal{O}_P(V)$ is a subring of $K(V)$ containing $\Gamma(V)$. Indeed, if $\frac{f}{g}, \frac{f'}{g'} \in \mathcal{O}_P(V)$, then $gg'(P) \neq 0$

$$\frac{f}{g} + \frac{f'}{g'} = \frac{fg' + f'g}{gg'} \in \mathcal{O}_P(V).$$

Furthermore,

$$\frac{f}{g}\frac{f'}{g'} = \frac{ff'}{gg'} \in \mathcal{O}_P(V).$$

Let $f \in \Gamma(V)$. Then $f = \frac{f}{1}$, and since $1(P) \neq 0$, $f \in \mathcal{O}_P(V)$. It thus also follows that $1, 0 \in \mathcal{O}_P(V)$.

An alternative way to go about defining the local ring of V at P, is to set $X := \{f \in \Gamma(V) : f(P) \neq 0\}$ and define

$$\mathcal{O}_P(V) := X^{-1}\Gamma(V).$$

One readily verifies that the first definition gives rise to ring that is isomorphic to second ring defined.

Note that X is saturated. Indeed, for $f \in \widehat{X}$ there is an $f'\Gamma(V)$ such that $f'f \in X$. Then $f'(P)f(P) \neq 0$, hence $f(P) \neq 0$.

Definition 3.3.47. Let $V \subset \mathbb{A}^n$ be a variety, $f \in K(V)$. We define *the pole set of f in V* to be the set

$$\mathcal{P}(f) := \{v \in V : f \text{ is not defined at } v\}$$

Definition 3.3.48. Let $f \in K(V)$. Define $J_f := \{g \in K[\mathbf{x}] : (g + I(V))f \in \Gamma(V)\} \subset K[\mathbf{x}]$.

Remark 3.3.49. The above set is an ideal containing $I(V)$. Indeed, if $g, h \in J_f$, then $(g + I(V))f, (h + I(V))f \in \Gamma(V)$, hence

$$((g + h) + I(V))f = (g + I(V))f + (h + I(V))f \in \Gamma(V) \implies g + h \in J_f,$$

let $r \in K[\mathbf{x}]$. Then

$$(rg + I(V))f = (r + I(V))(g + I(V))f \in \Gamma(V) \implies rg \in J_f.$$

Suppose $g \in I(V)$. Then $(g+I(V))f = (0+I(V))f = 0+I(V) \in \Gamma(V)$, hence $I(V) \subset J_f$.

Lemma 3.3.50. *Let $V \subset \mathbb{A}^n$ be a variety, $f \in K(V)$. $\mathcal{P}(f) = V(J_f)$, hence $\mathcal{P}(f)$ is algebraic.*

Proof. Let $v \in V(J_f)$. Let $(a + I(V)), (b + I(V)) \in \Gamma(v)$ such that $f = \frac{a+I(V)}{b+I(V)}$. Then $b \in J_f$, hence $b(v) = 0$, implying $(b + I(V))(v) = 0$, hence $v \in \mathcal{P}(f)$.
Let $v \in \mathcal{P}(f)$. Let $g \in J_f$, then $h + I(V) := (g + I(V))f \in \Gamma(V)$. If $g \in I(V)$, then clearly $(g + I(V))(v) = (0 + I(V))(v) = 0$. Otherwise $f = \frac{h+I(V)}{g+I(V)}$, and since f is not defined at v, $g(v) = (g + I(V))(v) = 0$, hence $v \in V(J_f)$. □

Example 3.3.51. Let $V = V(xw - yz) \subset \mathbb{A}^4$. By a simple application of for instance Eisenstein's criterion $xw - yz$ is indeed irreducible, hence we can consider $\Gamma(V) = K[x, y, z, w]/\langle xw - yz\rangle$. Denote every $a + I(V) \in \Gamma(V)$ by $\overline{a}$. Define $f := \frac{\overline{x}}{\overline{z}} = \frac{\overline{y}}{\overline{w}}$.
Note the following fact. Let $a, b \in K[x, y, z, w]$ be given such that $f = \frac{\overline{a}}{\overline{b}}$. Then $az - bx \in I(V) = \langle xw - yz\rangle$. We can thus find $q \in K[x, y, z, w]$ with $az - bx = q(xw - yz)$, hence $z(a + qy) = x(qw + b)$, implying $z \mid qw + b$. One therefor finds that $b = sz - qw$ for some $s \in K[x, y, z, w]$.
A feature of the rational function f is that given a representation of f there is a point $P \in V$ where f is defined at which the denominator vanishes. Indeed, suppose for a contradiction, $f = \frac{\overline{a}}{\overline{b}}$ for some $a, b \in K[x, y, z, w]$ with $\overline{b}(P) \neq 0$ for every $P \in V \setminus \mathcal{P}(f)$. Note that $P_{\alpha,\beta} = (0, 0, \alpha, \beta) \in V \setminus \mathcal{P}(f)$ for every $\alpha, \beta \in K$ with $\alpha \neq 0$ or $\beta \neq 0$. Hence in particular $b(0, 0, \alpha, \beta) \neq 0$ for every $(\alpha, \beta) \in \mathbb{A}^2$ with $\beta \neq 0$, hence by Proposition write proposition later! $b(0, 0, z, w) \in K[w]$. A symmetric argument

shows that $b(0,0,z,w) \in K[z]$, hence b is constant. However note that the fact noted at the beginning implies $b(0,0,0,0) = 0$. Then $b = 0$, leading to a contradiction with b not vanishing on points at which f is defined.
One can show that $J_f = \langle z, w \rangle$, which implies that f is defined exactly at the points where $\overline{z}$ or $\overline{w}$ do not vanish. Clearly $z, w \in J_f$. Let $b \in J_f$, then setting $\overline{a} = \overline{b}f$, $f = \frac{\overline{a}}{\overline{b}}$. The fact proven in the beginning of this section shows the other inclusion.

Example 3.3.52. Let $V := V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$. Since the generating polynomial is irreducible, $I(V) = \langle y^2 - x^2(x+1) \rangle$ by HNS. Set $\overline{a} := a + I(V)$ for every $a + I(V) \in \Gamma(V)$. Let $f = \frac{\overline{y}}{\overline{x}}$. We determine $\mathcal{P}(f)$. Note that

$$f = \frac{\overline{y}}{\overline{x}} = \frac{\overline{y}^2}{\overline{xy}} = \frac{\overline{x^2(x+1)}}{\overline{xy}} = \frac{\overline{x(x+1)}}{\overline{y}} \implies x, y \in J_f.$$

If we can prove that $(0,0) \in \mathcal{P}(f)$ it follows that $\mathcal{P}(f) = V(x,y) = \{(0,0)\}$. Let $\frac{\overline{a}}{\overline{b}} = f$. Then

$$\overline{b}\overline{y} = \overline{ax} \implies \overline{b}(0,c) = \overline{b}(0,c)c = \overline{a}(0,c)0 = 0,$$

for every $c \in K \setminus 0$, hence $b(0,y)$ has infinitely many roots, meaning $b(0,y) = 0$ for a representative $b \in K[x,y]$ of $\overline{b}$. This means that b has no terms not divisible by x. It follows that $\overline{b} = \overline{qx}$ for some $q \in K[x,y]$, hence $\overline{b}(0,0) = \overline{q}(0,0)0 = 0$.
Secondly, let's determine $\mathcal{P}\left(f^2\right)$. Then

$$f^2 = \frac{\overline{y}^2}{\overline{x}^2} = \frac{\overline{x^2(x+1)}}{\overline{x}^2} = \frac{x+1}{1},$$

hence $1 \in J_{f^2}$, meaning $\mathcal{P}\left(f^2\right) = V(J_{f^2}) = V(1) = \emptyset$ by WNS.

Proposition 3.3.53. *Let $V \subset \mathbb{A}^n$ be a variety. Then*

$$\Gamma(V) = \bigcap_{P \in V} \mathcal{O}_P(V).$$

Proof. As $\Gamma(V) \subset \mathcal{O}_P(V)$ for every $P \in V$, the first inclusion is obvious.
Let $f \in \bigcap_{P \in V} \mathcal{O}_P(V)$. Then f is defined at every $P \in V$, hence $V(J_f) = \mathcal{P}(f) = \emptyset$. By the weak Nullstellensatz, we then have that $1 \in J_f$, hence $f = (1 + I(V))f \in \Gamma(V)$. □

Definition 3.3.54. Let $V \subset \mathbb{A}^n$ be a variety, $P \in V$ and $f \in \mathcal{O}_P(V)$. Pick $a, b \in \Gamma(V)$ with $f = \frac{a}{b}$ and $b(P) \neq 0$. We define *the value of f at P* to be

$$f(P) := \frac{a(P)}{b(P)}$$

Remark 3.3.55. This is well-defined. Indeed, if there additionally are $a', b' \in \Gamma(V)$ with $f = \frac{a'}{b'}$ and $b'(P) \neq 0$, then

$$\frac{a}{b} = \frac{a'}{b'} \implies ab' = a'b \implies a(P)b'(P) = a'(P)b(P) \implies \frac{a(P)}{b(P)} = \frac{a'(P)}{b'(P)}.$$

Definition 3.3.56. Let $V \subset \mathbb{A}^n$ be a variety $P \in V$. *The maximal ideal of* V *at* P. Is the set

$$\mathfrak{m}_P(V) := \{f \in \mathcal{O}_P(V) : f(P) = 0\}$$

Remark 3.3.57. Consider the map

$$\begin{aligned} \mathrm{ev}_P : \mathcal{O}_P(V) &\to K \\ f &\mapsto f(P) \end{aligned}$$

This is easily verfied to be a K-algebra homomorphism. Note that

$$\mathfrak{m}_P(V) = \ker \mathrm{ev}_P,$$

hence $\mathfrak{m}_P(V)$ is an ideal.

One can furthermore show that $\mathfrak{m}_P(V)$ is the maximal ideal containing every proper ideal of $\mathcal{O}_P(V)$, hence $\mathcal{O}_P(V)$ will be a local ring (hence the name given to this ring is well chosen). The following proposition will be sufficient to prove this.

Proposition 3.3.58. *Consider* $V \subset \mathbb{A}^n$ *a variety and* $P \in V$. *Let* $f \in \mathcal{O}_P(V)$.

$$f \in \mathcal{O}_P(V)^* \iff f(P) \neq 0,$$

Hence,

$$\mathfrak{m}_P(V) = \{f \in \mathcal{O}_P(V) : f \notin \mathcal{O}_P(V)^*\}$$

Proof. Consider the set $X := \{f \in \Gamma(V) : a(P) \neq 0\}$. Note that X is saturated and $\mathcal{O}_P(V) = X^{-1}\Gamma(V)$ (cf. Remark 3.3.46). Let $f = \frac{a}{b}$ with $b(P) \neq 0$. Then

$$0 \neq f(P) = \frac{a(P)}{b(P)} \iff a(P) \neq 0 \iff a \in X = \widehat{X} \iff f = \frac{a}{b} \in \mathcal{O}_P(V)^*.$$

The last bi-implication is due to Lemma 1.6.95. □

Corollary 3.3.59. *Consider* $V \subset \mathbb{A}^n$ *a variety and* $P \in V$. $\mathfrak{m}_P(V)$ *is the maximal ideal containing every proper ideal of* $\mathcal{O}_P(V)$, *hence* $\mathcal{O}_P(V)$ *is local. In addition* $\mathcal{O}_P(V)$ *is Noetherian.*

Proof. By the above proposition, $\mathcal{O}_P(V)\backslash\mathcal{O}_P(V)^*$ is an ideal, hence by Proposition 1.6.110 $\mathcal{O}_P(V)$ is local where $\mathfrak{m}_P(V)$ is the unique maximal ideal.
Hilbert's basis theorem shows that $K[\mathbf{x}]$ is Noetherian, by Lemma 1.4.61 $\Gamma(V)$ is Noetherian. Let $I \subset \mathcal{O}_P(V)$. Let $J := I \cap \Gamma(V) \subset \Gamma(V)$. Then by Theorem 1.4.59 there are $f_1, \ldots, f_m \in \Gamma(V)$ such that $J = \langle f_1, \ldots, f_m \rangle$. Let $f \in I$ and pick $f = \frac{a}{b}$ with $b(P) \neq 0$. Then $bf \in \Gamma(V)$, hence $bf = \sum_1^m \lambda_i f_i$ for suitable $\lambda_1, \ldots, \lambda_m \in \Gamma(V)$, hence $f = \sum_1^m \frac{\lambda_i}{b} f_i$. Then I is generated by $f_1, \ldots, f_m$ in $\mathcal{O}_P(V)$. □

Corollary 3.3.60. *Consider $V \subset \mathbb{A}^n$ a variety and $P \in V$. Let $I \subset \mathcal{O}_P(V)$ be a proper ideal. Then $I \subset$ ker $(\mathrm{ev}_P : \mathcal{O}_P(V) \to K)$, hence in addition $I \cap \Gamma(V) \subset$ ker $(\mathrm{ev}_P : \Gamma(V) \to K)$*

Proof. By the above corollary $I \subset \mathfrak{m}_P(V) =$ ker $(\mathrm{ev}_P : \mathcal{O}_P(V) \to K)$. One readily verifies that ker $(\mathrm{ev}_P : \mathcal{O}_P(V) \to K) \cap \Gamma(V) =$ ker $(\mathrm{ev}_P : \Gamma(V) \to K)$. □

Lemma 3.3.61. *Let V be an affine variety and $P \in V$. There is a one-to-one correspondence between prime ideals in $\mathcal{O}_P(V)$ and prime ideals in $\Gamma(V)$ contained in* ker $(\mathrm{ev}_P : \Gamma(V) \to K)$

Proof. Let $I \subset \mathcal{O}_P(V)$ be prime. Then $I \cap \Gamma(V) \subset \Gamma(V)$ is prime by Proposition 1.6.30. By Corollary 3.3.60 $I \cap \Gamma(V) \subset$ ker $(\mathrm{ev} : P : \Gamma(V) \to K)$.
Let $J \subset \Gamma(V)$ be a prime ideal whose elements all vanish on P. Then $X^{-1}J$ is a proper ideal of $\mathcal{O}_P(V)$. Let $\frac{a}{x}, \frac{b}{x} \in \mathcal{O}_P(V)$ with $\frac{ab}{xy} \in X^{-1}I$, hence for some $\frac{c}{z} \in X^{-1}I$, $\frac{ab}{xy} = \frac{c}{z}$. Then $abz \in I$, hence $ab \in I$ or $z \in I$. Since $z(P) \neq 0$, $z \notin I$. Then $ab \in I$, meaning $a \in I$ or $b \in I$. It thus follows that $\frac{a}{x} \in X^{-1}I$ or $\frac{b}{y} \in X^{-1}I$. □

Lemma 3.3.62. *Let V be an affine variety and $P \in V$. There is a one-to-one correspondence between prime ideals in $\Gamma(V)$ contained in* ker ev_P *and subvarieties W of V containing P.*

Proof. Let $I \subset \Gamma(V)$ be prime with $I \subset$ ker ev_P. By Proposition 3.2.32 there is a subvariety $W \subset V$ such that $I = I_V(W)$. Let $f \in I(W)$, then $f(P) = \mathrm{ev}_P(f + I(V)) = 0$, hence $P \in V(I(W)) = W$.
Let $W \subset V$ be a subvariety containing P. Then $I_V(W)$ is prime and $\mathrm{ev}_P(f + I(V)) = 0$ for every $f + I(V) \in I_V(W)$. □

Proposition 3.3.63. *Let V be an affine variety and $P \in V$. There is a one-to-one correspondence between prime ideals in $\mathcal{O}_P(V)$ and subvarieties of V containing P.*

Proof. This follows directly from the two above lemmas. □

Proposition 3.3.64. *Let V be an affine variety. Denote $h + I(V)$ by $\overline{h}$ for $h \in K[\mathbf{x}]$. Consider $f = \frac{\overline{a}}{\overline{b}} \in K(V)$. Let $U = \{P \in V : f \text{ defined at } P\}$. Consider the function $\mathrm{ev}_\bullet(f) : U \to K$. f is uniquely determined by this function.*

Proof. Let $g = \frac{\overline{c}}{\overline{d}} \in K(V)$ be given such that $\mathrm{ev}_\bullet(g) = \mathrm{ev}_\bullet(f)$. Let $P \in U$. Then $\overline{a}(P)\overline{d}(P) = \overline{c}(P)\overline{b}(P)$ hence $(ad - cb)(P) = \overline{a}(P)\overline{d}(P) - \overline{c}(P)\overline{b}(P) = 0$. We thus see that $(ad - cb)(v) = 0$ for every $v \in V$, hence $ad - cb \in I(V)$, meaning $\overline{a}\overline{d} - \overline{c}\overline{b} = 0$, hence $f = g$. □

Proposition 3.3.65. *Let $\varphi : V \to W$ a polynomial map and $P \in V$. $\widetilde{\varphi} : \Gamma(W) \to \Gamma(V)$ extends uniquely to a K-algebra homomorphism $\widetilde{\varphi} : \mathcal{O}_{\varphi(P)}(W) \to \mathcal{O}_P(V)$. Furthermore $\widetilde{\varphi}(\mathfrak{m}_{\varphi(P)}(W)) \subset \mathfrak{m}_P(V)$.*

Proof. Let $X = \{a \in \Gamma(W) : a(\varphi(P)) \neq 0\}$. Let $a \circ \varphi \in \widetilde{\varphi}(X) = \{\widetilde{\varphi}(a) = a \circ \varphi \in \widetilde{\varphi}(X) : a \in X\}$. Then $a \circ \varphi(P) = a(\varphi(P)) \neq 0$, hence the first statement follows from Lemma **??**.
Let $f \in \mathfrak{m}_{\varphi(P)}(W) = \{f \in \mathcal{O}_{\varphi(P)}(W) : f(\varphi(P)) = 0\}$. Then

$$\widetilde{\varphi}(f)(P) = (f \circ \varphi)(P) = f(\varphi(P)) = 0 \implies \widetilde{\varphi}(f) \in \mathfrak{m}_P(V).$$

□

Corollary 3.3.66. *Let V, W be varieties, $P \in V$ and $\varphi : V \to W$ an isomorphism. Then $\mathcal{O}_{\varphi(P)}(W) \simeq \mathcal{O}_P(V)$.*

Proof. This follows from the above proposition and Corollaries 3.3.17, **??**. □

Corollary 3.3.67. *Let $V \subset \mathbb{A}^n$ be a subvariety and $P \in V$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^n, v \mapsto Av + w$ be an affine change of coordinates. Then $\widetilde{\varphi} : \mathcal{O}_{\varphi(P)}(\mathbb{A}^n) \to \mathcal{O}_P(\mathbb{A}^n)$ is an isomorphism. Furthermore, $\widetilde{\varphi}$ induces an isomorphism between $\mathcal{O}_{\varphi(P)}(V) \simeq \mathcal{O}_P(V^\varphi)$.*

Proof. The first statement follows from the above corollary. One notes that $V^\varphi \overset{\varphi'}{\simeq} V$, where φ' is the restriction of φ to V. Hence $\Gamma(V) \overset{\widetilde{\varphi'}}{\simeq} \Gamma(V^\varphi), v + I(V) \mapsto \varphi(v) + I(V^\varphi)$. The above corollary shows that $\varphi' : \mathcal{O}_{\varphi(P)}(V) \to \mathcal{O}_P(V^\varphi), \frac{a}{b} \mapsto \frac{\widetilde{\varphi'}(a)}{\widetilde{\varphi'}(b)}$ is an isomorphism. □

Proposition 3.3.68. *Let $P = \mathbf{0} \in \mathbb{A}^n$. Set $\mathcal{O} := \mathcal{O}_P(\mathbb{A}^n)$ and $\mathfrak{m} := \mathfrak{m}_P(\mathbb{A}^n)$. Let $I := \langle x_1, \ldots, x_n \rangle \subset K[x_1, \ldots, x_n]$. Then $I\mathcal{O} = \mathfrak{m}$, hence $I^r\mathcal{O} = \mathfrak{m}$.*

Proof. Firs note that for $f = \sum_1^n \frac{a_i}{b_i} x_i \in I\mathcal{O}$,

$$f(P) = \sum_1^n 0 \frac{a_i(P)}{b_i(P)} = 0 \implies f \in \mathfrak{m}.$$

Hence $I\mathcal{O}$ is a proper ideal. Let $f = \frac{a}{b} \in \mathfrak{m}$. Then $a(P) = 0$, hence $a \in \langle x_1, \ldots, x_n \rangle = I$ (cf. Proposition 1.8.47). □

Proposition 3.3.69. *Let $V \subset \mathbb{A}^n$ be an affine variety, set $I := I(V) \subset K[x_1, \ldots, x_n]$, pick $P \in V$ and let $J \subset K[\mathbf{x}]$ be an ideal containing I. Let $J' = J/I$. Then $\mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n) \simeq \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$ as K-algebras. In particular $\mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n) \simeq \mathcal{O}_P(V)/0\mathcal{O}_P(V) = \mathcal{O}_P(V)$.*

Proof. Consider the surjective ring homomorphism, $\sigma : \mathcal{O}_P(\mathbb{A}^n) \twoheadrightarrow \mathcal{O}_P(V)$ given by restriction, i.e $\frac{a}{b} \mapsto \frac{a+I}{b+I}$. Note that for $j\frac{a}{b} \in J\mathcal{O}_P(\mathbb{A}^n)$,

$$\sigma(j\frac{a}{b}) = (j + I)\frac{a + I}{b + I} \in (J/I)\mathcal{O}_P(V) = J'\mathcal{O}_P(V).$$

Conversely if $\sigma(\frac{g}{h}) \in J'\mathcal{O}_P(V)$, then $\frac{g+I}{h+I} = (j + I)\frac{a+I}{b+I}$, $gb + I = jha + I$, hence $gb + I \in J'\mathcal{O}_P(V)$, hence $gb \in J$, implying $\frac{g}{h} = \frac{bg}{bh} \in J\mathcal{O}_P(\mathbb{A}^n)$. This means

$$\varsigma : \mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n) \to \mathcal{O}_P(V)/J'\mathcal{O}_P(V), f + J\mathcal{O}_P(\mathbb{A}^n) \mapsto f + J'\mathcal{O}_P(V),$$

is an isomorphism of K-algebras. □

Lemma 3.3.70. ***I feel that this lemma should be added earlier*** *Consider ideal $I, J \subset K[x_1, \ldots, x_n]$. I, J are comaximal iff $V(I) \cap V(J) = \emptyset$*

Proof. This follows immediately from the weak Nullstellensatz:

$$I + J = \langle 1 \rangle \iff \emptyset = V(I + J) = V(I) \cap V(J).$$

□

3.3.4 Rational Functions and DVR's

Example 3.3.71. Let $V = \mathbb{A}^1$. Then $K(V) = K(X)$. Let $a \in V$. Then $\mathcal{O}_a(V)$ is a non-field integral domain which is local and Noetherian. We prove that $\mathfrak{m}_a(V) = \langle x - a \rangle$. Indeed, $\text{ev}_a(x - a) = 0$, proving the first inclusion. If $f \in \mathfrak{m}_a(V)$, then $f(a)$, hence $x - a \mid f$. The maximal ideal of $\mathcal{O}_a(V)$ is thus principal ideal with uniformizing parameter $x - a$.

Definition 3.3.72. We define the local ring at infinity to be the ring

$$\mathcal{O}_\infty := \left\{ \frac{f}{g} \in K(X) : \deg\ f \leq \deg\ g \right\}.$$

Remark 3.3.73. One readily verifies that this is a subring of $K(X)$ by noting that if $a, b, c, d \in K[x]$ with $\deg\ a \leq \deg\ b$ and $\deg\ c \leq \deg\ d$, then $\deg\ ac = \deg\ a + \deg\ c \leq \deg\ b + \deg\ d$ and $\deg\ ad + cb \leq \max(\deg\ ad, \deg\ cb) \leq bd$. Furthermore, $1, 0 \in \mathcal{O}_\infty$ Change def of degree?, since $1 = \frac{1}{1}$ and $0 = \frac{0}{1}$. It is a subring of $K(X)$, hence it is an integral domain. It is not a field since $\frac{1}{x} \in \mathcal{O}_\infty$ but $x \notin \mathcal{O}_\infty$. One can check that the units of $\mathcal{O}_\infty$ are the fractions $\frac{f}{g}$ with $\deg\ f = \deg\ g$. Indeed, $\frac{g}{f}$ is the inverse of such elements. Conversely, if $\frac{f}{g}$ is invertible, then $\frac{g}{f} \in \mathcal{O}_\infty$.

Proposition 3.3.74. *$\mathcal{O}_\infty$ is a DVR.*

Proof. We prove that $t := \frac{1}{x}$ is a uniformizing parameter. Suppose $\frac{1}{t} = \frac{a}{b}\frac{c}{d}$. Then

$$1 + \deg\ a + \deg\ c = \deg\ t + \deg\ a + \deg\ d = \deg\ b + \deg\ d.$$

Then $\deg\ a = \deg b$ or $\deg\ c = \deg\ d$, hence $\frac{a}{b}$ is a unit or $\frac{c}{d}$ is a unit. Consequently, t is irreducible. Let $\frac{f}{g} \in \mathcal{O}_\infty \setminus 0$ with $n = \deg\ f$ and $m := \deg\ g$. Set $u = \frac{fx^{m-n}}{g}$. This has a mutual inverse $\frac{g}{fx^{m-n}}$ in $\mathcal{O}_\infty$. Then

$$\frac{f}{g} = \frac{fx^{m-n}}{g}\frac{1}{x^{m-n}}.$$

The uniqueness really just follows from t being irreducible. Since if $ut^{m-n} = vt^l$ for some other unit v and $l \geq 0$, then $m - n - l = 0$ and hence $u = v$. By Proposition 1.6.121, $\mathcal{O}_\infty$ is a DVR. □

Proposition 3.3.75. *The only DVR's with quotient field $K(X)$ (recall K is assumed algebraically closed) containing K are $\mathcal{O}_\infty$ and $\mathcal{O}_a(\mathbb{A}^1) = \left\{ \frac{f}{g}(x-a)^n : \frac{f}{g} \in K(x), (x-a) \nmid g, n \geq 0 \right\}$, where $a \in K$.*

Proof. $K \subset R \subsetneq K(x)$ be a DVR. Let $\mathfrak{m} = \left\langle \frac{f}{g} \right\rangle$ be the maximal ideal of R with $\gcd(f, g) = 1$.

Suppose first that $K[x] \subset R$. Then $I := K[x] \cap \mathfrak{m} = \langle h \rangle$ (recall that $K[x]$ is a PID) is a prime ideal in $K[x]$ by Proposition 1.6.30. Since $\frac{f}{g}$ is irreducible, it is non-zero. Then $f = g\frac{f}{g} \in \mathfrak{m} \cap K[x]$, meaning $h \neq 0$. Then $h = x - a$ for some $a \in K$ using the assumption that K is algebraically closed. We prove that $R = \mathcal{O}_a(\mathbb{A}^1)$. Let $\frac{\lambda}{\mu} \in R \setminus 0$ with $\gcd(\lambda, \mu) = 1$. Suppose for a contradiction that $h \mid \mu$. Then $\mu \in \mathfrak{m}$, but then $\lambda = \mu\frac{\lambda}{\mu} \in \mathfrak{m} \cap K[x] = \langle h \rangle$, contracting the assumption $\gcd(\lambda, \mu) = 1$. Then

$\mu(a) \neq 0$, which implies $\frac{\mu}{\lambda} \in \mathcal{O}_a(\mathbb{A}^1)$, hence $R \subset \mathcal{O}_a(\mathbb{A}^1)$. In particular we get that $\frac{1}{g} \in \mathcal{O}_a(\mathbb{A}^1)$. This means that $\mathfrak{m} \subset \mathcal{O}_a(\mathbb{A}^1)\mathfrak{m} = \mathcal{O}_a(\mathbb{A}^1)f \subset \mathcal{O}_a(\mathbb{A}^1)h$. It follows from Proposition 1.6.125 that $\mathcal{O}_a(\mathbb{A}^1) = R$.

Suppose now that $K[x] \not\subset R$. In particular we must then have that $x - a \notin R$ for any $a \in K$. Then $x - a \in K(x) \setminus R$, which implies that $\frac{1}{x-a} \in \mathfrak{m}$ by Lemma 1.6.124. Note that for $a, b \in K$,

$$\frac{x-a}{x-b} = 1 + \frac{b-a}{x-b} \in R.$$

This means that $\frac{x-b}{x-a} \in R^*$. Now, let $\frac{h}{k} \in R \setminus 0$ with $\gcd(h,k) = 1$. Suppose for a contradiction that $\deg\ h > \deg\ k$. Then we can write $\frac{h'}{k}h''$ with $h', h'' \in K[x]$ such that $\deg\ h' = \deg\ k$ and $\deg\ h'' \geq 1$. But then

$$h'' = \frac{k}{h''}\frac{h}{k} \in R.$$

Then for some $\alpha \in K$ and $l \in K[x]$, $(x - \alpha) = \frac{h''}{l} \in R$, but then $K[x] \subset R$, leading to a contradiction. We thus conclude that $R \subset \mathcal{O}_\infty$. By irreducibility of $\frac{f}{g}$ it is clear that since $\frac{f}{g} \mid \frac{1}{x}$, $\frac{f}{g} = u\frac{1}{x}$ for some unit $u \in R$. We thus get that $\mathfrak{m} \subset \mathcal{O}_\infty\frac{1}{x}$. It follows from Proposition 1.6.125 that $\mathcal{O}_\infty = R$. □

3.3.5 Ideals with a Finite Number of Zeroes

Theorem 3.3.76. *Let $I \subset K[x_1, \ldots, x_n]$ and suppose $\#V(I) < \infty$. Denote the points of $V(I)$ by $P_1, \ldots, P_m$. Set $\mathcal{O}_i := \mathcal{O}_{P_i}(\mathbb{A}^n)$. Then*

$$K[\mathbf{x}]/I \simeq \prod_1^m \mathcal{O}_i/I\mathcal{O}_i$$

Proof. Set $I_i := I(\{P_i\}) = \langle x_1 - P_{i1}, \ldots, x_n - P_{in}\rangle$. We can construct

$$\sigma_i : K[\mathbf{x}]/I \to \mathcal{O}_i/I\mathcal{O}_i$$
$$f + I \mapsto f + I\mathcal{O}_i$$

We thus get a ring homomorphism

$$\sigma : K[\mathbf{x}]/I \to \prod_1^m \mathcal{O}_i/I\mathcal{O}_i$$
$$f + I \mapsto (\sigma_1(f + I), \ldots, \sigma_m(f + I))$$

We aim to prove that σ is an isomorphism. We need to make few constructions before we are able to do so. Note that by HNS, $\mathrm{rad}(I) = I(V(I)) = I(P_1, \ldots, P_m) = \bigcap_1^m I_i$. Note

also that $V(I_i) \cap V(\bigcap_{j\neq i} I_j) = \emptyset$, hence by Lemma 3.3.70 I_i and $\bigcap_{j\neq i} I_j$ are comaximal. Thus by Lemma 1.6.42 and Lemma 1.6.60,

$$\bigcap_1^m I_i^d = \left(\prod_1^m I_i\right)^d = \left(\bigcap_1^m I_i\right)^d \subset I,$$

for some $d \geq 0$. For each $i \in \{1, \ldots, m\}$ pick $F_i \in K[\mathbf{x}]$ such that $F_i(P_j) = 0$ and $F_i(P_i) = 1$ for $j \neq i$ (such a polynomial exist do to Corollary 3.1.43). Set $E_i := 1 - (1 - F_i^d)^d$ for each i. Note that

$$E_i + \langle F_i^d \rangle = 1 - (1 - F_i^d)^d + \langle F_i^d \rangle = 1 - 1 + \langle F_i^d \rangle = 0 + \langle F_i^d \rangle \implies E_i \in \langle F_i^d \rangle \subset \bigcap_{j\neq i} I_j^d,$$

hence $E_i E_j \in \bigcap_1^m I_i^d \subset I$. Furthermore,

$$1 - \sum_1^m E_j + I_i^d = (1 - E_i) - \sum_{j\neq i} E_j + I_i^d = 0 + I_i^d,$$

for each i, hence $1 - \sum_1^m E_j \in \bigcap_1^m I_j^d$. Note also that

$$E_i^2 - E_i = E_i(E_i - 1) = E_i(1 - (1 - F_i^d)^d - 1) = -E_i(1 - F_i^d)^d \in I_i^d \bigcap_{j\neq i} I_j^d = \bigcap_1^m I_j^d \subset I.$$

Set $e_i := E_i + I$. Then $e_i e_j = 0$ for each $i \neq j$, $e_i^2 = e_i$ for each i and $\sum_1^m e_i = 1$.

Claim: If $G \in K[\mathbf{x}]$ and $G(P_i) \neq 0$. Then there is a $t \in K[\mathbf{x}]/I$ such that $tg = e_i$ where $g := G + I$.

Proof of Claim: WLOG $G(P_i) = 1$. Set $H := 1 - G$ and $h := H + I$. Since $H(P_i) = 0$, $H_i \in I_i$, hence $H_i^d E_i \in I_i^d \bigcap_{j\neq i} I_j^d = \bigcap_1^m I_j^d \subset I$. Note that

$$(1 - H) \sum_0^{d-1} E_i H^k = E_i - E_i H^d.$$

Considering the image of the left- and right-hand side in $K[\mathbf{x}]/I$ we therefor get that

$$g \underbrace{\left(-\sum_0^{d-1} e_i h^k\right)}_{t} = e_i.$$

Equipped with the above claim we can proceed by proving σ is injective. Suppose $f := F + I \in K[\mathbf{x}]/I$ is given such that $f \in I\mathcal{O}_i$ for each i. For each i we can then pick a $G_i \in K[\mathbf{x}]$ such that $G_i(P_i) \neq 0$ and $G_i F \in I$. Set $g_i = G_i + I$. Then there is a $t_i g_i = e_i$. Then

$$f = f \sum_1^m e_i = \sum_1^m t_i g_i f = 0 \implies \ker \sigma = 0.$$

Before we prove that σ is surjective we record a few facts. Since $E_i(P_i) \neq 0$, E_i is a unit in $\mathcal{O}_i$, hence e_i is a unit in $\mathcal{O}_i/I\mathcal{O}_i$. This in particular means,

$$\sigma_i(e_i)\sigma_i(e_j) = \sigma_i(e_ie_j) = 0 \implies \sigma_i(e_j) = 0.$$

This means

$$\sigma_i(e_i) = \sigma_i(e_i) + \sigma_i\left(\sum_{j\neq i} e_j\right) = \sigma_i\left(\sum_1^m e_j\right) = \sigma_i(1) = 1.$$

Consider an arbitrary element $z = \left(\frac{a_1}{s_1}, \ldots, \frac{a_m}{s_m}\right) \in \prod_1^m \mathcal{O}_i/I\mathcal{O}_i$. For a suitable t_i, $s_it_i = e_i$. Then

$$\frac{a_i}{s_i} = \frac{a_it_i}{s_it_i} = \frac{a_it_i}{e_i} = a_it_i\sigma_i(e_i)^{-1} = a_it_i.$$

Then we get that

$$\sigma_i\left(\sum_1^m a_jt_je_j\right) = \sigma_i(a_it_i) = \frac{a_i}{s_i} \implies$$
$$\sigma\left(\sum_1^m a_jt_je_j\right) = \left(\sigma_1\left(\sum_1^m a_jt_je_j\right), \ldots, \sigma_m\left(\sum_1^m a_jt_je_j\right)\right) = \left(\frac{a_1}{s_1}, \ldots, \frac{a_m}{s_m}\right) = z,$$

meaning σ is surjective. □

Corollary 3.3.77. *Let $I \subset K[x_1, \ldots, x_n]$ and suppose $\#V(I) < \infty$. Denote the points of $V(I)$ by $P_1, \ldots, P_m$. Set $\mathcal{O}_i := \mathcal{O}_{P_i}(\mathbb{A}^n)$. Then*

$$\dim_k\ K[\mathbf{x}]/I = \dim_k\left(\prod_1^m \mathcal{O}_i/I\mathcal{O}_i\right) = \sum_1^m \dim_k\ \mathcal{O}_i/I\mathcal{O}_i$$

Corollary 3.3.78. *In the setup above suppose $m = 1$ then*

$$K[\mathbf{x}]/I \simeq \mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n).$$

Corollary 3.3.79. *Suppose $R \supset K$ and $\dim_k\ R < \infty$. Then $R \simeq \prod_1^m R_i$ where R_i are local.*

Proof. Since R is finite dimensional over K it is also ring finite over K generated by some linear basis $\{a_1, \ldots, a_n\}$ of R over K, hence $R \simeq K[x_1, \ldots, x_n]/I$ (cf. Proposition 1.9.3). Then by a suitable corollary of HNS,

$$V(I) \leq \dim_K\ K[\mathbf{x}]/I = \dim_K\ R < \infty.$$

It follows by the theorem of this subsection that $R \simeq K[\mathbf{x}]/I \simeq \prod_1^m R_i$, where R_i is local for each i. □

3.4 Local Properties of Affine Plane Curves

Fix again an algebraically closed field K. Sometimes we will need that char $K = 0$ (we will have to make some considerations with derivatives, so to make things more simple we makes this assumption about characteristic). We first define an equivalence relation of polynomials in $K[x, y]$.

Definition 3.4.1. We say that a pair of polynomials $f, g \in K[x, y]$ are *(factor) equivalent* (the "factor"-part is non-standard) if $f = \lambda g$ for some $\lambda \in K \setminus 0$. (This is the definition of $\sim$ from Definition 1.7.25 in the special case $V = K[x, y]_{\geq 1}$).

Definition 3.4.2. *An affine plane curve* C is an equivalence class under factor equivalence, i.e. an element of $K[x, y]/\sim$. The *degree* of C (denoted deg C) is the degree of a representative of C (This is notion is clearly independent of choice of representative). A degree 1 curve is called a *line*. The irreducible factors of any representative of C up to multiplication by a scalar are called *components* of C. A component with multiplicity 1 in the factorization of a representative of C is called a *simple* components. A non-simple component is called a *multiple* component.

Remark 3.4.3. We already now that $I(V(C)) = f_1 \cdots f_m$ where $f_1, \ldots, f_m$ are the components of C. We thus lose information about multiplicities. For an irreducible polynomial $f \in K[x, y]$, $V(f)$ is a variety. We define $\Gamma(f) := \Gamma(V(f))$, $K(f) := K(V(f))$, $\mathcal{O}_P(f) := \mathcal{O}_P(V(f))$. We know that each curve in $\mathbb{A}^2$ arises as $V(f)$ for some $f \in K[x, y]$. It thus follows that there is a bijection between algebraic curves in $\mathbb{A}^2$ and algebraic curves in $(K[x, y]_{\geq 1} \setminus 0)/\sim$, so the choice of terminology is well-chosen.

Definition 3.4.4. let f be a curve and $P = (a, b) \in f$. P is called *simple* if $f_x(P) \neq 0$ or $f_y(P) \neq 0$. The *tangent line* at a simple point is the line

$$f_x(P)(x - a) + f_y(P)(y - b).$$

If P is not simple it is called *multiple/singular*. If every point on a curve is simple, then the curve is called *non-singular*.

Example 3.4.5. Each of the described curves will be in an algebraically closed field of characteristic 0 (think $\mathbb{C}$).

1. A *parabola*, $f = y - ax^2 - bx - c$, $a, b, c \in K$ with $a \neq 0$ is a non-singular curve since $f_y = 1$

2. Consider the curve $f = y^2 - x^3 + x$, an example of an *elliptic curve*. In this instance f is non-singular. Indeed, $f_x = -3x^2 + 1$, $f_y = y$. If $f_y(a,b) = 0$, then $0 = f(x,b) = -x^3 + x = -x(x^2-1) = x(x-1)(x+1)$, has solution $a \in \{0, \pm 1\}$. Then $f_x(a,b) \in \{1, -2\}$. On the other hand, if $f_x(a,b) = 0$, then $a = \pm\frac{1}{\sqrt{3}}$. Then $0 = f(\pm a, y)$ has solutions $b \in \{\pm\sqrt{-a^3+a}, \pm\sqrt{a^3-a}\}$, hence $b \neq 0$. It follows that $f_y(b) \neq 0$.

3. Consider the curve $f = y^2 - x^3$. It has its only singular point at $(0,0)$. Indeed, $f_x = -3x^2$ and $f_y = 2y$. Therefor $-3a^2 = f_x(a,b) = 0$ and $2b = f_y(a,b) = 0$ if and only if $(a,b) = (0,0)$.

4. Consider the curve $f = y^2 - x^3 - x^2$. It has its only singular point at $(0,0)$. Indeed, $f_x = -3x^2 - 2x = -3x(x + 2/3)$ and $f_y = 2y$. Thus the only common zeroes of f_x and f_y are $(0,0)$ and $(-2/3, 0)$. The only common zero of f_x and f_y lying on f is thus $(0,0)$.

Definition 3.4.6. For a curve f write $f = \sum_m^n f_i$, where $f_m, \ldots, f_n$ are homogeneous polynomials with degree corresponding to their index and $f_m \neq 0$. We call the value m the *multiplicity of f at* $(0,0)$, we denote it $m_0(f)$. If $m = 2$ we call $(0,0)$ a *double point*, if $m = 3$ we call $(0,0)$ a *triple point*, etc.

Lemma 3.4.7. *Let f be a curve. Then $m_0(f) > 0$ if and only if $(0,0) \in f$.*

Proof. $m_0(f) > 0$ is equivalent to f having constant term zero which is equivalent to $(0,0) \in f$. □

Lemma 3.4.8. *Let f be a curve with $(0,0) \in f$. Then $m_0(f) = 1$ if and only if $(0,0)$ is a simple point in f. In this case f_1 is the tangent line at 0.*

Proof. " $\Longrightarrow$:" If $m_0(f) = 1$. Then $f_1 = ax + by$ where $a \neq 0$ or $b \neq 0$. Then $(f_1)_x = a \neq 0$ or $(f_1)_y = b \neq 0$, hence $(0,0)$ is a simple point of f.
" $\Longleftarrow$ ": Let $f_1 = ax + by$ the degree 1 homogeneous polynomial in the term expansion of f. We write $f = f_1 + g$ where deg $g > 1$. Given a term t in g we thus have that deg $t_x \geq 1$ resp. deg $t_y \geq 1$ or $t_x = 0$ resp. $t_y = 0$. Thus any term in g_x and g_y is divisible by x or y. Then $a = f_x(0,0) \neq 0$ or $b = f_y(0,0) \neq 0$, hence $m_0(f) = 1$.
Writing $f_1 = ax + by \in K[x,y]$, we find that the degree 0 homogeneous polynomial terms in f_x is exactly $(f_1)_x = a$ and $(f_1)_y = b$, hence $f_x(0,0)x + f_y(0,0)y = ax + by = f_1$. □

Remark 3.4.9. In the case $m_0(f) > 1$, f_m has multiple tangent lines corresponding to those obtained from its factorization into linear forms (cf Corollary 1.8.121). Each of

these tangent lines have a multiplicity in the factorization of f_m. If the multiplicity of each tangent line is 1 and $m > 1$ we call $(0,0)$ an *ordinary multiple point* of f. An ordinary double point is called a *node*.

Lemma 3.4.10. *Consider a curve and its factorization into irreducible factors* $f = \prod_1^r p_i^{e_i}$. *Then* $m_0(f) = \sum_1^m m_0(p_i)e_i$.

Proof. Indeed, write $p_i = \sum_{m_i}^{n_i} p_{ij}$, as the representation of p_i as a sum of homogeneous polynomials where $p_{im_i} \not\equiv 0$. We get that

$$\sum_m^n f_i = f = \prod_1^m p_i^{e_i} = \prod_1^m \left(\sum_{m_i}^{n_i} p_{ij} \right)^{e_i} = \prod_1^m p_{im_i}^{e_i} + \underbrace{\cdots}_{\text{strictly higher degree terms}} \implies f_m = \prod_1^m p_{im_i}^{e_i},$$

hence $m_0(f) = m = \deg \prod_1^m p_{im_i}^{e_i} = \sum_1^m m_i e_i = \sum_1^m m_0(p_i)e_i$ □

Definition 3.4.11. Let $P = (a,b) \in \mathbb{A}^2 \setminus 0$. Let $T_P : \mathbb{A}^2 \to \mathbb{A}^2$ be translation by P. We extend all above to definitions pertaining to a $(0,0)$ and a curve f to involve an arbitrary point P by considering $f^T = f(x + a, y + b)$. Then $m_P(f) := m_0(f^T)$. Writing $f^T = \sum_{m_P(f)}^n g_i$. We define the tangent lines at P to be $l_i^{T^{-1}}$, where l_i are the linear factors of $g_{m_P(f)}$. Moreover, we assert that $m_P(0) = \infty$.

Remark 3.4.12. Note that $f_x(P) = f_x^T(P)$ and $f_y(P) = f_y^T(0)$, so the choice of definition of tangent line is sensible, since then P is simple if and only if $m_P(f) = 1$ and in this case the tangent line at P is exactly $g_{m_P(f)}^{T^{-1}}$.

Example 3.4.13. Show some examples of multiple points and tangent lines for a few different points for a few different curves

Proposition 3.4.14. *If a curve* f *of degree* d *has a point* $P \in f$ *of multiplicity* d. *Then*

$$f = \prod_1^d l_i,$$

where l_i *are (possibly non-distinct) lines.*

Proof. Suppose $P = (a,b)$ is a point of multiplicity d. Then $f(x + a, y + b)$ is a non-zero homogeneous polynomial of degree d, since the lowest degree term in the homogeneous polynomial expansion of $f(x + a, x + b)$ is d (as $m_P(f) = d$) and deg $f(x + a, x + b) \leq$ deg $f = d$. It follows from Corollary 1.8.121 that $f(x + a, x + b) = \prod_1^d l_i$, where l_i are lines and hence $f = f(x + a, x + b)^{T^{-1}} = \prod_1^d l_i(x - a, x - b)$, note that $l_i(x - a, x - b)$ are lines, since otherwise $f = 0$. □

Proposition 3.4.15. *Let P be a double point on a curve f. P is a node if and only if*

$$f_{xy}(P)^2 \neq f_{xx}(P)f_{yy}(P).$$

Proof. By Lemma 1.8.143 it is sufficient to prove the result for $P = (0,0)$. We translate the statement into one of linear algebra, by noting that since f has a double point, $f_2 = l_1 l_2$ where $l_1 = ax + by$, $l_2 = cx + dy$. Furthermore, the constant term of f_{xx}, f_{yy}, f_{xy} is respectively $(l_1 l_2)_{xx} = 2ac, (l_1 l_2)_{yy} = 2bd$ and $(l_1 l_2)_{xy} = ad + bc$. We thus notice that

$$f_{xy}(P)^2 - f_{xx}(P)f_{yy}(P) = (ad+bc)^2 - 4adbc = (ad-bc)^2 = \left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^2.$$

Note that l_1 and l_2 are distinct (i.e. not factor equivalent) if and only if (a,b) and (c,d) are linearly independent. Hence P is a node if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \iff f_{xy}(P)^2 - f_{xx}(P)f_{yy}(P) \neq 0.$$

□

Proposition 3.4.16. *Suppose* char $K = 0$. *For a point P on a curve f, $m_P(f)$ is the smallest integer such that for some i, j with $i + j = m_P(f)$,*

$$\left(\frac{\partial^{m_P(f)}}{\partial x^i \partial y^j} f\right)(P) \neq 0.$$

The lowest degree homogeneous polynomial of f at P in terms of these partial derivatives in the following way

$$f_{m_P} = \sum_{i,j:i+j=m} \frac{1}{i!j!} \left(\frac{\partial^m}{\partial x^i \partial y^j} f\right)(P) x^i y^j.$$

Proof. Again we may assume WLOG that $P = (0,0)$, since it again follows from Lemma 1.8.143 that $\left(\frac{\partial^{m_P(f)}}{\partial x^i \partial y^j} f\right)(P) = \left(\frac{\partial^{m_P(f)}}{\partial x^i \partial y^j} f^{T_P}\right)(0,0)$. We prove the result by induction in the multiplicity at P. (we define the 0'th partial derivative at any variable to be the identity map). If $m_P(f) = 0$, then $P \notin f$, hence $f(P) \neq 0$. If this is isn't satisfactory, when $m_P(f) = 0$, then $f_x(P) \neq 0$ or $f_y(P) \neq 0$. Suppose the statement is true for every polynomial of multiplicity $m \geq 0$. Consider a curve f of multiplicity $m + 1$ at P. Then f_x or f_y is curve of multiplicity m and the result follows by applying the induction hypothesis. Note here that we implicitly use that char $K = 0$. Indeed, from this assumption it follows that $\deg((f_{m+1}))_x = m$ or $\deg (f_{m+1})_y = m$.

An alternative way (perhaps a more algorithmic approach) is to write $f = \sum_m^n f_i$ in the

homogeneous polynomial expansion. Then a term in f_m is of the form $a_{ij}x^iy^j$ where $i+j = m$. Then $\frac{\partial^m}{\partial x^i y^j} a_{ij}x^iy^j = a_{ij}j!i!$ and $\frac{\partial^m}{x^k y^l} a_{k,l}x^k y^l = 0$, meaning $\left(\frac{\partial^m}{\partial x^i \partial y^j} f\right)(P) = i!j!a_{ij}$. Hence,

$$f_m = \sum_{i,j:i+j=m} \frac{1}{i!j!} \left(\frac{\partial^m}{\partial x^i \partial y^j} f\right)(P) x^i y^j.$$

□

Proposition 3.4.17. *Let $l_1, \ldots, l_n$ be lines all vanishing at $(0,0)$ in $\mathbb{A}^2$ and $r_1, \ldots, r_n$ a sequence of positive integers. Set $m := \sum_1^n r_i$ and $f_m = \prod_1^n l_i^{r_i}$, pick a form f_{m+1} of degree $m+1$ such that $\gcd(f_m, f_{m+1}) = 1$. Then $f_m + f_{m+1}$ is an irreducible curve with tangent lines $l_1, \ldots, l_n$ with respective multiplicities r_i*

Proof. This follows immediately from Proposition 1.8.122. □

Definition 3.4.18. For a polynomial map $\varphi = (\varphi_1, \ldots, \varphi_m) : \mathbb{A}^n \to \mathbb{A}^m$ and a point $P \in \mathbb{A}^n$ we define the *jacobian of φ at P* to be the matrix

$$J_P T := \left(\frac{\partial \varphi_i}{\partial x_j}(P)\right) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(P) & \cdots & \frac{\partial \varphi_1}{\partial x_n}(P) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m}{\partial x_1}(P) & \cdots & \frac{\partial \varphi_m}{\partial x_n}(P) \end{pmatrix} \in M_{m\times n}(K)$$

Proposition 3.4.19. *Let $\varphi = (\varphi_1, \varphi_2) : \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial map such that f^φ is non-constant. Let f be a curve and Q a point. Set $P = \varphi(Q)$. We then have the following:*

1. *$m_Q(f^\varphi) \geq m_P(f)$.*
2. *the above is true with equality if $J_Q\varphi$ is invertible.*

Proof. 1. We consider first the case $Q = P = (0,0)$. Then $\varphi_i(0,0) = 0$ for $i = 1, 2$, hence the constant term of φ_i is 0 for $i = 1, 2$. Then the degree of any monomial under composition with φ does not decrease. Thus writing $f = \sum_m^n$ and $f^\varphi = \sum_l^k g_i$ as sums of homogeneous polynomials we have that $m_0(f^\varphi) = \deg g_l \geq \deg\ f_m = m_0(f)$. Note that any translation T is a polynomial such that h^T is a non-constant whenever h is non-constant. Suppose now φ is an arbitrary polynomial map restricted to the assumption. Let $\varphi' = T_P^{-1}\varphi T_Q$ Note that $\varphi'(0,0) = (T_P^{-1}\varphi T_Q)(0,0) = T_P^{-1}(P) = (0,0)$, hence

$$m_Q(f^\varphi) = m_0(f^{\varphi T_Q}) = m_0(f^{T_P T_P^{-1}\varphi T_Q}) = m_0((f^{T_P})^{\varphi'}) \geq m_0(f^{T_P}) = m_P(f).$$

2. Consider again first the case of $Q = P = (0,0)$. We have that $\frac{\partial \varphi_1}{\partial x}(Q), \frac{\partial \varphi_2}{\partial y}(Q) \neq 0$ or $\frac{\partial \varphi_1}{\partial y}(Q), \frac{\partial \varphi_2}{\partial x}(Q) \neq 0$. Meaning the jacobian of φ at $(0,0)$ is given by

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix},$$

where $\lambda_1 = \alpha_1 x + \beta_1 y$ and $\lambda_2 = \alpha_2 x + \beta_2 y$ are the linear homogeneous forms of φ_1 resp. φ_2 such that $\alpha_1, \beta_2 \neq 0$ or $\alpha_2, \beta_1 \neq 0$. There is a natural invertible, linear map on linear forms in 2 variables given by

$$J_0\varphi : V(1,2) \to V(1,2)$$
$$l \mapsto l(\alpha_1 x + \beta_1 y, \alpha_2 x + \beta_2 y).$$

In particular we thus have that $J_0\varphi(l) = 0 \iff l = 0$. Write $\varphi_i = \lambda_i + \mu_i$, where μ_i is 0 or of degree>1. By Corollary 1.8.121 $f_i = \prod_1^{m_0(f)+i} l_j$ for suitable (non-zero!) linear forms $l_j := c_{1j}x + c_{2j}y$. Then

$$l_j(\varphi_1, \varphi_2) = \underbrace{J_0\varphi(l_j)}_{=:l_j' \neq 0} + c_{1j}\mu_1 + c_{2j}\mu_2.$$

We thus get that

$$f_i^{\varphi} = \prod_1^{m_0(f)+i} l_j' + h$$

where h is 0 or a polynomial of degree>1. It thus follows that the lowest degree term of f^{φ} is equal to the lowest degree term $f_{m_0(f)}$, which by the above (maybe overly thorough) considerations (that therefor should be reorganized and rewritten in smaller lemmas at a later point) that $m_0(f) = m_0(f^{\varphi})$. Let P, Q be arbitrary, and define φ' as in 1. It is easy to check that

$$J_v T_w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for any $v, w \in \mathbb{A}^n$ By the chain rule for polynomial maps we find that

$$J_{(0,0)}\varphi' = J_P T_P^{-1} \varphi T_Q = (J_{\varphi(Q)} T_P^{-1})(J_Q\varphi)(J_{(0,0)} T_Q) = J_Q\varphi.$$

Hence $J_{(0,0)}\varphi'$ is invertible, meaning

$$m_Q(f^{\varphi}) = m_0((f^{T_p})^{\varphi'}) = m_0(f^{T_p}) = m_P(f).$$

□

Example 3.4.20. The converse of 2. in the above proposition is not true in general. Consider $\varphi = (x^2, y)$, $f = y - x^2$, $P = Q = (0,0)$. We note that $f^{\varphi} = y - x^4$, hence $m_0(f) = m_0(f^{\varphi})$. But the jacobian of φ at $(0,0)$ is equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This result maybe is not in the right place

Lemma 3.4.21. *Let* char $K = p > 0$ *(and K algebraically closed). If $f \in K[x_1, \ldots, x_n]$ is non-constant with $f_{x_i} = 0$ for each i, then $f = h^p$ for some polynomial h. In particular f is not irreducible.*

Proof. By lemma 1.8.145 $f = g(x_1^p, \ldots, x_n^p)$ for some $g = \sum_{v \in \mathbb{N}^n} a_v \mathbf{x}^v \in K[x_1, \ldots, x_n]$. Then set $h = \sum_{v \in \mathbb{N}^n : a_v \neq 0} a_v^{1/p} \mathbf{x}^v$. Then using Freshman's dream we get that

$$h^p = \left(\sum_{v \in \mathbb{N}^n : a_v \neq 0} a_v^{1/p} \mathbf{x}^v \right)^p = \sum_{v \in \mathbb{N}^n : a_v \neq 0} a_v \mathbf{x}^{(pv_1, \ldots, pv_n)} = g(x_1^p, \ldots, x_n^p) = f.$$

□

Proposition 3.4.22. *An irreducible curve f has only finitely many multiple points.*

Proof. A point of f is multiple at a point P if and only if $f_x(P) = f_y(P) = 0$. It is thus sufficient to prove that $V(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is finite. It is again sufficient to prove that $\gcd(f, f_x) = 1$ or due to Theorem 3.2.13. We prove the result in the char $K = 0$ case first. WLOG f_x is non-constant and deg $f_x <$ deg f, hence $f \nmid f_x$, meaning $\gcd(f, f_x) = 1$, proving the statement in this case. For the char $K = p > 0$, we use the assumption that f is irreducible in conjunction with the contrapositive of the above lemma to see that f_x and f_y cannot both be identically 0. Hence WLOG $f_x \neq 0$, hence deg $f_x <$ deg f, meaning $\gcd(f_x, f) = 1$. □

3.4.1 Aside on Hypersurfaces and Tangent Spaces

Definition 3.4.23. Let $f \in K[x_1, \ldots, x_n]$ be a non-constant polynomial and $P \in \mathbb{A}^n$. We define *the multiplicity of f at P* to be the degree of the lowest degree term in f^{T_P}, where $T_P : \mathbb{A}^n \to \mathbb{A}^n$ is translation by P. If $m_P(f) = 1$, then we can write f^{T_p} as a sum of homogeneous polynomial $\sum_1^n f_i$, where $f_1 = \sum_1^n a_i x_i \neq 0$. We define the *tangent hyperplane* at P to be the vanishing set of f_1. A simple point of a hypersurface is a point at which all the partial derivatives vanish.

Remark 3.4.24. It is easy to see that a point is simple if and only if $m_P(f) = 1$.

Example 3.4.25. Let's examine $f = x^2 + y^2 - z^2$ have tangent hyperplanes at $(0, 0)$. Can we write $f = l_1 l_2$ for homogeneous planes $l_1 := a_1 x + b_1 y + c_1 y, l_2 = a_2 x + b_2 y + c_2 z$. We have that

$$l_1 l_2 = a_1 a_2 x^2 + b_1 b_2 y^2 + c_1 c_2 z^2 + (a_1 b_2 + b_1 a_2) xy + (a_1 c_2 + c_1 a_2) xz + (b_1 c_2 + c_1 b_2) yz.$$

To satisfy the identity we thus have that

$$\begin{cases} a_1a_2 = 1 \\ b_1b_2 = 1 \\ c_1c_2 = -1 \\ a_1b_2 + b_1a_2 = 0 \\ a_1c_2 + c_1a_2 = 0 \\ b_1c_2 + c_1b_2 = 0 \end{cases} \iff \begin{cases} a_1 = \frac{1}{a_2} \\ b_1 = \frac{1}{b_2} \\ c_1 = \frac{-1}{c_1} \\ \frac{a_1}{b_1} + \frac{b_1}{a_1} = 0 \\ -\frac{a_1}{c_1} + \frac{c_1}{a_1} = 0 \\ -\frac{b_1}{c_1} + \frac{c_1}{b_1} = 0 \end{cases} \iff \begin{cases} a_2 = \frac{1}{a_1} \\ b_2 = \frac{1}{b_1} \\ c_2 = \frac{-1}{c_1} \\ \frac{a_1^2+b_1^2}{b_1a_1} = 0 \\ \frac{a_1^2-c_1^2}{a_1c_1} = 0 \\ \frac{b_1^2-c_1^2}{b_1c_1} = 0 \end{cases}.$$

It is thus sufficient to find $(a_1, b_1, c_1) \in V(X^2 + Y^2) \cap V(X^2 - Z^2) \cap V(Y^2 - Z^2)$ such that $a_1, b_1, c_1 \neq 0$. However for any given point in the intersection, $a_1^2 = -b_1^2$ and $c_1^2 = b_1^2$, implying $b_1^2 = c_1^2 = a_1^2 = b_1^2$, hence $c_1 = 0$. It is therefor not possible to write f as a product of linear homogeneous polynomials. The theory, therefor does not extend naively to a theory of the behavior of hypersurfaces.

Example 3.4.26. Proposition 3.4.22 is not true for general irreducible hypersurfaces. Indeed, consider a curve f with a multiple point $P = (a, b)$. Let $g = zf \in K[x, y, z]$. Then $V(g)$ has infinitely many multiple points since (a, b, c) is a multiple point of g for each $c \in K \setminus 0$.

Definition 3.4.27. Let $V \subset \mathbb{A}^n$ be an affine variety, $P \in V$. We define *the tangent space of* V *at* P to the linear subspace

$$T_P(V) := \{v \in \mathbb{A}^n : \forall g \in I(V), (\nabla g)(P) \cdot v = 0\}$$

Lemma 3.4.28. *Let* $f \in K[x_1, \ldots, x_n]$ *be irreducible,* $P \in \mathbb{A}^n$*, set* $V := V(f)$*. Then*

$$T_P(V) = \{v \in \mathbb{A}^n : (\nabla f)(P) \cdot v = 0\}.$$

Proof. If f is irreducible $I(V) = \langle f \rangle$ by HNS. It is clear that $T_P(V) \subset \{v \in \mathbb{A}^n : (\nabla f)(P) \cdot v = 0\}$. Note

$$(\nabla gf)_i = (gf)_{x_i} = gf_{x_i} + g_{x_i}f \implies (\nabla gf)_i(P) = g(P)f_{x_i}(P) \quad \forall g \in K[\mathbf{x}], P \in V.$$

Therefor, if $v \in \mathbb{A}^n$ such that $(\nabla f)(P) \cdot v = 0$, then $(\nabla h)(P) \cdot v = 0$ for any $h \in \langle f \rangle$, implying the converse inclusion. □

Remark 3.4.29. One readily sees that if P is a multiple point, then $T_P(V) = \mathbb{A}^n$. If P is simple then dim $T_P(V) < n$ it seems that one show that for simple points dim $T_P(V)$ = dim V. The tools for this is not developed at this point in fultons curves, so let's leave it

3.4.2 Multiplicities & Local Rings of Rational Function

For a pair of a point and a curve $P \in f$ and $g \in K[x,y]$, $\overline{g} := g + I(f) \in \Gamma(f) \subset \mathcal{O}_P(f)$.

Lemma 3.4.30. *Let $P \in f$ where f is an irreducible curve. If P is a simple point of f, then $\mathcal{O}_P(f)$ is a DVR.*

Proof. WLOG $P = (0,0)$. Since An affine transformation has invertible jacobian, hence by Proposition 3.4.19 we don't lose generality either by assuming the tangent line of f at P is y (recall that the tangent line has (affine) dimension 1). Then $l = x$ is a line passing through P which is not tangent to f at P. If we can prove that $\mathfrak{m}_P(f) = \langle \overline{x} \rangle$ in $\mathcal{O}_P(f)$, then we are done by definition of a DVR, since the local ring of any non-trivial variety at any point is a non-field integral domain that is Noetherian and local (cf. Corollary 3.3.59). By Proposition 3.3.68 $\mathfrak{m}_P(\mathbb{A}^2) = \mathcal{O}_P(\mathbb{A}^2)\langle x, y \rangle$. By Proposition 3.3.69, $\mathcal{O}_P(\mathbb{A}^2)/\langle f \rangle \mathcal{O}_P(\mathbb{A}^2) \simeq \mathcal{O}_P(f)$, hence $\mathfrak{m}_P(f) = \overline{\mathfrak{m}_P(\mathbb{A}^2)} = \langle \overline{x}, \overline{y} \rangle$. Now write $f = gy - hx^2$, where $g = 1 + g'$, where $g' = 0$ or deg $g' > 0$ (we can do this since y is the linear term of f) and where $h \in K[x]$. Since g has constant term 1, we note that $g(P) \neq 0$. Note also that we obtain a factor of x^2 by the fact that no linear term of f is divisible by x. Then

$$\overline{yg} = \overline{x^2 h} \implies \overline{y} = \frac{\overline{x^2 h}}{\overline{g}} \in \langle \overline{x} \rangle \implies \mathfrak{m}_P(f) = \langle \overline{x}, \overline{y} \rangle = \langle \overline{x} \rangle.$$

□

Remark 3.4.31. To spell out the assumption of $P = (0,0)$. Note that f is simple at P if and only if f^{T_P} is simple at $(0,0)$ and that $\mathcal{O}_{(0,0)}(f^{T_P}) \simeq \mathcal{O}_P(f)$ by Corollary 3.3.66, since $f^{T_P} \overset{T_P^{-1}}{\simeq} f$.

Definition 3.4.32. Let P be a simple point on an irreducible curve f. Then ord_P^f is the order function on $K(f)$ induced by the DVR $\mathcal{O}_P(f)$.

Remark 3.4.33. It readily follows that $\text{ord}_P^f \circ \varphi^{-1} = \text{ord}_0^{f^\varphi}$, where φ is the affine change of coordinates mapping P to 0 and the tangent of f at $(0,0)$ to Y, again since $\mathcal{O}_{(0,0)}(f^\varphi) \simeq \mathcal{O}_P(f)$ Perhaps add little diagram. Note then that given a non-tangent line l of f passing through P we have that $\text{ord}_P^f(\bar{l}) = \text{ord}_P^f(\varphi^{-1}(\overline{x + by})) = \text{ord}_0^{f^\varphi}(\overline{x + by}) = 1$. If l is tangent to f at P, then

$$\text{ord}_P^f(\bar{l}) = \text{ord}_0^{f^\varphi}(\overline{y}) = \text{ord}_0^{f^\varphi}(\overline{x}^2) + \text{ord}_0^{f^\varphi}\left(\frac{\overline{h}}{\overline{g}}\right) \geq 2 > 1.$$

Theorem 3.4.34. *Let $P \in f$ where f is an irreducible curve. Then for sufficiently large n,*

$$m_P(f) = \dim\ \mathfrak{m}_P(f)^n/\mathfrak{m}_P(f)^{n+1},$$

where we recall that $\mathfrak{m}_P(f)$ is the maximal ideal of $\mathcal{O}_P(f)$.

Proof. WLOG $P = (0,0)$.Set $R := \mathcal{O}_P(f)$ and $\mathfrak{m} := \mathfrak{m}_P(f)$. For each n we get an exact sequence

$$0 \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n \longrightarrow 0$$

We thus note that

$$\dim\ \mathfrak{m}^n/\mathfrak{m}^{n+1} = \dim\ R/\mathfrak{m}^{n+1} - \dim\ R/\mathfrak{m}^n,$$

hence if we can prove that $\dim\ R/\mathfrak{m}^n = nm_P(f) + s$ for some $s \in \mathbb{Z}$ for every $n \geq m_P(f)$ we get for such n that

$$\dim\ \mathfrak{m}^n/\mathfrak{m}^{n+1} = (n+1)m_P(f) + s - (nm_P(f) + s) = m_P(f).$$

Set $I := \langle x, y\rangle$. Fix $n \geq m_P(f)$. Then $\mathfrak{m}^n = I^n R$ by Proposition 3.3.68. $V(I^n) = \{P\}$, hence

$$K[x,y]/\langle I^n, f\rangle \simeq \mathcal{O}_P(\mathbb{A}^2)/(I^n, f)\mathcal{O}_P(\mathbb{A}^2) \simeq R/I^n R \simeq R/\mathfrak{m}^n$$

by Theorem 3.3.76, Proposition 3.3.69. We will be using the fact that

$$K[x,y]/I^d \simeq K_{\leq d-1}[x,y],$$

repeatedly ref. Recall that $\dim\ K_{\leq d-1}[x,y] = \frac{d(d+1)}{2}$ (cf. Example 1.8.34. Since the lowest degree term of f is of degree $m := m_P(f)$ we have that $f \in I^m$. Hence for $g \in K[x,y]$, $g \in I^{n-m}$ if and only if $fg \in I^n$. Let $\sigma : K[x,y]/I^n \twoheadrightarrow K[x,y]/\langle f, I^n\rangle$ and $\tau : K[x,y]/I^{n-m} \hookrightarrow K[x,y]/I^n, g + I^{n-m} \mapsto fg + I^n$. We then get an exact sequence

$$0 \longrightarrow K[x,y]/I^{n-m} \xhookrightarrow{\tau} K[x,y]/I^n \xrightarrow{\sigma\ \ } K[x,y]/\langle f, I^n\rangle \longrightarrow 0$$

Then

$$\begin{aligned}
\dim\ K[x,y]/\langle f, I^n\rangle &= \dim\ K[x,y]/I^n - \dim\ K[x,y]/I^{n-m} = \frac{n(n+1)}{2} - \frac{(n-m)(n-m+1)}{2} \\
&= \frac{n(n+1)}{2} - \frac{n^2 - nm + n - nm + m^2 + m}{2} \\
&= \frac{n(n+1)}{2} + nm - \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\
&= nm - \underbrace{\frac{m(m+1)}{2}}_{s}.
\end{aligned}$$

□

Remark 3.4.35. Consider an arbitrary curve f with components $f_1, \ldots, f_m$ such that $f = \prod_i^m f_i^{e_i}$. Then the multiplicity f at a point $P \in f$ depends only on the local rings of the components at P and the positive integers $e_1, \ldots, e_m$.

Theorem 3.4.36. *Let $P \in f$ where f is curve. P is a simple point of f if and only if $\mathcal{O}_P(f)$ is a DVR. In this case, if $l = ax + by + c$ is a line through P such that l is not tangent to f at P, then $l + \Gamma(f) \in \mathcal{O}_P(f)$ is a uniformizing parameter of $\mathcal{O}_p(f)$.*

Proof. The first implication follows from Lemma 3.4.30. If $\mathcal{O}_P(f)$ is a DVR, then by ref and the prior theorem for sufficiently large n,

$$m_P(f) = \dim\ \mathfrak{m}_P(f)^n/\mathfrak{m}_P(f)^{n+1} = 1.$$

The remaining statement was already proven in Lemma 3.4.30. □

Remark 3.4.37. For a local ring R, the function

$$\chi_R : \mathbb{N} \to \mathbb{N}$$
$$n \mapsto \dim\ R/\mathfrak{m}$$

is called the *Hilbert-Samuel polynomial of R.*

Example 3.4.38.

1. Let $R = \mathcal{O}_P(\mathbb{A}^2)$, $P \in \mathbb{A}^2$. Then $\chi_R(n) = \dim\ R/\mathfrak{m}^n = \dim\ K[x,y]/\langle x, y\rangle^n = \frac{n(n+1)}{2}$ for each n.

2. We determine χ for $R := \mathcal{O}_P(\mathbb{A}^k)$ at a point $P \in \mathbb{A}^k$. We then have that

$$\chi_R(n) = \dim\ R/\mathfrak{m}^n = \dim\ K[x_1, \ldots, x_k]/\langle x_1, \ldots, x_k\rangle^n = \dim\ K_{\leq n-1}[x_1, \ldots, x_k] = \binom{k+n-1}{k}$$
$$= \frac{(n-1+k)!}{(n-1)!k!} = \frac{1}{k!}\prod_{h=n-1}^{n-1+k} h = \frac{1}{k!}n^k + \cdots$$

The n'th Hilbert-Samuel polynomial for R is thus a polynomial over $\mathbb{Q}$ of degree k with leading coefficient $\frac{1}{r!}$.

Proposition 3.4.39. *Let $f \in K[x_1, \ldots, x_k]$ define an irreducible hypersurface containing $P = 0 \in \mathbb{A}^k$. Let $f_{m_P(f)}$ be the polynomial of lowest degree terms in f. Set $R := \mathcal{O}_P(V(f))$ and denote its maximal ideal by $\mathfrak{m}$. For sufficiently large n, $\chi(n)$ is a polynomial n of degree $k-1$ with leading coefficient $m_P(f)/(k-1)!$*

Proof. DO AT SOME POINT □

Remark 3.4.40. Given a local ring R we are enclined to define $m(R)$ (the *Hilbert-Samuel multiplicity of R*) as $\chi(1)$.

Definition 3.4.41. A point simple point P on a curve f is called a *flex* if $\text{ord}_P^f(l) \geq 3$, where l is tangent of f at P. It is called *ordinary* if the inequality is true with equality, otherwise it is called *higher*.

Example 3.4.42. 1. Let $f_n = y - x^n$, $n \geq 1$ and $P = (0,0)$.

For $n = 1$, $l_1 = y - x$ is tangent to f_1 at P. x is a non-tangent line to f_1 at P. Hence

$$\overline{l_1} = 0 \implies \text{ord}_P^{f_1}(\overline{l_1}) = \infty,$$

and hence P is a higher flex.

For $n \geq 2$, $l_n = y$ is the tangent of f_n at P, hence x is a non-tangent to f_n at P, meaning $\overline{x}$ is a uniformizing parameter of $\mathcal{O}_P(f_n)$. We thus get that

$$\text{ord}_P^{f_n}(\bar{l}) = \text{ord}_P^{f_n}(\overline{x}^n) = n,$$

so for $n = 2$, P is not a flex, for $n = 3$, P is an ordinary flex and for $n \geq 4$, P is a higher flex.

2. Let $f = y - \sum_2^n a_i x^i$, where the polynomial in x is non-zero. Then $P = (0,0)$ is a flex of P if and only if $a_2 = 0$. Indeed, y is the tangent of f at P and $\overline{x}$ is a uniformizing parameter of $\mathcal{O}_P(f)$, hence

$$\text{ord}_P^f(\overline{y}) = \text{ord}_P^f\left(\sum_2^n a_i \overline{x}^i\right) = \min(\{i : a_i \neq 0\}).$$

Hence if $a_2 = 0$, $\text{ord}_P^f(\overline{y}) \geq 3$; otherwise $\text{ord}_P^f(\overline{y}) = 2$.

Proposition 3.4.43. *Let P be a point on an irreducible curve f. Then*

$$\dim\ \mathfrak{m}_P(f)^n/\mathfrak{m}_P(f)^{n+1} = n+1, \quad (0 \leq n < m_P(f)),$$

hence P is simple if and only if $\dim\ \mathfrak{m}_P(f)/\mathfrak{m}_P(f)^2 = 1$*; otherwise* $\dim\ \mathfrak{m}_P(f)/\mathfrak{m}_P(f)^2 = 2$.

Proof. With the notation used in Theorem 3.4.34 we see that $R/\mathfrak{m}^k \simeq K[x,y]/\langle f, I^k\rangle$ for each $k \geq 0$. In particular for $k \leq m$, $\langle f, I^k\rangle = I^k$ since $f \in I^m$, hence $R/\mathfrak{m}^k \simeq K[x,y]/I^k$. For each $n \in \{0,\ldots,m-1\}$ we get an exact sequence

$$0 \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n \longrightarrow 0$$

Hence,

$$\dim\ \mathfrak{m}^n/\mathfrak{m}^{n+1} = \dim\ R/\mathfrak{m}^{n+1} - \dim\ R/\mathfrak{m}^n = \frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = \frac{(n+1)(n+2-n)}{2} = n+1.$$

□

Example 3.4.44. Set $f_1 := x^2 - y^3, f_2 := y^2 - z^3 \in K[x,y,z]$, $P = (0,0,0)$ and $V := V(f_1, f_2) \subset \mathbb{A}^3$. Furthermore set $\mathcal{O} := \mathcal{O}_P(V)$ and $\mathfrak{m} := \mathfrak{m}_P(V)$. We aim to compute $\mathfrak{m}/\mathfrak{m}^2$. Note that dim $\mathcal{O}\mathfrak{m} = 1$. Elements of $\mathcal{O}/\mathfrak{m}^2$ are on the form $\frac{a\overline{x}+b\overline{y}+c\overline{z}+d}{\beta}$. An element of $\mathfrak{m}/\mathfrak{m}^2$ is of the form $a\overline{x} + b\overline{y} + c\overline{z} + d\beta$, since $\overline{t} \in \mathfrak{m}$ for any non-constant monomial, meaning $\overline{s} \in \mathfrak{m}^2$ for any monomial of degree > 2. One readily verifies that if $\alpha + \mathfrak{m}$ is such that $\alpha(P) = 0$, then

$$\frac{\alpha}{\beta} + \mathfrak{m}^2 = \frac{\alpha}{\beta(P)} + \mathfrak{m}^2,$$

by checking the statement for the image of a mononimal in $\mathfrak{m}$ modulo $\mathfrak{m}^2$. So $\mathfrak{m}/\mathfrak{m}^2 = K(\overline{x} + \mathfrak{m}^2) + K(\overline{y} + \mathfrak{m}^2) + K(\overline{z} + \mathfrak{m}^2)$. It remains to argue that these elements are linearly independent. Indeed if $a\overline{x} + b\overline{y} + c\overline{z} = \overline{\mu}\overline{\lambda}$ where μ and λ are linear polynomials in x, y, z vanishing at P, then $ax + by + cz - \mu\lambda \in \langle f, g\rangle$, which means $a = b = c = 0$, since any term of a polynomial in $\langle f, g\rangle$ has x-degree > 2 or y-degree > 2 or z-degree > 3 and there can be no cancellation of ax, by, cz with terms in $\mu\lambda$, since these are all degree 2 monomials. We conclude that dim $\mathfrak{m}/\mathfrak{m}^2 = 3$.

3.4.3 Intersection Numbers

For this subsection, we begin by fixing curves f and g together with a point $P \in \mathbb{A}^2$. We allow that f or g be non-zero constants. We will see that allowing this will make definitions simpler and that it will inconsequential for counting intersections of curves.

Definition 3.4.45. f and g *intersect properly* at P, if f and g have no common components passing through P. I.e. for f and g to intersect properly at P we require that for any common factor h of f and g, $h(P) \neq 0$

Definition 3.4.46. If P is simple for both f and g such that f and g have distinct tangents at P we say that f and g intersect *transversally* at P.

Definition 3.4.47. An *intersection number of f and g at P* is a value $I(P, f \cap g) \in \mathbb{Z} \cup \{\infty\}$ satisfying the following 7 axioms:

1. When f and g intersect properly at P, $I(P, f \cap g) \geq 0$; otherwise $I(P, f \cap g) = \infty$.
2. $I(P, f \cap g) = 0 \iff P \notin F \cap G$ and $I(P, f \cap g)$ depends only on the components of f and g passing through P, i.e if $f_1, \ldots, f_m$ and $g_1 \ldots, g_l$ are these components, then
$$I(P, f \cap g) = I\left(P, \prod f_i \cap \prod g_i\right)$$

3. If T is an affine change of coordinates on $\mathbb{A}^2$ with $T(Q) = P$, then

$$I(P, f \cap g) = I(Q, f^T \cap g^T).$$

4. $I(P, f \cap g) = I(P, g \cap f)$.

5. $I(P, f \cap g) \geq m_P(f)m_P(g)$ with equality holding if and only if f and g have no tangents in common. In particular $I(P, f \cap g) = 1$ if and only if f and g intersect transversally.

6. If $f = \prod_1^m f_i^{r_i}$ and $g = \prod_1^l g_i^{s_i}$, then

$$I(P, f \cap g) = \sum_1^m \sum_1^l r_i s_j I(P, f_i \cap g_j).$$

7. $I(P, f \cap g)$ depends only on the image of g in $\Gamma(f)$, i.e. for every $h \in K[x, y]$,

$$I(P, f \cap g) = I(P, f \cap g + hf).$$

The intuition for the above definition is that we want to be able count the number of intersections f and g. We detect an intersection if $P \in f \cap g$. We want this way of counting to be sensitive to the fact that intersections may occur with multiplicity. For example if P vanishes on two components of f and on one component of g, then an intersection in a sense actually occurs twice. To be more concrete take $f = x$ and $g = y^2 - x$. Then g intersects f only at $(0, 0)$, however $g(0, y)$ has a double root at $(0, 0)$, so this intersection should really be counted twice. This is indeed the case: $I((0, 0), f \cap g) = I((0, 0), f \cap g + f) = I((0, 0), x \cap y^2 = 2I((0, 0), x \cap y) = 2$. We want to describe this number in terms of rational functions on the affine plane at each P. To do this we need some technical lemmas.

Lemma 3.4.48. *Set $m := m_P(f)$ and $n := m_P(g)$. Suppose f and g intersect properly, have no common tangents and pass through P. Then for $I := \langle x, y \rangle$, we have*

$$I^t \subset \langle f, g \rangle \mathcal{O}_P(\mathbb{A}^2),$$

For sufficiently large $t \geq 0$.

Proof. Note that $\gcd(f, g) = 0$, hence $f \cap g = \{Q_1, \ldots, Q_k, P\}$. Then there is an $h \in K[x, y]$ such that $h(Q_i) = 0$ and $h(P) = 1$. Then $xh, yh \in I(f \cap g)$, hence for a suitably large N, $(xh)^N, (yh)^N \in \langle f, g \rangle$ (cf. HNS). In $\mathcal{O}_P(\mathbb{A}^2)$, h is a unit, since it doesn't pass through P, hence $x^N, y^N \in \langle f, g \rangle$, implying the result for $t \geq 2N$ □

Lemma 3.4.49. *With the same setup as the above lemma we have that*

$$I^t \subset \langle f, g\rangle \mathcal{O}_P(\mathbb{A}^2),$$

whenever $t \geq m + n - 1$

Proof. Let $l_1, \ldots, l_m$ and $\lambda_1, \ldots, \lambda_n$ be the tangents of f resp. g. for $i > m$ define $l_i := l_m$ and for $j > n$ define $\lambda_j := \lambda_n$ Define $a_{ij} := \left(\prod_1^i l_i\right)\left(\prod_1^j \lambda_i\right)$ for every $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. Then $\{a_{ij} : i + j = t\}$ forms a basis for $V_K(t, 2) = I^t$ by do exercise!. Hence it is sufficient to prove that $a_{ij} \in \langle f, g\rangle \mathcal{O}_P(\mathbb{A}^2)$ for every i, j with $i + j \geq m + n - 1$. If we are given i, j such that $i + j \geq m + n - 1$. Then $i \geq m$ or $j \geq n$. By the lemma above there is a $T \geq 0$ such that $I^T \subset \langle f, g\rangle$. Hence if we can prove that $a_{ij} \in \langle f, r\rangle$ where deg $r \geq T$ in the case where $i \geq m$ and symmetrically that $a_{ij} \in \langle g, s\rangle$ when $j \geq n$ we are done. Indeed in the case $i \geq m$ we would have that $a_{ij} = bf + cr \in \langle f, g\rangle$. The other case follows by an identical argument. Suppose $i \geq m$. Then $a_{ij} = a_{m0}b$ for some homogeneous b of degree $t := i + j - m$. Writing $f = A_{m0} + f'$ where deg $f' \geq m + 1$ we get that

$$a_{ij} = bf - bf'$$

where $bf' = 0$ or the terms of bf' are of degree$\geq i + j + 1$. In the second case write all terms in bf' as an appropriate linear combination of elements in $\{a_{kh} : k + h \geq m'\}$. Then $a_{ij} = cf - ch'$ where $h' = 0$ or has terms of degree $\geq m' + 1$. Repetition of this argument yields an expression $a_{ij} = uf + r$ with $r \in I^T$. □

Lemma 3.4.50. *With the same setting as above, the map* $\rho : K[x, y]/I^m \times K[x, y]/I^n \to K[x, y]/I^{n+m}, (\lambda + I^m, \mu + I^n) \mapsto \lambda f + \mu g + I^{n+m}$ *is injective if and only if* f *and* g *have distinct tangents at* P.

Proof. " $\Longrightarrow$ ": If f and g had a common tangent, l say, write $f_m = lf'_{m-1}$ and $g_n = lg'_{n-1}$. Let F and G, denote term of degree$> m$ resp. n. Then $f'_{m-1} \notin I^m$ and $g'_{n-1} \notin I^n$ and

$$g'_{n-1}f + f'_{m-1}g = f'_{m-1}g_n + g'_{n-1}F - f'_{m-1}g_n - f'_{m-1}G = g'_{n-1}F - f'_{m-1}G \in I^{m+n}.$$

implying $\rho(g'_{n-1} + I^m, -f'_{m-1} + I^n) = 0$, which means ρ is not injective. " $\Longleftarrow$ ": Suppose the tangents are distinct. Suppose then that $af + bg \in I^{n+m}$. Suppose for a contradiction that $m_P(a) := r < m$ or $m_P(b) := s < n$. Write $a = a_r + \ldots$ and $b = b_s + \ldots$. The terms in $a_r f_m$ and $b_s g_n$ must cancel, hence $r + m = s + n$ and $a_r f_m = -b_s g_n$. Then, since f_m and g_n have no common factors. Then $f_m \mid a_r$ and $g_n \mid b_s$, leading to a contradiction. This means $r \geq m$ and $s \geq n$, hence $a \in I^m$ and $b \in I^n$. □

Remark 3.4.51. We describe an algorithm $J(_, _ \cap _)$ for computing intersection numbers given that an intersection number exists for every $P \in \mathbb{A}^2$ and every pair of curves f and g. We first describe it in the special case $P = (0,0)$:

(I) If $P \neq (0,0)$, return $J((0,0), f^{T_P} \cap g^{T_P})$.

(II) If f and g do not intersect properly, then return ∞.

(III) If $P \notin f \cap g$, return 0.

(IV) Set $r := \deg_x f$ and $s := \deg_x g$. If $r \leq s$, proceed, otherwise swap the role of f and g in the next steps

(V) If $r = 0$, write $f = yh$ for a suitable unique $h \in K[x,y]$ and return $m_P(g(x,0)) + J(P, h \cap g)$.

(VI) If $r > 0$ scale f and g such that $f(x,0)$ and $g(x,0)$ become monic, and set $h := g - x^{s-r} f$. Return $J(P,h)$.

We will prove that the algorithm will always terminate in the theorem below.

Theorem 3.4.52. $\dim \mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle$ *is the unique intersection number of* f *and* g *at* P.

Proof. **Uniqueness:** We prove that if there is an intersection number $I(P, f \cap g)$ for each $P \in \mathbb{A}^2$ and each pair of curves f, g, then the algorithm above returns that intersection value. Trivially, if there were two candidates for intersection numbers $I(_, _ \cap _)$, $I'(_, _ \cap _)$, then

$$I(_, _ \cap _) = J(_, _ \cap _) = I'(_, _ \cap _).$$

By property 3, if $P \neq (0,0)$, then $I(P, f \cap g) = I((0,0), f^{T_P} \cap g^{T_P})$, hence, if the algorithm terminates correctly in the $P = (0,0)$-case so will it in the general case. So assume $P = (0,0)$. If $I(P, f \cap g) = \infty$, then f and g do not intersect properly by property 1, hence $J(P, f \cap g) = \infty$. Suppose $I(P, f \cap g) < \infty$. Suppose $I(P, f \cap g) = 0$. Then $P \notin f \cap g$ by property 2, hence $J(P, f \cap g) = 0$. Suppose $I(P, f \cap g) > 0$. Suppose $J(P, A \cap B) = I(P, A \cap B)$, whenever $I(P, A \cap B) < n$ for some $n \geq 0$. If $I(P, f \cap g) = n$, set $r := \deg_x f$ and $s := \deg_x g$, where WLOG $r \leq s$ and $\deg 0 = 0$.

When r = 0, we land in (V). We get that $y \mid f$, since $P = (0,0) \in f$, hence we can write $f = yh$ for a unique $h \in K[x,y]$. Then

$$n = I(P, f \cap g) = I(P, y \cap g) + I(P, h \cap g).$$

Write $g = \lambda + yk$ for some unique $\lambda \in K[x]$ and $k \in K[x,y]$. Note that $\lambda \neq 0$ for otherwise g and f would not intersect properly. We also have that $m_P(\lambda) > 0$ since $\lambda(P) = \text{ev}_P(\lambda + yk) = g(P) = 0$. Then

$$I(P, y \cap g) = I(P, y \cap \lambda + yk) = I(P, y \cap \lambda) = m_P(y)m_P(\lambda) = m_P(\lambda) > 0.$$

For the second equality we use property 7. For the third equality we use property 5. We thus have that

$$I(P, h \cap g) = n - m_P(\lambda) < n.$$

Note that $\lambda = g(x, 0)$. Then by induction hypothesis

$$I(P, f \cap g) = m_P(g(x,0)) + I(P, h \cap g) = m_P(g(x,0)) + J(P, h \cap g) = J(P, f \cap g).$$

When $r > 0$ we land in (VI). WLOG f and g are monic. We set $a_0 := \frac{1}{\text{lc}(g)}g$, $b_0 := \frac{1}{\text{lc}(f)}f$ and define

$$h_i := a_{i-1} - x^{\deg_x\ a_{i-1} - \deg_x\ b_{i-1}} b_{i-1}.$$

We pick a_i to be the polynomial with largest x-degree of b_{i-1} and h_i scaled by the inverse of its leading coefficient and let b_i be the polynomial with smaller x-degree of the two scaled by the inverse of it leading coefficient. We thus note that $\deg_x\ h_1 < \deg_x\ g$ and that $\deg_x\ h_i < \deg_x h_{i+1}$. Hence for some $N \geq 1$, $\deg_x\ b_N = 0$ with $\deg_x\ a_N \geq \deg\ b_N$. We repeatedly use property 7 and 4 to find that

$$I(P, f \cap g) = I(P, a_0 \cap b_0) = \ldots = I(P, a_N \cap b_N) = J(P, a_N \cap b_N),$$

where the last equality follows from case $r = 0$. Applying the algorithm to P, f and g clearly yields

$$J(P, f \cap g) = J(P, a_0 \cap b_0) = \ldots = J(P, a_N \cap b_N) = I(P, f \cap g),$$

finishing the proof of uniqueness Check if can be written better later!.

Existence: We first show that property 2 is fulfilled. Note that $\mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle = 0$ if and only if f or g is a unit which is equivalent to f or g not passing through P. Note that if $f_1, \ldots, f_m$ and $g_1, \ldots, g_l$ are the components of f resp. g. passing through O, then in $\mathcal{O}_P(\mathbb{A}^2)$, $\langle f, g\rangle = \langle \prod_1^m f_i, \prod_1^l g_i\rangle$ since every other component is a unit in $\mathcal{O}_P(\mathbb{A}^2)$. Property 4 is obviously also fulfilled. As for property 7 note that clearly $\langle f, g\rangle = \langle f, g + hf\rangle$ for every curve h. For an isomorphism φ an affine change of coordinates on $\mathbb{A}^2$ taking Q to P is an isomorphism it follows that

$$\mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle \simeq \mathcal{O}_{T(Q)}(\mathbb{A}^2)/\langle f^T, g^T\rangle$$

by Corollary 3.3.67, which shows that property 3 holds. From this point forward we therefor need only consider the case where $P = (0,0)$ and P passes through every component of f and g. We now check that property 1 is satisfied. If f and g have no common components then $f \cap g = \{P_1, \ldots, P_n\}$ is finite, hence if $P_i = (a_i, b_i)$,

$$\dim\ \mathcal{O}_{P_i}(\mathbb{A}^2)/\langle f,g\rangle = \dim\ K[x,y]/\langle f,g\rangle < \infty$$

by Theorem 3.3.76 and a Corollary of HNS. If f and g have a common component h, then $\langle f,g\rangle \subset \langle h\rangle$, hence we get that $\dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle \geq \mathcal{O}_P(\mathbb{A}^2)/\langle h\rangle$ since we have a K-algebra surjection

$$\mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle \longrightarrow\!\!\!\!\!\rightarrow\ \mathcal{O}_P(\mathbb{A}^2)/\langle h\rangle$$

Note that $\mathcal{O}_P(\mathbb{A}^2)/\langle h\rangle \simeq \mathcal{O}_P(h)$ by Proposition 3.3.69. Note also that $\mathcal{O}_P(h) \supset \Gamma(f)$ and that $\dim\ \Gamma(f) = \infty$ by a corollary of HNS, hence $\dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle \geq \dim\ \Gamma(f) = \infty$. We move on to prove that property 6. holds. It is sufficient to prove that $I(P, f \cap gh) = I(P, f \cap g) + I(P, f \cap h)$ since 6 then follows by induction arguments. When f and gh have a common component, the result is trivial. So suppose this is not the case. We get an injective K-algebra map $\mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle \to \mathcal{O}_P(\mathbb{A}^2)/\langle f,gh\rangle, \lambda + \langle f,g\rangle \mapsto \lambda h + \langle f,gh\rangle$. It is well-defined K-algebra homomorphism, since if $\lambda = af + bg$, then $h\lambda = afh + bgh \in \langle f,gh\rangle$. Suppose $z \in \mathcal{O}_P(\mathbb{A}^2)$ is given such that $zh \in \langle f,gh\rangle$. Then for suitable $b, c \in \mathcal{O}_P(\mathbb{A}^2)$,

$$hz = bf + cgh$$

For some $\mu \in K[x,y] \setminus 0$, $\mu b, \mu c \in K[x,y]$, hence

$$\mu zh = \mu bf + \mu cgh \implies \mu bf = (\mu z - \mu cg)h.$$

Then since $\gcd(f,h) = 1$, we get that $f \mid \mu z - \mu cg$ in $K[x,y]$, hence $\mu z - \mu cg = qf$, meaning

$$z = \frac{q}{\mu} f + cg \in \langle f,g\rangle$$

We thus get a short exact sequence

$$0 \longrightarrow \mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle \hookrightarrow \mathcal{O}_P(\mathbb{A}^2)/\langle f,gh\rangle \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_P(\mathbb{A}^2)/\langle f,h\rangle \longrightarrow 0\ ,$$

hence $\dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f,gh\rangle = \dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f,g\rangle + \dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f,h\rangle$ which is what we wanted to show.

We know just need to prove that property 5 holds. Set $m := m_P(f)$, $n := m_P(g)$ and

$I = \langle x, y\rangle \subset K[x, y]$. Consider the diagram

$$\begin{array}{ccccccc} K[x,y]/I^m \times K[x,y]/I^n & \xrightarrow{\rho} & K[x,y]/I^{m+n} & \xrightarrow{\sigma} & K[x,y]/\langle I^{n+m}, f, g\rangle & \longrightarrow & 0 \\ & & & & \downarrow \alpha & & \\ & & \mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle & \xrightarrow{\pi} & \mathcal{O}_P(\mathbb{A}^2)/\langle I^{n+m}, f, g\rangle & \longrightarrow & 0 \end{array}$$

Where $\rho(\lambda + I^m, \mu + I^n) = \lambda f + \mu g + I^{m+n}$, σ and π are the canonical surjections. Note that $V(I^{n+m}, f, g) \subset V(I) = \{P\}$, hence α is the isomorphism $K[x, y]/\langle I^{n+m}, f, g\rangle \simeq \mathcal{O}_P(\mathbb{A}^2)/\langle I^{n+m}, f, g\rangle$ (cf. Theorem 3.3.76). Clearly $\sigma \circ \rho = 0$, hence the top sequence is exact

$$\dim\ K[x,y]/I^m + \dim\ K[x,y]/I^n = \dim\,(\ker\ \rho) + \dim\,(\operatorname{im}\rho) = \dim\,(\ker\ \rho) + \dim\,(\ker\ \sigma),$$

hence $\dim\ K[x,y]/I^m + \dim\ K[x,y]/I^n \geq \dim\ (\ker\ \sigma)$, which is true with equality if and only if ρ is injective. Furthermore,

$$\dim\ K[x,y]/I^{m+n} = \dim\,(\operatorname{im}\sigma) + \dim\,(\ker\ \sigma) = \dim\ K[x,y]/\langle I^{n+m}, f, g\rangle + \dim\ \ker\ \sigma,$$

hence $\dim\ K[x,y]/\langle I^{m+n}, f, g\rangle = \dim\ K[x,y]/I^{m+n} - \dim\ \ker\ \sigma$. It thus follows that

$$\begin{aligned} I(P, f \cap g) &= \dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle \geq \dim \mathcal{O}_P(\mathbb{A}^2)/\langle I^{m+n}, f, g\rangle \\ &= \dim\ K[x,y]/\langle I^{m+n}, f, g\rangle = \dim\ K[x,y]/\langle I^{m+n}\rangle - \dim\,(\ker\ \sigma) \\ &\geq \dim\ K[x,y]/\langle I^{m+n}\rangle - \dim\ K[x,y]/I^m - \dim\ K[x,y]/I^n \\ &= mn \end{aligned}$$

which holds with equality if and only if ρ is injective and π is an isomorphism. The last equality follows from verifying the computation,

$$\frac{(m+n)(m+n+1)}{2} - \frac{m(m+1)}{2} - \frac{n(n+1)}{2} = mn.$$

That π is an isomorphism is ensured by the assumption that f and g have no common tangents due to Lemma 3.4.49. Indeed, $m+n \geq m+n-1$, meaning $\langle I^{n+m}, f, g\rangle = \langle f, g\rangle$. That ρ is injective also follows from the assumption that f and g have no common tangents due to Lemma 3.4.50. □

Example 3.4.53. 1. Over $\mathbb{C}$, let $e = (x^2 + y^2)^2 + 3x^2y - y^3$, $f = (x^2 + y^2)^3 - 4x^2y^2$. Then

$$f - e(x^2 + y^2) = y(\underbrace{(x^2 + y^2)(y^2 - 3x^2) - 4x^2y}_{g}) = yg.$$

Note now that

$$g + 3e = y(\underbrace{5x^2 - 3y^2 + 4y^3 + 4x^2y}_{h}) = yh.$$

The tangent lines of e are two copies of y, $(\sqrt{3}x + y)$ and $\sqrt{3}x - y$ and the tangent lines of h are two copies of x, hence e and h have no tangent lines in common. Then for $P = (0, 0)$.

$$\begin{aligned} I(P, e \cap f) &= I(P, e \cap f - e(x^2 + y^2)) = I(P, e \cap yg) = I(P, e \cap y) + I(P, e \cap g) \\ &= I(P, e \cap y) + I(P, e \cap g + 3e) = I(P, e \cap y) + I(P, e \cap yh) \\ &= 2I(P, e \cap y) + I(P, e \cap h) = 2I(P, y \cap e - y^4 - 2x^2y^2 - 3x^2y + y^3) + m_P(e)m_P(h) \\ &= 2I(P, y \cap x^4) + 6 = 8I(P, y \cap x) + 6 = 8 + 6 = 14. \end{aligned}$$

Note that instead of religiously following the steps of the algorithm presented it is often better to intuitively use the properties of the intersection numbers.

2. Consider $a = y - x^2$, $b = y^2 - x^3 + x$. Then $I((0, 0), a \cap b) = m_0(a)m_0(b) = 1$. We then see that a and b intersect once in $(0, 0)$. Note that $a \cap b = \{c \in \mathbb{C} : c^4 - c^3 + c\} = \{(c, c^2) : c \in V(x, x^3 - x^2 + 1)\}$. Hence we get an intersection at $(0, 0)$ and another real intersection (c, c^2)

$$c = \frac{1}{3}\left(1 - \sqrt[3]{\frac{2}{25 - \sqrt{69}}} - \sqrt[3]{\frac{1}{2}(25 - 3\sqrt{69})}\right),$$

together with a pair of distinct non-real solutions. We thus have that a and b have 4 intersections each of multiplicity 1.

3. Consider a as before and $c = y^2 - x^3$. It is clear that $a \cap c = \{(\alpha, \alpha^2) : \alpha V(x, x - 1)\} = \{(0, 0), (1, 1)\}$.

$$\begin{aligned} I((0, 0), a \cap c) &= I((0, 0), a \cap c - ay) = I((0, 0), a \cap x^3 + x^2y) = I((0, 0), a \cap x^2(y + x)) \\ &= 2I((0, 0), a \cap x) + I((0, 0), a \cap y + x) = 2m_0(a)m_0(x) + m_0(a)m_0(y + x) = 3. \end{aligned}$$

We also have that $a(x + 1, y + 1) = -x^2 + y - 2x$ and $b(x + 1, y + 1) = y^2 + 2y + 1 - x^3 - 3x^2 - 3x - 1 = -x^3 + y^2 - 3x^2 + 2y - 3x$. These polynomials clearly intersect transversally, hence $I((1, 1), a \cap b) = I((0, 0), a(x + 1, y + 1) \cap b(x + 1, y + 1)) = 1$. a and b therefor have 4 intersections; 3 at $(0, 0)$ and 1 at $(1, 1)$.

Proposition 3.4.54. *If P is a simple point of f, then $I(P, f \cap g) = \text{ord}_P^f(g)$.*

Proof. We note that when writing f as a product of it's components $p_1, \dots, p_n$. Then $m_P(f) = \sum_1^n m_P(p_i)$, hence WLOG, P is passed through by only one p_i. We thus have that it is sufficient to consider the case where f is irreducible by property 2. and 6. We thus get that $\text{ord}_P^f(g) = \dim\ \mathcal{O}_P(f)/\langle g\rangle$ by Lemma 1.6.135. We note that $\langle f\rangle \subset \langle f, g\rangle \subset K[x, y]$. Note also that

$$\langle f, g\rangle/\langle f\rangle = \langle g\rangle/\langle f\rangle \subset \Gamma(f).$$

Then

$$\mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle \simeq \mathcal{O}_P(f)/\langle g\rangle,$$

by Proposition 3.3.69. We thus conclude that

$$\text{ord}_P^f(g) = \dim\ \mathcal{O}_P(f)/\langle g\rangle = \dim\ \mathcal{O}_P(\mathbb{A}^2)/\langle f, g\rangle = I(P, f \cap g),$$

where the second equality is due to Lemma 1.6.135 □

Remark 3.4.55. Deduce only using 1-7

Proposition 3.4.56. *If* $\gcd(f, g) = 1$*, then* $\sum_{P\in\mathbb{A}^2} I(P, f \cap g) = \dim\ K[x, y]/\langle f, g\rangle$

Proof. Note that $f \cap g$ only pass through finitely many $P \in \mathbb{A}^2$ by Theorem 3.2.13. It follows from Corollary 3.3.77 that

$$\sum_{P\in\mathbb{A}^2} I(P, f \cap g) = \dim\ K[x, y]/\langle f, g\rangle$$

□

Corollary 3.4.57. *A line* l *is tangent to* f *if and only if* $I(P, f \cap l) > m_P(f)$.

Proof. This follows immediately from property 5. Indeed, l is a tangent of f if and only if l and f has a tangent in common, which by 5. is equivalent to

$$I(P, f \cap l) > m_P(f)m_P(l) = m_P(f).$$

□

Proposition 3.4.58. *If* P *is a simple point on* f*, then*

$$I(P, f \cap g + h) \geq \min(I(P, f \cap g), I(P, f \cap h)).$$

Proof. Indeed, it follows from Proposition 3.4.54 that

$$I(P, f \cap g + h) = \text{ord}_P^f(g + h) \geq \min(\text{ord}_P^f(g), \text{ord}_P^f(h)) = \min(I(P, f \cap g), I(P, f \cap h)).$$

□

Example 3.4.59. Show bound is not true when point is multiple

Proposition 3.4.60. *Let f be a curve and l a line that is not a component of f. If $l = \{(a + tb, c + td) : t \in K\}$. Define $g := f(a + bz, c + dz) \in K[x]$. For suitable distinct $\alpha_i \in K$, $r_i \geq 1$ we may write*

$$g = \alpha \prod_1^m (z - \alpha_i)^{r_i}.$$

Then there is a one-to-one correspondence between roots of g and points in $l \cap f$ given by $\alpha_i \mapsto P_i := (a + b\alpha_i, c + d\alpha_i)$. Moreover $I(P_i, f \cap l) = r_i$ and

$$\sum_{P \in \mathbb{A}^2} I(P, l \cap f) \leq \deg\ f.$$

Proof. The one-to-one correspondence is established in Proposition 3.1.22. WLOG $l = x$ and $P_i = (0, 0)$. Then

$$f(0, z) = g = z^{r_i} \underbrace{\prod (z - \lambda_j)^{r_j}}_{h}.$$

We can therefor write $f = y^{r_i} h + h'$ where $h'(0, y) = 0$. Note that $h' = xh''$ $h(P_i) \neq 0$ by assumption. Then

$$I(P_i, l \cap f) = I(P_i, x \cap f - xh'') = I(P_i, x \cap y^{r_i} h) = I(P_i, x \cap y^{r_i}) + I(P_i, x \cap h) = r_i I(P_i, x \cap y) = r_i.$$

It then readily follows that.

$$\sum_P I(P, l \cap f) = \sum_1^m e_i = \deg\ g \leq \deg\ f.$$

The last inequality follows from the fact evaluating f in linear polynomials cannot increase the degree (when we use the convention that $\deg\ 0 = 0$). □

Definition 3.4.61. Let f be a curve with only one tangent l at a double point P. f is said to have an *(ordinary) cusp* at P if $I(P, f \cap l) = 3$.

Lemma 3.4.62. *Let f be a curve with a tangent l at a double point P. Then $I(P, f \cap l) \geq 3$.*

Proof. It follows from Corollary 3.4.57 that

$$I(P, f \cap l) > m_P(f) \geq 2.$$

□

Lemma 3.4.63. *Suppose* char $K \neq 2, 3$. *Let $P = (0, 0)$ and $l = y$ be the only tangent of f at a double point P. Then P is a cusp if and only if $f_{xxx}(P) \neq 0$*

Proof. " $\Longrightarrow$ ": $f = y^2 + h$, where every term in h has degree greater than 2. Then $3 = I(P, y \cap f) = I(P, y \cap h) \geq m_P(h) = 3$, which means y is not tangent to h. Then x^3 must divide the lowest degree term of h, hence $f_{xxx}(P) = h_{xxx}(P) = (cx^3 + \dots)_{xxx} = 6c \neq 0$.
" $\Longleftarrow$ ": If $f_{xxx}(P) = 0$, then y is tangent to h (using the same notation as for the first implication). Then $I(P, l \cap f) = I(P, l \cap h) > m_P(h) = 3$ □

Remark 3.4.64. Since every technique used above is indifferent to an affine change of coordinates we get the above result for general tangents and points.

Proposition 3.4.65. *If P is a cusp on f, then only one component of f passes through P.*

Proof. □

Definition 3.4.66. A point P on a curve f is called a *hypercusp* if $m_P(f) > 1$, f has a single tangent l and $I(P, f \cap l) = m_P(f) + 1$.

Lemma 3.4.67.

Proof. □

Proposition 3.4.68.

Proof. □

3.4.4 The Dimension of an Affine Variety

Definition 3.4.69. Let $V \subset \mathbb{A}^n$ be an affine variety. We define the *dimension* of V to be

$$\dim\ V = \text{trdeg}_K\ K(V)$$

Remark 3.4.70. Note that dim is just the functor $\text{trdeg}_K \circ Q \circ \Gamma$. It is thus an integer invariant. Hence if $V \simeq W$, then $\dim\ V = \dim\ W$. Note that $K(\mathbb{A}^n) = K(x_1, \dots, x_n)$, hence $\dim\ \mathbb{A}^n = n$. If V is a linear subvariety, then $V \simeq V(x_{d+1}, \dots, x_n) \simeq \mathbb{A}^d$ for some d, hence $\dim\ V = d$, hence this generalizes the notion of dimension in definition 3.3.31. By Lemma 1.9.78 every variety has finite dimension.

Lemma 3.4.71. *If $V \subset W$ is a subvariety of an affine variety W, then* $\dim\ V \leq \dim\ W$. *Furthermore,* $\dim\ V = \dim\ W \iff V = W$.

Proof. Note that $\Gamma(W) \to \Gamma(V)$ is given by restriction of a polynomial function on W to V, which is clearly surjective. Let $\widetilde{\varphi} : K(W) \twoheadrightarrow K(V)$ be the surjection between rational functions on W to a rational functions on V induced by this map. Since this is a

surjective K-algebra homomorphism, dim W = trdeg $K(W) \geq$ trdeg $K(V)$ = dim V by Lemma 1.9.42. Suppose $n :=$ dim V = dim W. Let $f \in I(V)$. Take $\overline{g} \in \gamma(W)$ such that $\overline{g}|_V = \overline{f} \in \Gamma(V)$. There is a monic, non-zero $H \in K[x_1, \ldots, x_n][y]$ ($x_1, \ldots, x_n$ are polynomial variables over K) such that $H(\overline{g}) = 0$. Then $H(\overline{f}) = 0$, and the minimal monic polynomial in $K[x_1, \ldots, x_n][y]$ vanishing on $\overline{f}$ is y, hence $H = y$, meaning $g \in I(W)$. It follows that $I(V) = I(W)$, hence $V = W$ by Lemma 3.1.42. □

3.4.5 Finite Polynomial Maps

Definition 3.4.72. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be affine varieties and $\varphi : V \to W$ a polynomial map. φ is said to be *finite* if $\widetilde{\varphi} : \Gamma(W) \to \Gamma(V)$ is finite.

Lemma 3.4.73. *Let $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ and $Z \subset \mathbb{A}^l$ be affine varieties and $\varphi : V \to W$, $\psi : W \to Z$ be finite polynomial maps. Then $\psi\varphi$ is finite*

Proof. This follows from functoriality of $\Gamma(\bullet)$ and Lemma 1.9.59. □

Lemma 3.4.74. *Let $V \subset \mathbb{A}^n$ be a d-dimensional, affine variety. Then there is a finite polynomial map $\pi = (y_1, \ldots, y_d) : V \to \mathbb{A}^d$ where* deg $y_i = 1$.

Proof. By Corollary 1.9.79 there is a finite K-algebra homomorphism $\iota : K[y_1, \ldots, y_d] \hookrightarrow \Gamma(V)$, where $y_1, \ldots, y_n \in \Gamma(v)$ is a Noether normalization of $\Gamma(V)$. By the Noether normalization theorem for infinite fields we may pick $y_i = \sum_1^n a_{ij} x_j$ for suitable $a_{ij} \in K$. Then upon defining

$$\pi : V \to \mathbb{A}^d$$
$$v \mapsto (y_1(v), \ldots, y_d(v))$$

we see that $\widetilde{\pi} = \iota$, hence π is finite. □

3.4.6 A Second Approach to an ENS

Theorem 3.4.75. *Consider non-constant $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_m) = \emptyset$ and $d_1 \geq \cdots \geq d_m$. Suppose $m \leq n$. Then there are $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$ such that*

1. $\sum_1^m \lambda_i f_i = 1$,
2. deg $\lambda_i f_i \leq \prod_1^m d_i \leq D^m \quad (i \in \{1, \ldots, m\}, D :=$ max$(d_1, \ldots, d_m)$.

Proof. Using WNS we can find $\mu_1, \ldots, \mu_m \in K[x_1, \ldots, x_n]$ such that

$$\sum_1^n \mu_i f_i = 1.$$

Consider the polynomial map

$$\varphi : \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n \times \mathbb{A}^m$$
$$(v, a) \mapsto (v, af_1(v), \ldots, af_m(v)).$$

. We prove that $X = Y$. It is clear that $X \subset Y$. Let $(v, w) \in Y$. WLOG $f_1(v) \neq 0$. Set $a = \frac{w_1}{f_1(v)}$. For each i we $w_i f_1(v) = w_1 f_i(v)$, hence $w_i = af_i(v)$. It follows that

$$(v, w) = (v, af_1(v), \ldots, af_m(v)) = \varphi(v, a) \in X.$$

Since $\varphi^{-1}(X) = \mathbb{A}^n \times \mathbb{A}^1$ is a variety and $X = \text{im } \varphi$ it follows that X is a variety by Lemma 3.3.4. Then $\phi : \mathbb{A}^n \times \mathbb{A}^1 \to X$ is a polynomial map with inverse

$$\phi^{-1} : X \to \mathbb{A}^n \times \mathbb{A}^1$$
$$(v, w) = (v, af_1(v), \ldots, af_m(v)) \mapsto \left(v, a \sum_1^m g_i(v) f_i(v) \right) = (v, a).$$

Then ϕ is finite by Lemma **??** and dim X = dim $\mathbb{A}^n \times \mathbb{A}^1 = n + 1$. Then it follows by Lemma 3.4.74 that there is a finite affine change of coordinates

$$\pi : X \to \mathbb{A}^{n+1}$$

Defined by linear forms $l_1 + \sum_1^m a_{1j} y_j, l_2 + \sum_2^m a_{2j} y_j, \ldots, l_m + a_{mm} y_m, l_{m+1}, \ldots, l_{n+1} \in K[\mathbf{y}, \mathbf{y}]$ where $l_i \in K[\mathbf{x}]$ are linear forms for each i. Then $\psi = \pi \circ \phi$ is a finite polynomial map by Lemma 3.4.73. This polynomial map is defined by $h_1 := l_1 + \sum_1^m a_{1j} z f_j, h_2 := l_2 + \sum_2^m a_{2j} z f_j, \cdots, h_m := l_m + a_{mm} z f_m, h_{m+1} := l_{m+1}, \ldots, h_{n+1} := l_{n+1} \in K[\mathbf{x}][z]$. Set $L := K(z)$. When viewing $h_i \in L[\mathbf{x}]$, we see that deg $h_i = d_i$ for $i \leq m$ and deg $h_i = 1$ for $i > m$. By finiteness the map

$$\widetilde{\psi} : K[\mathbf{x}][z] \to K[\mathbf{x}][z], f \mapsto f(h_1, \ldots, h_{n+1})$$

is finite. Then $K[\mathbf{x}][z]$ is finitely generated as a $K[h_1, \ldots, h_n, h_{n+1}]$-module, hence $K[\mathbf{x}][z] \supset K[h_1, \ldots, h_n, h_{n+1}]$ is integral. Then there is a monic polynomial $P_z \in K[Y_1, \ldots, Y_{n+1}, t] \setminus 0$ of minimal degree in t such that $P_z(h_1, \ldots, h_{n+1}, z) = 0$ (cf. Remark 1.9.13). By Perron's theorem there is a $Q \in L[Y_1, \ldots, Y_{n+1}] \setminus 0$ such that

a. $Q(h_1, \ldots, h_{n+1}) = 0$

b. deg $Q(Y_1^{d_1}, \ldots, Y_m^{d_m}, Y_{m+1}, \ldots, , Y_{n+1}) \leq \prod_1^m d_i$

After scaling Q by an appropriate power of z, we get a polynomial P such that

a. $P(h_1, \ldots, h_{n+1}, z) = 0$

b. $\deg_{\mathbf{Y}}\ P(Y_1^{d_1}, \ldots, Y_m^{d_m}, Y_{m+1}, \ldots, , Y_{n+1}, z) \leq \prod_1^m d_i$

Then $P_z \mid P$, hence $\deg_{\mathbf{Y}}\ P_z(Y_1^{d_1}, \ldots, Y_m^{d_m}, Y_{m+1}, \ldots, , Y_{n+1}, z) \leq \prod_1^m d_i$. Write $P_z(Y_1, \ldots, Y_{n+1}, z) = z^N + \sum_0^{N-1} b_i z^i$, $b_i \in K[\mathbf{Y}]$. Set $b_N := 1$ and write

$$b_i(h_1, \ldots, h_{n+1}) z^i = \mu_i z^N + \sum_{j \neq N} \nu_{ij} t^j,$$

for suitable $\mu_i, \nu_j \in K[\mathbf{x}]$. Then

$$0 = P_z(h_1, \ldots, h_{n+1}, z) = z^N + \sum_0^{N-1} b_i(h_1, \ldots, h_n) t^i = (1 + \sum_0^{N-1} \mu_i) z^N + \sum_{j \neq N} \left[\sum_0^{N-1} \nu_{ij} \right] z^j$$

which implies that $1 + \sum_0^{N-1} \mu_i = 0$ hence $1 = -\sum_0^{N-1} \mu_i$. Note that $\deg_z\ b_i(h_1, \ldots, h_{n+1}) z^i \geq N$, implies $\deg_z\ b_i(h_1, \ldots, h_{n+1}) > 0$. Now note that each $h_j = \omega_j + \kappa_j z$, where $\omega_j, \kappa_j \in K[\mathbf{x}]$ and $\kappa_j \in \langle f_1, \ldots, f_m \rangle$, hence $\mu_i \in \langle f_1, \ldots, f_m \rangle$, hence for suitable $\lambda_1, \ldots, \lambda_m \in K[\mathbf{x}]$,

$$1 = -\sum_0^{N-1} \mu_i = \sum_1^m \lambda_i f_i.$$

Secondly using Lemma 1.8.53,

$$\begin{aligned} \deg\ \lambda_i f_i &\leq \max_{1,\ldots,N-1} \deg\ \mu_i \leq \deg_{\mathbf{x}}\ P_z(h_1, \ldots, h_{n+1}, z) \\ &\leq \deg_{\mathbf{Y}}\ P_z(Y_1^{d_1}, \ldots, Y_m^{d_m}, Y_{m+1}, \ldots, Y_{n+1}, z) \leq \prod_1^m d_i. \end{aligned}$$

□

Remark 3.4.76. Suppose $m > n$. Then we still get the bound by considering $f_1, \ldots, f_m$ as elements of $K[x_1, \ldots, x_n, \ldots, x_m]$.

4 Projective & Multiprojective Algebraic Geometry

4.1 Projective Algebraic Sets & Projective Varieties

4.1.1 Basic Definitions

Definition 4.1.1. Let $S \subset K[x_1, \ldots, x_{n+1}]$ then we define *the projective vanishing of S over K* to be the set

$$V^{\mathbb{P}}(S) := V(S) := \{[v] \in \mathbb{P}^n : [v] \text{ is a zero for every } f \in S\}$$

Definition 4.1.2. A set $X \subset \mathbb{P}^n$ is said to be a *projective algebraic set* if

$$X = V(S)$$

for some $S \subset K[x_1, \ldots, x_{n+1}]$. X is called a *(projective) hypersurface* if $X = V(F)$ for some homogeneous $F \in K[\mathbf{x}]$. If deg $F = 1$ we call it a *(projective) hyperplane*.

Remark 4.1.3. One notes that the hyperplanes $V(x_i)$ are the i'th hyperplanes at infinity.

Proposition 4.1.4. *Let K be an infinite field, $S \subset K[x_1, \ldots, x_{n+1}]$ and $I := \langle S \rangle = \langle f_1, \ldots, f_m \rangle \subset K[x_1, \ldots, x_{n+1}]$. Write*

$$f_i = \sum_0^{d_i} f_{ij},$$

where f_{ij} are forms of degree j. Then

$$V(S) = V(I) = V(\{f_{ij} : 1 \leq i \leq m, 0 \leq j \leq d_i\}).$$

Proof. The first equality is trivial and the second equality follows from Lemma 1.8.129. □

Remark 4.1.5. It thus follows that if $X \subset \mathbb{P}^n$ is a projective algebraic set, we may assume that $X = V(F_1, \ldots, F_m)$ for homogeneous polynomials $F_1, \ldots, F_m \in K[x_1, \ldots, x_{n+1}]$. As a consequence if given a homogeneous ideal $I \subset K[\mathbf{x}]$, then $V^{\mathbb{A}}(I) \setminus 0 \neq \emptyset$ implies $V^{\mathbb{P}}(I) \neq \emptyset$. Indeed, we may write $I = \langle F_1, \ldots, F_m \rangle$ for suitable homogeneous polynomials $F_1, \ldots, F_m \in K[\mathbf{x}]$. Meaning if $v \in \mathbb{A}^{n+1} \setminus 0$ is a zero of each F_i, then so is $[v]$.

Lemma 4.1.6. *If $K[x_1, \ldots, x_{n+1}] \supset M \supset M'$, then $V^{\mathbb{P}}(M) \subset V^{\mathbb{P}}(M')$.*

Proof. The proof is almost identical to the proof of Lemma 3.1.5. □

Lemma 4.1.7. *We collect the following results which are projective analogues of Lemma 3.1.10*

(i) Let A be some indexing set. Consider a family of algebraic sets $\{X_\alpha\}_{\alpha \in A}$ in $\mathbb{A}^n$. Then $\bigcap_\alpha X_\alpha$ is an algebraic set.

(ii) Consider algebraic sets $X, Y \subset \mathbb{A}^n$. Then $X \cup Y$ is algebraic. It follows by induction that $\bigcup_1^k X_i$ is algebraic for any finite sequence of algebraic sets $X_1, \ldots, X_k$ in $\mathbb{A}^n$.

Proof. 1. Writing $X_\alpha = V(I_\alpha)$ for homogeneous I_α, we prove the statement in the same manner as for the affine case, i.e. by proving $\bigcap_\alpha X_\alpha = V\left(\bigcup_\alpha I_\alpha\right) = V\left(\langle \bigcup_\alpha I_\alpha \rangle\right) = V\left(\sum_\alpha I_\alpha\right)$. The proof of this is identical to the one given in Lemma 3.1.10 1.
2. For homogeneous ideals $I, J \subset K[\mathbf{x}]$. We prove that $V(I) \cup V(J) = V(IJ)$ for which the proof is identical to that of Lemma 3.1.10 2. □

Example 4.1.8. 1. $V^{\mathbb{P}}(0) = \mathbb{P}^n$. Indeed $0([v]) = 0$ for every $[v] \in \mathbb{P}^n$.

2. $V^{\mathbb{P}}(1) = \emptyset$. Indeed $1(v) = 1 \neq 0$ for every $v \in \mathbb{A}^{n+1} \setminus 0$.

3. $V^{\mathbb{P}}(x_1 - x_{n+1}a_1, \ldots, x_n - x_{n+1}a_n) = \{[(a_1, \ldots, a_n, 1)]\}$. Indeed, set $v = a_1, \ldots, a_n, 1) \in \mathbb{A}^{n+1}$. Then evaluating each of the n polynomials in λv for $\lambda \in K \setminus 0$, we get 0. Conversely if $w_i = w_{n+1}a_i$ for each i, note that necessarily $w_{n+1} \neq 0$, hence $(w_1, \ldots, w_{n+1}) = w_{n+1}v$, hence $[w] = [v]$. After permutation of indices we see that any point in $\mathbb{P}^n$ is a projective algebraic set, hence any finite subset of $\mathbb{P}^n$ is algebraic

Remark 4.1.9. The system

$$\tau_{\mathcal{Z}} := \{\mathbb{P}^n \setminus X : X \subset \mathbb{P}^n \text{ is an algebraic set}\}$$

(analogously to the affine case) defines a topology on $\mathbb{P}^n$ which we also call *Zariski topology*.

Definition 4.1.10. For a set $X \subset \mathbb{P}^n$ we form

$$I^{\mathbb{P}}(X) := I(X) := \{f \in K[x_1, \ldots, x_n] : f([v]) \text{ for every } [v] \in X\}$$

which we call the *(homogeneous) ideal of* X

Remark 4.1.11. The above set is a homogeneous ideal. That it is an ideal is trivial. Again by Lemma 1.8.129 it is homogeneous. We also have that $I(X)$ is generated by a finite set of homogeneous polynomials due to Lemma 1.8.27.

Lemma 4.1.12. *If $X, Y \subset \mathbb{P}^n$ are algebraic such that $X \subset Y$, then $I(X) \supset I(Y)$.*

Proof. The proof is identical to the affine case □

Lemma 4.1.13. *Let $M \subset K[x_1, \ldots, x_{n+1}]$ and $X \subset \mathbb{P}^n$. Then we have the following*

1. $I(V(M)) \supset M$.
2. $V(I(X)) \supset X$.
3. $V(I(V(M))) = V(M)$. *Hence if X is algebraic $X = V(I(X))$.*
4. $I(V(I(X))) = I(X)$. *Hence if M is an ideal of some algebraic set, $M = I(V(M))$*

Proof. The proof is identical to the affine case. □

Lemma 4.1.14. *Let $X \subset \mathbb{A}^n$. Then $I(X)$ is radical.*

Proof. The proof is identical to the affine case. □

Lemma 4.1.15. *Let $X, Y \subset \mathbb{A}^n$ be algebraic subsets. Then*

$$X = Y \iff I(X) = I(Y).$$

Proof. The proof is identical to the affine case. □

Example 4.1.16. 1. $I(\emptyset) = K[x_1, \ldots, x_{n+1}]$

2. If $\#K = \infty$, $I(\mathbb{P}^n) = 0$. Indeed, if $f([v]) = 0$ for every $[v]$, then $f_0([v]) = 0$, hence $f(0) = 0$. It thus follows that $f(v) = 0$ for every $v \in \mathbb{A}^{n+1}$, hence $f = 0$.

3. $I(\{[v_1, \ldots, v_n, 1]\}) = \langle x_1 - x_{n+1}v_1, \ldots, x_n - x_{n+1}v_n \rangle$.

Definition 4.1.17. An algebraic set is *reducible* if can be written as the union of two strictly smaller algebraic set. It is called *irreducible* or *a (projective) variety*, if it is not reducible.

Lemma 4.1.18. *An algebraic set $V \subset \mathbb{P}^n$ is a variety if and only if $I(V)$ is prime if and only if $K[x_1, \ldots, x_{n+1}]/I(V)$ is an integral domain.*

Proof. " $\implies$ ": Suppose $I(V)$ is not prime. Then there are homogeneous $F_1, F_2 \in K[x_1, \ldots, x_{n+1}] \setminus I(V)$ such that $F_1F_2 \in I(V)$, by Lemma 1.8.28. Then $V \cap V(F_i) \subsetneq V$ and

$$V = V \cap V(F_1F_2) = (V \cap V(F_1)) \cup (V \cap V(F_2)).$$

" $\impliedby$ ": Conversely if $V = V_1 \cup V_2$ where $V_i \subsetneq V$, then $I(V_i) \supsetneq I(V)$. Pick $F_i \in I(V_i) \setminus I(V)$, then $F_1F_2 \in I(V_1 \cup V_2) = I(V)$, hence $I(V)$ is not prime. □

Again using that $K[x_1, \ldots, x_{n+1}]$ is noetherian, we can prove that any projective algebraic set has unique decomposition into a union of projective varieties.

4.1.2 The Cone of a Projective Algebraic Set

Definition 4.1.19. Let $X \subset \mathbb{P}^n$ be algebraic. We define *the cone over X* to be the set

$$C(X) := \{v \in \mathbb{A}^{n+1} : [v] \in X \text{ or } v = 0\}.$$

Lemma 4.1.20. *Let $V \subset \mathbb{P}^n$ be a non-empty algebraic set. Then*

$$I^{\mathbb{A}}(C(V)) = I^{\mathbb{P}}(V).$$

Proof. "$\subset$": Let $f \in I^{\mathbb{A}}(C(V))$ and $[v] \in V$. Then $\lambda v \in C(V)$ for every $\lambda \in K \setminus 0$, meaning $f(\lambda v) = 0$. Then $f([v]) = 0$, hence $f \in I^{\mathbb{P}}(V)$.
"$\supset$": Let $f \in I^{\mathbb{P}}(V)$ and $v \in C(V)$. If $[v] \in V$, then $f([v]) = 0$, hence in particular $f(v) = 0$. If $v = 0$, note that writing $f = \sum_0^d f_i$, we get that $f_i(w) = 0$ for a $[w] \in V$, hence $f_i = 0$. Then trivially $f(0) = 0$. □

Lemma 4.1.21. *If $I \subset K[x_1, \ldots, x_{n+1}]$ is homogeneous, then $V^{\mathbb{P}}(I) \neq \emptyset$ implies $C(V^{\mathbb{P}}(I)) = V^{\mathbb{A}}(I)$*

Proof. "$\subset$": Let $v \in C(V^{\mathbb{P}}(I))$ and $f = \sum_0^d f_i \in I$. Then if $[v] \in V^{\mathbb{P}}(I)$, $f([v]) = 0$, hence $f(v) = 0$. If $v = 0$, then for a given $[w] \in V^{\mathbb{P}}(I)$, $f_i([w]) = 0$, hence $f_i = 0$, meaning $f_i(0) = 0$. In any case $v \in V^{\mathbb{A}}(I)$.
"$\supset$": Let $v \in V^{\mathbb{A}}(I)$ and $f = \sum_0^d f_i \in I$. If $v = 0$, then $v \in C(V^{\mathbb{P}}(I))$ trivially. So suppose $v \neq 0$. Note that $f_i(v) = 0$ for each i since $f_i \in I$, hence $f_i(\lambda v) = 0$ for each $\lambda \in K \setminus 0$. It thus follows that $f([v]) = 0$, hence $v \in C(V^{\mathbb{P}}(I))$. □

Lemma 4.1.22. *If $I \subset K[x_1, \ldots, x_{n+1}]$ is homogeneous and not contained in $\langle x_1, \ldots, x_{n+1}\rangle$, then $V^{\mathbb{P}}(I) \neq \emptyset$ if and only if $C(V^{\mathbb{P}}(I)) = V^{\mathbb{A}}(I)$. In the case $I \not\subset \langle x_1, \ldots, x_{n+1}\rangle$, $V^{\mathbb{A}}(I) = \emptyset$. In the case $I = \langle x_1, \ldots, x_{n+1}\rangle^d$, then $C(V^{\mathbb{P}}(I)) = \{0\} = V^{\mathbb{A}}(I)$.*

Proof. "$\implies$": Follows immediately from the prior lemma.
"$\impliedby$": If $V^{\mathbb{P}}(I) = \emptyset$, then $C(V^{\mathbb{P}}(I)) = \{0\}$ and $V^{\mathbb{A}}(I) \setminus 0 = \emptyset$ (cf. Remark 4.1.5). Note also that "$\supset$" in the prior lemma holds true in any case. Then $V^{\mathbb{A}}(I) \in \{\emptyset, \{0\}\}$. If $V^{\mathbb{A}}(I) = \{0\}$, then $I \subset I(V^{\mathbb{A}}(I)) = \langle x_1, \ldots, x_n\rangle$, so we conclude that $V^{\mathbb{A}}(I) = \emptyset \subsetneq \{0\} = C(V^{\mathbb{P}}(I))$.
If $I = \langle x_1, \ldots, x_{n+1}\rangle^d$, then $v \in V^{\mathbb{A}}(I) \iff v = 0$, hence $V^{\mathbb{P}}(I) = \emptyset$ □

Proposition 4.1.23. *Each irreducible component of a cone $C(V^{\mathbb{P}}(I))$ is itself a cone.*

Proof. **Claim:** Let $W \subset V^{\mathbb{P}}(I)$ be a component of $V^{\mathbb{P}}(I)$. Then $C(W) \subset C(V^{\mathbb{P}}(I))$ is a component. Suppose $V := V^{\mathbb{P}}(I) = V_1 \cup V_2$ with $V_i \subsetneq V$. Then there is a $[v_i] \in V$ such that $[v_i] \notin V_i$. Then $v_i \in C(V) \setminus C(V_i)$.
Let U be a component of $C(V)$. By uniqueness of decomposition into components $U = C(W)$ for some component of V. □

4.1.3 The Projective Nullstellensatz

Lemma 4.1.24. *Let $I \subset K[x_1, \ldots, x_{n+1}]$ be an homogeneous ideal. The following are equivalent.*

1. $V^{\mathbb{P}}(I) = \emptyset$.

2. $V^{\mathbb{A}}(I) \subset \{0\}$.

3. $\text{rad}(I) = I^{\mathbb{A}}(V^{\mathbb{A}}(I)) \supset \langle x_1, \ldots, x_{n+1} \rangle$.

4. $\langle x_1, \ldots, x_{n+1} \rangle^d \subset I$ *for a suitably large* $d \geq 0$

Proof. "1. $\iff$ 2.": This follows from the prior lemma. "2. $\iff$ 3.": This follows from HNS. "3. $\iff$ 4.": This follows from Lemma 1.6.42. □

Theorem 4.1.25. *(Projective Nullstellensatz/PNS)*
Let $I \subset K[x_1, \ldots, x_{n+1}]$ *be a homogeneous ideal. Then*

1. $V^{\mathbb{P}}(I) = \emptyset$ *if and only if* $V(d, n+1) \subset I$ *for some* $d \geq 0$.

2. *If* $V^{\mathbb{P}}(I) \neq \emptyset$, *then* $I^{\mathbb{P}}(V^{\mathbb{P}}(I)) = \text{rad}(I)$.

Proof. 1. Is an immediate consequence of the above lemma.
2. By Lemma 4.1.21 $V^{\mathbb{A}}(I) = C(V^{\mathbb{P}}(I))$ and by Lemma 4.1.20

$$I^{\mathbb{P}}(V^{\mathbb{P}}(I)) = I^{\mathbb{A}}(C(V^{\mathbb{P}}(I))) = I^{\mathbb{A}}(V^{\mathbb{A}}(I)) = \text{rad}(I),$$

where the last equality is just HNS. □

write relevant corollaries to the PNS

4.1.4 Rational Functions and Local Rings of Projective Varieties

Definition 4.1.26. For a projective variety, $V \subset \mathbb{P}^n$ we define the *homogeneous coordinate ring of* V to be the integral domain (recall that $I(V)$ is prime)

$$\Gamma^h(V) := \Gamma(V) := K[\mathbf{x}]/I^{\mathbb{P}}(V).$$

Remark 4.1.27. In general it is not true that the homogeneous coordinate ring can be thought of as a subring of $\text{Fun}(V, K)$, indeed consider for example $f = \sum_0^d f_i \in K[x_1, \ldots, x_n]$, then for a $v \in \mathbb{A}^{n+1} \setminus 0$, $f(\lambda v) = \sum_0^d \lambda^i f_i(v)$. Note then that the function $\lambda \mapsto f(\lambda v)$ is constant if and only if $f_i = 0$ for $i > 0$. Hence for some $\lambda, \lambda' \in K \setminus 0$, $f(\lambda v) \neq f(\lambda' v)$, meaning $f([v])$ is not well-defined unless f is constant. What then can be said is that for $V \neq \emptyset$, $\{c + I(V) : c \in K\} \simeq \{([v] \mapsto c) \in \text{Fun}(V, K) : c \in K\}$, which is trivial.

Definition 4.1.28. For a projective variety, $V \subset \mathbb{P}^n$ we define the *homogeneous function field of* V to be the field

$$K^h(V) := Q(\Gamma^h(V).$$

Remark 4.1.29. It is also clear that the homogeneous function field is not isomorphic to some subring of $Q(\text{Fun}(V, K))$. We can however make an amendment to this consider $\phi = \frac{f+I(V)}{g+I(V)} \in K^h(V)$, where $f + I(V), g + I(V)$ are homogeneous of degree $d \geq 0$. Then for every $v \in V$ with $(g + I(V))(v) = g(v) \neq 0$ and every $\lambda \in K \setminus 0$,

$$\frac{f(\lambda v)}{g(\lambda v)} = \frac{\lambda^d f(v)}{\lambda^d g(v)} = \frac{f(v)}{g(v)}.$$

We thus get that $\phi([v]) := \frac{f(v)}{g(v)}$ is well-defined. It is also independent of choice of representative of ϕ, since evaluation of $\frac{f+I(V)}{g+I(V)}$ is independent of representatives.

Inspired by the above remark we introduce the following definition.

Definition 4.1.30. For a projective variety, $V \subset \mathbb{P}^n$ we define the *function field of* V to be

$$K(V) := \left\{ z \in K^h(V) : \exists d \geq 0 f, g \in V(d, n+1), g \neq 0, z = \frac{f + I(V)}{g + I(V)} \right\}.$$

Remark 4.1.31. This is a subfield of $K^h(V)$ and a K-algebra. Indeed, consider $f, g \in V(d, n+1)$ and $\lambda, \mu \in V(e, n+1)$. Then $f\lambda, g\mu, f\mu, g\lambda$ and there sums are in $V(d+e, n+1)$. This implies that $\frac{f+I(V)}{g+I(V)} + \frac{\lambda+I(V)}{\mu+I(V)}, \frac{f+I(V)}{g+I(V)} \frac{\lambda+I(V)}{\mu+I(V)} \in K(V)$. If $f \neq 0$, then clearly $\left(\frac{f+I(V)}{g+I(V)}\right)^{-1} = \frac{g+I(V)}{f+I(V)}$. We see that an element of this ring can be viewed as a function $[v] \mapsto \frac{f}{g}(v)$ away from the points where $g([v]) = 0$. We therefor also call these elements *rational functions on* V.

Definition 4.1.32. Let $V \subset \mathbb{P}^n$ be a projective variety, $\phi \in K(V)$. We define *the pole set of* ϕ to be the set

$$\mathcal{P}(\phi) := \{P \in \mathbb{P}^n : \phi \text{ not defined at } P\}$$

Definition 4.1.33. For $\phi \in K(V)$, we define $J_\phi := \{g \in K[\mathbf{x}] : (g + I(V))\phi \in \Gamma^h(V)\}$, which we know to be an ideal already (cf. Remark 3.3.49).

Remark 4.1.34. The ideal is also homogeneous. Indeed, write $g = \sum_0^d g_i \in J_\phi$ and $\phi = \frac{\alpha}{\beta}$ where $\alpha, \beta \in \Gamma^h(V)$ are homogeneous of degree δ. Note that $\sum_0^d (g_i + I(V))\alpha$ is a decomposition into a sum of forms of degree $\delta + i$ for $i \leq d$. We have that $(g + I(V))\phi = \sum_0^e h_i$ for suitable homogeneous $h_i \in \Gamma^h(V)$. Then

$$\sum_0^d (g_i + I(V))\alpha = \sum_0^e h_i \beta$$

hence $e = d$ and $(g_i + I(V))\alpha = h_i\beta$ for each i, implying $(g_i + I(V))\phi \in \Gamma^h(V) \implies g_i \in J_\phi$.

Lemma 4.1.35. *Let V be a variety and $\phi \in K(V)$. Then $\mathcal{P}(\phi) = V(J_\phi)$.*

Proof. The proof is identical to the affine case. □

Definition 4.1.36. Let $V \subset \mathbb{P}^n$ be a projective variety. Consider $\phi = \frac{\alpha}{\beta} \in K(V)$, where α and β are homogeneous of degree $d \geq 0$ and a point $P \in V$. We say that ϕ *is defined at* P if P is not a zero of g.
In continuation of this we define *the local ring of* V *at* P to be

$$\mathcal{O}_P(V) := \{z \in K(V) : z \text{ is defined at } P\}.$$

Remark 4.1.37. One readily verifies that this is a subring of $K(V)$ containing K and that $\mathfrak{m}_P(V) := \mathcal{O}_P(V) \setminus \mathcal{O}_P(V)^* = \{z \in \mathcal{O}_P(V) : z = \frac{\alpha}{\beta}, \alpha(P) = 0\}$ is an ideal, meaning $\mathcal{O}_P(V)$ is a local ring.

Definition 4.1.38. For a variety V with a point $P \in V$ we define *evaluation at* P to be the function

$$\begin{aligned} \text{ev}_P : \mathcal{O}_P(V) &\to K \\ \phi &\mapsto \phi(P) \end{aligned}$$

Remark 4.1.39. One readily verifies that this is a K-algebra homomorphism.

4.1.5 (Projective) change of coordinates

Definition 4.1.40. Consider an affine change of coordinates $\varphi : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}, v \mapsto Av$, i.e. an affine change of coordinates that maps 0 to 0. We define *the projective change of coordinates* induced by φ to be the function

$$\begin{aligned} \varphi : \mathbb{P}^n &\to \mathbb{P}^n \\ [v] &\mapsto [Av] \end{aligned}$$

Remark 4.1.41. The above function is well-defined: Since A is invertible $Av \neq 0$, hence $[Av]$ is an element of $\mathbb{P}^n$. For any $[v] \in \mathbb{P}^n$, $[v] = L(v, 0) \setminus 0$. Hence by Lemma 3.3.36 and using A being invertible,

$$A(L(\lambda v, 0)) \setminus 0 = A(L(v, 0) \setminus 0) = L(Av, 0) \setminus 0.$$

Note that the inverse of φ is the affine change of coordinates induced by A^{-1}.

Lemma 4.1.42. *If $V \subset \mathbb{P}^n$ is a projective variety and $\varphi : \mathbb{P}^n \to \mathbb{P}^n, [v] \mapsto [Av]$ is a projective change of coordinates, then $\varphi^{-1}(V)$ is a projective variety.*

Proof. One readily verifies that if $V = V(F_1, \ldots, F_m)$ for homogeneous $F_1, \ldots, F_m$, and φ is induced by linear forms $\varphi_1, \ldots, \varphi_{n+1}$, then $F_i(\varphi_1, \ldots, \varphi_{n+1})$ is homogeneous and $\varphi^{-1}(V) = V(F_1(\varphi_1, \ldots, \varphi_{n+1}), \ldots, F_m(\varphi_1, \ldots, \varphi_{n+1})$ is therefor a projective algebraic set. Suppose $\varphi^{-1}(V) = V_1 \cup V_2$. Then $V = \varphi(V_1) \cup \varphi(V_2) = (\varphi^{-1})^{-1}(V_1) \cup (\varphi^{-1})^{-1}(V_2)$, hence WLOG $V = \varphi(V_1)$. Then $\varphi^{-1}(V) = V_1$. □

Lemma 4.1.43. *For a projective variety V and a projective change of coordinates φ induced by linear forms φ_i, we get induced isomorphisms K-algebras*

$$\widetilde{\varphi} : \Gamma^h(V) \to \Gamma^h(\varphi^{-1}(V))$$
$$f + I(V) \mapsto f(\varphi_1, \ldots, \varphi_{n+1}) + I(\varphi^{-1}(V))$$

(It has inverse $f + I(\varphi^{-1}) \mapsto f(\varphi_1^{-1}, \ldots, \varphi_{n+1}^{-1}) + I(V)$). We thus get an induced isomorphism

$$\widetilde{\varphi} : K(V) \to K(\varphi^{-1}(V))$$

and if $\varphi(Q) = P$ an isomorphism

$$\widetilde{\varphi} : \mathcal{O}_P(V) \to \mathcal{O}_Q(\varphi^{-1}(V))$$

Proof. $\widetilde{\varphi} : \Gamma^h(V) \to \Gamma^h(\varphi^{-1}(V))$ is well-defined for the same reason it is in the affine case (i.e. $f(\varphi_1, \ldots, \varphi_{n+1}) \in I(\varphi^{-1}(V))$ for every $f \in I(V)$). The inverse is $\widetilde{\varphi^{-1}}$. We get an induced isomorphism $K^h(V) \to K^h(\varphi^{-1}(V))$ □

Example 4.1.44. Let $V = \mathbb{P}^1$ and set $t = \frac{x}{y} \in K(V)$. Then $K(V) = K(t) \simeq K(z)$ (i.e. $K(t)$ can be viewed as the quotient field of the polynomial ring in 1 variable over K, since t is algebraically independent over K). It remains to check that $K(V) \subset K(t)$. It is sufficient to check that $\frac{f}{g} \in K(V)$ where f, g are homogeneous of degree $d \geq 1$ are elements of $K(t)$. Consider $\frac{\alpha x + \beta y}{\lambda x + \mu y} \in K(V)$ where $(\alpha, \beta), (\lambda, \mu) \neq (0, 0)$. Then

$$\begin{aligned}\frac{\alpha x + \beta y}{\lambda x + \mu y} &= \frac{\alpha x}{\lambda x + \mu y} + \frac{\beta y}{\lambda x + \mu y} \\ &= \left(\frac{\lambda}{\alpha} + \frac{\mu}{\alpha} t^{-1}\right)^{-1} + \left(\frac{\lambda}{\beta} t + \frac{\mu}{\beta}\right)^{-1} \in K(t)\end{aligned}$$

Since f and g can be written as product of linear forms it follows that $\frac{f}{g} \in K(t)$. Using Proposition 3.3.75 we obtain the following one-to-one correspondence

$$\mathbb{P}^1 = K \cup \{\infty\} \to \{\text{DVRs containing } K \text{ with quotient field } K(t)\}$$
$$P \mapsto \begin{cases} \mathcal{O}_P(\mathbb{A}^1) & \text{if } P \in K \\ \mathcal{O}_\infty & \text{if } P = \infty \end{cases}$$

Definition 4.1.45. A *projective linear subvariety* is a projective algebraic set of the form $V(L_1, \ldots, L_m)$ where $L_1, \ldots, L_m \in K[x_1, \ldots, x_{n+1}] \setminus 0$ are linear homogeneous polynomials.

Lemma 4.1.46. *The pre-image of a projective linear subvariety under a projective change of coordinates is again a projective linear subvariety.*

Proof. This follows from Lemma 4.1.42 the fact that a non-zero linear homogeneous polynomial evaluated in non-zero linear homogeneous polynomial is itself a non-zero linear homogeneous polynomials. □

Proposition 4.1.47. *Let $V := V(L_1, \ldots, L_m)$ be a projective linear subvariety. Then $\varphi^{-1}(V) = V(x_{d+2}, \ldots, x_{n+1})$ for some projective change of coordinates $\varphi : \mathbb{P}^n \to \mathbb{P}^n$, $-1 \leq d \leq n-1$ (in particular V is a projective variety). This d is independent of choice of change of coordinates.*

Proof. In the projective we have the ability to handle the case where $V = \emptyset$. Indeed setting $d = -1$ we are done since $V^{\mathbb{P}}(x_1, \ldots, x_{n+1}) = \emptyset$. In this case any projective change of coordinates will do the trick, by bijectivity.
Suppose $V \neq \emptyset$. In the $m = 1$, we use the same construction as in the proof of Proposition 3.3.29. Note that this yields an invertible linear transformation and hence a projective change of coordinates. We then get that $\varphi^{-1}(V) = V(x_{n+1})$ and there for $d = n - 1$ works. We proceed by induction in m. So consider $V = V(L_1, \ldots, L_{m+1})$ and $W = V(L_1, \ldots, L_m)$. Then there is an affine change of coordinates $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ induced by some invertible linear transform of the same name such that $\varphi^{-1}(W) = V(x_{d+2}, \ldots, x_{n+1})$. Then

$$V^{\varphi} = V(x_{d+2}, \ldots, x_{n+1}) \cap V(h),$$

where $h := (L_{m+1} \circ \varphi)(x_1, \ldots, x_{d+1}, \mathbf{0})$. As with the affine case (cf. Proposition 3.3.29), $h = 0$, in which case we are done, or $\deg\ h = 1$ and h is homogeneous. In the second case as there is linear transform $\phi : \mathbb{A}^{d+1} \to \mathbb{A}^{d+1}$ inducing a projective change of coordinates $\psi : \mathbb{P}^d \to \mathbb{P}^d$ such that $\psi^{-1}(V(h)) = V(x_{d+1})$. Taking

$$\phi : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$$
$$v \mapsto (\psi(v_1, \ldots, v_{d+1}), v_{d+2}, \ldots, v_{n+1})$$

it follows that $(\varphi \circ \phi)^{-1}(V) = V(x_{d+1}, \ldots, v_{n+1})$. The uniqueness follows from the same linear algebra as it did in the affine case. □

Remark 4.1.48. This integer invariant d, will be denoted dim V and will be referred to as the *dimension of V (over K)*. Note that upon writing $L_i = \sum_1^{n+1} a_{ij}x_j$, the dimension of a projective variety under this definition is the same as dim ker $(a_{ij}) - 1$. In fact $V^{\mathbb{P}}(L_1, \ldots, L_m)$ is the projection of (ker $(a_{ij})) \setminus 0$ into $\mathbb{P}^{n+1}$. One therefor sees that a d-dimension linear subvariety is just the projective span of some $[v_1], \ldots, [v_{d+1}]$ where $v_1, \ldots, v_{d+1}$ are linearly independent, since for a suitable projective change of coordinates, A,

$$V(L_1, \ldots, L_m) = A(V(x_{d+2}, \ldots, x_{n+1}) = A(\text{Span}([e_1], \ldots, [e_{d+1}])) = \text{Span}([Ae_1], \ldots, [Ae_{d+1}])$$

by lemma 4.1.50. Conversely, if $v_1, \ldots v_{d+1} \in \mathbb{A}^{n+1} \setminus 0$ are linearly independent, then Span$([v_1], \ldots, [v_{d+1}])$ is also a linear subvariety of dimension d.

Definition 4.1.49. Let $[v] \neq [w] \in \mathbb{P}^n$. We define *the line through* $[v]$ *and* $[w]$ to be

$$L([v], [w]) := \text{Span}([v], [w]).$$

If $[v_1], \ldots, [v_n] \in \mathbb{P}^n$ such that $[v_i] \notin \text{Span}([v_1], \ldots, \widehat{[v_i]}, \ldots, [v_n])$, we define the *hyperplane through* $[v_1], \ldots, [v_n]$ to be $H([v_1], \ldots, [v_n]) := \text{Span}([v_1], \ldots, [v_n])$

Lemma 4.1.50. *Let* $[v_1], \ldots, [v_m] \in \mathbb{P}^n$ *and* $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ *a projective change of coordinates. Then* $\varphi(\text{Span}([v_1], \ldots, [v_m])) = \text{Span}([\varphi(v_1)], \ldots, [\varphi(v_m)])$

Proof. Set $A \in M_{n+1}(K)$ to be the linear transform inducing ϕ. Then for any $(\lambda_1, \ldots, \lambda_m) \in K^m \setminus 0$,

$$\left[\sum_1^m \lambda_i(Av_i)\right] = \left[A\left(\sum_1^m \lambda_i v_i\right)\right] = \varphi\left(\left[\sum_1^m \lambda_i v_i\right]\right),$$

and the result follows. □

Lemma 4.1.51. *Consider* $[v_1], \ldots, [v_m] \in \mathbb{P}^n$ *with* $v_1, \ldots, v_m$ *linearly independent over* K. *Then* Span$([v_1], \ldots, [v_m])$ *is linear subvariety of dimension* $m-1$ *if* $m \leq n$, *of dimension* n *otherwise. In particular, we have that a line between distinct points* $[v], [w] \in \mathbb{P}^n$ *is a projective linear variety of dimension* 1. *And that the hyperplane through* $[v_1], \ldots, [v_m]$ *is of dimension* n

Proof. In the case $m \geq n + 1$ the result is obvious since then Span$([v_1], \ldots, [v_m]) = \mathbb{P}^n$. In other case, we complete the basis of $\mathbb{A}^{n+1}$, with $v_{m+1}, \ldots, v_{n+1}$ such that $v_1, \ldots, v_{n+1}$ spans $\mathbb{A}^{n+1}$. We set $A = (v_{ij})^T$ and using the above lemma find that

$$A^{-1}\text{Span}([v_1], \ldots, [v_m]) = \text{Span}([A^{-1}v_1], \ldots, [A^{-1}v_m]) = \text{Span}([e_1], \ldots, [e_m]) = V(x_{m+1}, \ldots, x_{n+1}),$$

hence Span$([v_1], \ldots, [v_m])$ is a linear subvariety and its dimension is $m - 1$. □

Lemma 4.1.52. *Let* $[v_1], \dots, [v_{n+1}], [w_1], \dots, [w_{n+1}] \in \mathbb{P}^n$ *such that* $[v_1], \dots, [v_{n+1}$ *resp.* $[w_1], \dots, [w_{n+1}]$ *do not lie on a hyperplane, i.e. for any* $j \in \{1, \dots, n+1\}$, $[v_j] \notin H([v_1], \dots \widehat{[v_j]}, \dots, [v_{n+1}]) := \left\{ \left[\sum_{i \neq j} \lambda_i v_i \right] : (\lambda_i) \in K^n \setminus 0 \right\}$ *(the same is true for the* $w_i's$*). Then there is a projective change of coordinates* $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ *such that* $\varphi[v_i] = [w_j]$

Proof. By Lemma 1.7.33, we have that $v_1, \dots, v_{n+1}$ resp. $w_1, \dots, w_{n+1}$ are linearly independent. Set $A = (v_{ij})^T$ and $B = (w_{ij})^T$, and pick $\varphi = BA^{-1}$. Then

$$\varphi([v_i]) = [BA^{-1}v_i] = [Be_i] = [w_i].$$

The result in other words amount to a linear change of basis. □

Lemma 4.1.53. *let* $L = L([v], [w])$ *and* $\Lambda = L([u], [r])$ *be distinct lines in* $\mathbb{P}^2$*. Then* $L \cap \Lambda$ *is a point.*

Proof. WLOG $[u] \notin L$, hence $[r] \in \text{Span}([v], [w], [u])$, implying that there exist a unique(!) solution $(\lambda, \mu, \nu) \in K^3 \setminus 0$ to the equation

$$Ax = r,$$

where $A \in M_3(K)$ has columns v, w, u. We thus find that

$$L \cap \Lambda = \{[\lambda v + \mu w]\} = \{[\nu u + r]\}.$$

□

Lemma 4.1.54. *Let* $H_1, \dots, H_m \subset \mathbb{P}^n$*, hyperplanes with* $m \leq n$*. Then* $\bigcap_1^m H_i \neq \emptyset$*. Write* $H_i = V(\sum_1^{n+1} a_{ij}x_j)$*. If* $(a_{i1}, \dots, a_{i,n+1})$ *are linearly independent, then* $\bigcap_1^m H_i$ *is an* $n-m$*-dimensional linear subvariety. In particular, the intersection of* n *such hyperplanes is a point.*

Proof. Since $(a_{ij}) : \mathbb{A}^{n+1} \to \mathbb{A}^m$ is a non-zero map it follows by rank-nullity that dim $\bigcap_1^m H_i \geq 0$, hence $\bigcap_1^m H_i \neq \emptyset$. Under the further assumptions that (a_{ij}) have linearly independent, it follows that dim $\bigcap_1^n H_i$ = dim ker$((a_{ij}) - 1 = n + 1 - m - 1 = n - m$ □

Lemma 4.1.55. *Let* $H_1, \dots, H_{n+1}$ *(defined by* $\sum_1^{n+1} a_{ij}x_j$*) and* $H_1', \dots, H_{n+1}'$ *be hyperplanes in* $\mathbb{P}^n$ *that respectively do not all pass through a common point. Then there is a projective change of coordinates* $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ *such that* $\varphi(H_i) = H_i'$*.*

Proof. The assumption that $H_1, \dots, H_{n+1}$ resp. $H_1', \dots, H_{n+1}'$ do not pass through a common point means that $\bigcap_1^{n+1} H_i = \bigcap_1^{n+1} H_i' = \emptyset$, hence (a_{ij}) has null space 0, hence any selection of n rows in this matrix will be linearly independent. Then for each $i \in \{1, \dots, n+$

$1\}$, $P_i := \bigcap_{j\neq i} H_i$ (cf. the above lemma) is a point not contained in $H(P_1, \ldots, \widehat{P_i}, \ldots, P_{n+1}) = H_i$, which is also true for $P_i' := \bigcap_{j\neq i} H_i'$. We apply Lemma 4.1.52 to find a projective change of coordinates $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\varphi(P_i) = P_i'$. By Lemma 4.1.50 it follows that

$$\varphi(H_i) = H(\varphi(P_1), \ldots, \widehat{\varphi(P_i)}, \ldots, \varphi(P_{n+1})) = H(P_1', \ldots, P_i', \ldots, P_{n+1}') = H_i'$$

□

Proposition 4.1.56. *There is a one-to-one correspondence*

$$\mathbb{P}^n \longleftrightarrow \{H \subset \mathbb{P}^n : H \textit{ is a hyperplane } \}$$

Proof. We map a point $[v_1, \ldots, v_{n+1}]$ to $V\left(\sum_1^{n+1} v_i x_i\right)$ which is well-defined since $I = \lambda I$ for any ideal $I \subset R$, for any ring R, $\lambda \in R^*$. Given a hyperplane $H = V(f)$ where f is a form of degree 1, then $f = \sum_1^{n+1} a_i x_i$ for suitable $a_1, \ldots, a_{n+1} \in K^{n+1}$ where at least one $a_i \neq 0$. Then $[a_1, \ldots, a_{n+1}] \mapsto V(f)$. If $V\left(\sum_1^{n+1} v_i x_i\right) = V\left(\sum_1^{n+1} v_i' x_i\right)$, then $\sum_1^{n+1} v_i' x_i = \lambda \sum_1^{n+1} v_i x_i$ for some $\lambda \in K \setminus 0$, hence $\lambda(v_1', \ldots, v_{n+1}') = (v_1, \ldots, v_{n+1})$, meaning $[v] = [v']$ □

Remark 4.1.57. Denote the hyperplane corresponding to a point P by $P^\star$ and the point corresponding to a hyperplane H by $H^\star$.

Corollary 4.1.58. *Given a point P and a hyperplane H in $\mathbb{P}^n$, $P^{\star\star} = P$ and $H^{\star\star} = H$. Furthermore $P \in H \iff H^\star \in P^\star$*

Proof. The first statement is obvious since the functions are mutual inverses. For second statement suppose $[v_1, \ldots, v_{n+1}] \in H = V\left(\sum_1^{n+1} w_i x_i\right)$. Then

$$\text{ev}_{[w_1,\ldots,w_{n+1}]}\left(\sum_1^{n+1} v_i x_i\right) = \sum_1^{n+1} w_i v_i = \text{ev}_{[v_1,\ldots,v_{n+1}]}\left(\sum_1^{n+1} w_i x_i\right) = 0 \implies H^\star = [w_1, \ldots, w_{n+1}] \in P^\star.$$

The converse implication follows from the first statement of this corollary and the first implication. □

4.1.6 Affine and Projective Varieties

For this section we denote the identification of $\mathbb{A}^n$ with $U_{n+1} \subset \mathbb{P}^n$ by φ_{n+1}. Recall that this is given by $v \mapsto [v_1, \ldots, v_n, 1]$

Definition 4.1.59. Given an affine algebraic set $V = V(I) = V(f_1, \ldots, f_m) \subset \mathbb{A}^n$ we define the *homogenization of* V to be the projective algebraic set

$$V^* := V(I^*) \subset \mathbb{P}^n.$$

Conversely given a projective algebraic set $V = V(I) = V(F_1, \ldots, F_m) \subset \mathbb{P}^n$, $I \subset K[x_1, \ldots, x_{n+1}]$, and $F_1, \ldots, F_m \in K[x_1, \ldots, x_{n+1}]$ are homogeneous, we define the *dehomogenization of* V to be the affine algebraic set

$$V_* := V(I_*) = V((F_1)_*, \ldots, (F_m)_*)$$

Example 4.1.60. With the same setup as Example 1.8.126 where $R = K$ and consider $V := V(I)$. Note that $V = \{(a, a^2, a^3) : a \in K\}$. Let $f \in I(V)$, Then $f(a, a^2, a^3) = 0$ for any $a \in K$, hence $f(x, x^2, x^3) = 0$. In other words $(x, x^2, x^3) \in K[x, y, z]^3 = K[x][y, z]^3$ is a zero of f and hence $f \in \langle x - x, y - x^2, z - x^3 \rangle = I$ by Proposition 1.8.47.
This illustrates that homogenization of a variety cannot be written as the vanishing set of the homogenization of the finite polynomials defining the affine variety.

Lemma 4.1.61. *Let $V \subset \mathbb{A}^n$ be algebraic. Then $I(V)^* \subset I(V^*)$. We also have that $\varphi_{n+1}(X_*) \subset X$ for $X = V(I) \subset \mathbb{P}^n$ algebraic.*

Proof. This follows from $V^* = V(I(V)^*)$. Let $[v, 1] \in \varphi_{n+1}(X_*) = \varphi_{n+1}(V(I_*))$ and $f \in I(X)$. Then $f_* \in I(X)_* \supset I_*$, hence $f(v, 1) = f_*(v)$ □

Lemma 4.1.62. *Let $V = V(I), W = V(J) \subset \mathbb{A}^n$, $X, Y \subset \mathbb{P}^n$ be algebraic sets. We have the following results:*

1. *$\varphi_{n+1}(V) = V^* \cap U_{n+1}$ and $(V^*)_* = V$, $(X_*)^* \subset X$.*
2. *If $V \subset W$, then $V^* \subset W^*$. If $X \subset Y \subset$, then $X_* \subset Y_*$.*
3. *If V is a variety so is V^*.*
4. *$V^* = \overline{\varphi_{n+1}(V)}$, where $\overline{\bullet}$ is the Zariski-closure in $\mathbb{P}^n$.*
5. *If $\bigcup_1^k V_i$ is the decomposition of V into affine varieties, then $\bigcup_1^k V_i^*$ is the decomposition of V^* into projective varieties.*
6. *If $\emptyset \neq V \subsetneq \mathbb{A}^n$, then no component of V^* lies in H_∞ or contains it.*
7. *If an algebraic $X \subset \mathbb{P}^n$ and no component of X is in or contains H_∞, then $X_* \subsetneq \mathbb{A}^n$ and $(X_*)^* = X$.*

Proof. 1. Let $v \in V$, then $\varphi(V) \in U_{n+1}$. Let $f^* \in I^*$, $f \in I$. Then for $\lambda \in K \setminus 0$,

$$f^*(\lambda(v_1, \ldots, v_n, 1)) = \lambda^d (f^*)_*(v) = \lambda^d f(v) = 0 \implies f^*(\varphi(v)) = 0 \implies \varphi(v) \in V^* \cap U_{n+1}.$$

Conversely, if $[v_1, \ldots, v_n, 1] \in V^*$, then by the same computation $f(v) = f^*([v_1, \ldots, v_n, 1]) = 0 \implies v \in V \implies \varphi(v) \in \varphi(V)$.
The second equality follows from $(I^*)_* = I$. The second inclusion follows from $f = x_{n+1}^r (f_*)^*$ for some $r \geq 0$ for every $f \in I(V)$. 2. Since $I(V) \supset I(W)$, we have $I(V)^* \supset I(W)^*$ implying $V^* = V(I(V)^*) \subset V(I(W)^*) = W^*$. The second statement follows from $I(X) \supset I(Y) \implies I(X)_* \subset I(Y)_*$. 3. Since I is prime I^* is prime, hence $V^* = V(I^*)$ is a variety.
4. Let $U \subset \mathbb{P}^n$ be algebraic with $\varphi_{n+1}(V) \subset W$. Let $f \in I(W)$, then $f \in I(\varphi_{n+1}(V))$, hence f_* vanishes on every $[v, 1]$ where $v \in V$, implying $f_* \in I(V)$. Then $(f_*)^* \in I(V)^*$, hence $f = x_{n+1}^r (f_*)^* \in I(V)^*$ for some $r \geq 0$. We therefor see that $I(U) \subset I(V)^*$, hence $V = V(I(V)^*) \subset V(I(U)) = U$.
5. 4. and 2. implies $V^* = (\bigcup V_i)^* = \bigcup V_i^*$. 3. implies each V_i^*. We have that $(V_i^*)_* = V_i \not\subset V_j = (V_i^*)_*$ implying that $V_i^* \not\subset V_j^*$ by the second statement of 2.
6. Let C be a component of V. Then $\neq C \subsetneq \mathbb{A}^n$. Then by 1. $C^* \cap U_{n+1} = \varphi_{n+1}(C) \neq \emptyset$, hence $C^* \not\subset H_\infty$. Suppose $\emptyset \neq B \subset \mathbb{A}^n$ is algebraic such that $B^* \supset H_\infty$. Then $I(B)^* \subset I(B^*) \subset I(H_\infty) = \langle x_{n+1} \rangle$ by PNS. Pick $f \in I(B)$. Then $f^*(v, 0) = 0$ for every $v \in \mathbb{A}^n$. This correspondence to the highest degree terms of f being 0, implying $f = 0$, hence $I(B) = 0$, meaning $B = \mathbb{A}^n$. We thus have that $V \not\supset H_\infty$.
7. If $X_* = \mathbb{A}^n$, $X \supset (X_*)^* = \mathbb{P}^n \supset H_\infty$, hence trivially $X_* \subsetneq \mathbb{A}^n$. Let $Z = V(I)$ be a component of X. Then no component of Z lies in H_∞. By 1. it is sufficient to check that $Z \subset (Z_*)^*$, meaning it is sufficient to check that $I(Z) \supset I(Z_*)^*$, since in general $I(V)^* \subset I(V^*)$ implies that $V(I(V)^*) \supset V^*$. Let $f \in I(Z_*)$. Then, since PNS tells us that $I(Z_*) = \text{rad}(I_*) = \text{rad}(I)_* = I(Z)_*$, we get that for some $N \geq 0$, $f^N \in I(Z)_*$, hence for some $r \geq 0$, $x_{n+1}^r (f^n)^* \in I(V)$. Note that $I(Z)$ is prime and $x_{n+1} \notin I(Z)$ since $Z \not\subset H_\infty$, we get that $f^* \in I(Z)$, hence $I(Z_*)^* \subset I(Z)$. □

Remark 4.1.63. V^* is called the *projective closure of* V.

Lemma 4.1.64. *Let $V \subset W \subset \mathbb{P}^n$ be varieties, where $V = V(f)$ is a hypersurface (where f is a form). Then $V = W$ or $W = \mathbb{P}^n$*

Proof. The case $f \in K$ is trivial. So suppose $\deg f \geq 1$. Since V is a variety $I(V) = \langle f \rangle$ and f is prime and therefor irreducible. Then $\langle f \rangle \supset I(W)$, hence if $g \in I(W)$, then $g = qf$. Then $I(W) = \langle f \rangle J$ for some $J \subset K[x_1, \ldots, x_{n+1}]$. Then $W = V(f) \cup V(J)$,

hence $W = V(f) = V$ or $W = V(J)$. Note that if J is non-trivial, then $V(J)$ and V are components of W, hence $J = 0$ or $J = K[\mathbf{x}]$. Only the first case is possible and in this case $W = \mathbb{P}^n$. □

Lemma 4.1.65. *Let $H_\infty \subset V \subset \mathbb{P}^n$ be a variety. Then $V = H_\infty$ or $V = \mathbb{P}^n$. If $\mathbb{P}^n_* = \mathbb{A}^n$ and if $(H_\infty)_* = \emptyset$.*

Proof. The first statement follows from the prior lemma. $\mathbb{P}^n_* = V^{\mathbb{A}}(0_*) = V^{\mathbb{A}}(0) = \mathbb{A}^n$ and $(H_\infty)_* = V^{\mathbb{A}}((x_{n+1})_*) = V^{\mathbb{A}}(1) = \emptyset$. □

Remark 4.1.66. There is a one-to-one correspondence between varieties in $\mathbb{P}^n$ that do not lie in H_∞ and non-empty varieties in $\mathbb{A}^n$, established via $V \mapsto V_*$ with mutual inverse $\emptyset \neq W \mapsto W^*$ (cf. statements 6. and 7. of the main lemma and the above lemma). The map

$$\Gamma(\mathbb{P}^n) = K[x_1, \ldots, x_{n+1}] \to K[x_1, \ldots, x_n] = \Gamma(\mathbb{A}^n), f \mapsto f_*,$$

is surjective with kernel $\langle x_{n+1} - 1 \rangle$. Therefor $\Gamma(\mathbb{P}^n)/\langle x_{n+1} - 1 \rangle \simeq \Gamma(\mathbb{A}^n)$. We thus establish that $K^h(V)/\langle x_{n+1} - 1 \rangle K^h(V) \simeq Q(\Gamma^h(V)/\langle x_{n+1} - 1 \rangle) \simeq K(V_*)$. Let $f \in K[x_1, \ldots, x_n]$. Then $x_{n+1}^d f^* + \langle x_{n+1} - 1 \rangle = f + \langle x_{n+1} - 1 \rangle$. We therefor get a surjective K-algebra map,

$$K(V) \to Q(\Gamma^h(V)/\langle x_{n+1} - 1 \rangle), \frac{f}{g} \mapsto \frac{f + \langle x_n - 1 \rangle}{g + \langle x_{n+1} - 1 \rangle}.$$

It is clearly also surjective, since for a form $F \in K[x_1, \ldots x_{n+1}]$, $\mathrm{ev}_{x_{n+1} \mapsto 1}(F) = 0$ if and only if $F = 0$, hence $F \in \langle x_{n+1} - 1 \rangle$ if and only if $F = 0$. Therefor $K(V) \simeq K^h(V)/K^h(V)\langle x_{n+1} - 1 \rangle \simeq K(V_*)$. Suppose V is a variety not containing or contained in H_∞. Consider the map

$$K[x_1, \ldots, x_{n+1}] \to K[x_1, \ldots, x_n]/I(V_*), f \mapsto f_* + I(V_*),$$

Since $I(V)_* = I(V_*)$ we establish an identification of $\Gamma^h(V) \simeq \Gamma(V_*)$. This induces an isomorphism $K^h(V) \simeq K(V_*)$. We claim that $K^h(V) = K(V)$. We note that $x_{n+1} + I(V) \mapsto 1 + I(V_*)$, implying $x_{n+1} + I(V) = 1 + I(V)$ hence $\langle x_{n+1} - 1 \rangle \subset I(V)$. If $f + I(V) \in \Gamma^h(V)$, then $f + I(V) = f_* + I(V) = x_{n+1}^r (f_*)^* + I(V)$ for any sufficiently large r. Then for $\frac{\alpha}{\beta} \in K^h(V)$, we may find representatives for α and β that are homogeneous of the same degree, meaning $\frac{\alpha}{\beta} \in K(V)$. This means $K(V) = K^h(V) \simeq K(V_*)$.
The above is just a way of saying that the map

$$K(V) \to K(V_*), \frac{a + I(V)}{b + I(V)} \mapsto \frac{a_* + I(V_*)}{b_* + I(V_*)}$$

is an isomorphism. Although not strictly necessary, I feel that this detour somewhat illuminating, simply due to the fact that along the way, we saw the exact relationship between

$\Gamma^h(V)$ and $\Gamma(V_*)$, $K^h(V)$, $K(V)$ and $K(V_*)$ in any possible case.
Let $P = [v_1, \dots, v_n, 1] \in V \cap U_{n+1}$. Then if $\frac{\alpha}{\beta} \in K(V)$ is defined at P, then $0 \neq \beta(P) = \beta_*(v_1, \dots, v_n)$, hence $\frac{\alpha_*}{\beta_*} \in K(V_*)$ is defined at $(v_1, \dots, v_n)$ as well. Therefor $\mathcal{O}_P(V) \simeq \mathcal{O}_(v_1, \dots, v_n)(V)$.

Example 4.1.67. 1. A quick note on the algebraic subsets of $\mathbb{P}^1$. Consider such a set $V = V(F_1, \dots, F_m)$ where F_i are non-zero forms. We may write $F_i = \prod_1^{d_i} L_{ij}$ for linear forms L_{ij}. Then

$$X = \bigcup_{(i_1,\dots,i_m)} V(L_{1i_1}, \dots, L_{mi_m}).$$

Note that the vanishing set of a linear form in $\mathbb{P}^1$ is a point. For each $(i_1, \dots, i_m)$, if $L_{1i_1}, \dots, L_{mi_m}$ are distinct (up to multiplication by a unit), then $V(L_{1i_1}, \dots, L_{mi_m}) = \emptyset$. If they are equal then $V(L_{1i_1}, \dots, L_{mi_m})$ is a point. The algebraic subsets of $\mathbb{P}^1$ are therefor $\mathbb{P}^1, \emptyset$ or finite union of points.

2. By the above, we conclude that the varieties of $\mathbb{P}^1$ are therefor $\mathbb{P}^1, \emptyset$ and hyperplanes, which in $\mathbb{P}^1$ are the same as points

3. Suppose $\emptyset \neq V \subset \mathbb{P}^2$ is a variety that is not contained or does not contain H_∞. Then $\emptyset \neq V_* \subsetneq \mathbb{A}^n$ and $(V_*)^* = V$. Then V_* is a variety since V_* has the same number of components as V. This means $V_* = V(f)$ where $f \in K[x, y]$ for deg $f \geq 1$ or $V_* = \{x - a, y - b\}$ for $a, b \in K$, hence $V = V(f^*)$ or $V = V(x - az, y - bz)$. If V contains H_∞, $V = H_\infty$ or $V = \mathbb{P}^2$. Suppose $V = V(I) \subsetneq H_\infty$. Note that $V(I)$ is in bijection with $W = \{[v_1, v_2] \in \mathbb{P}^1 : \forall f \in I, f(v_1, v_2, 0)\}$, which is a finite set for otherwise $V(I) = H_\infty$. Then in particular $\#V = 1$, hence V is a point $V(bx - ay, z)$ where $[a, b] \in \mathbb{P}^1$. In conclusion the varieties in $\mathbb{P}^2$ are $\mathbb{P}^2 = (\mathbb{A}^2)^*, \emptyset$, $V(f^*)$, where $f \in K[x, y]$ is irreducible, $V(I(P)^*)$ where P is a point in $\mathbb{A}^2$, $V = H_\infty$, or $V(bx - ay, z) \subsetneq H_\infty$ for $[a, b] \in \mathbb{P}^1$.

Example 4.1.68. 1. Let's classify the lines through $P = [0, 1, 0] \in \mathbb{P}^2$. Note that $\{P\} = V(x, z)$. Let $l = ax + by + cz$. Suppose $L := V(l)$ passes through P, i.e. that $l(0, 1, 0) = 0$. Then $b = 0$, hence $l = ax + cz$, where $[a, c] \in \mathbb{P}^1$. If $a = 0$, then $L = L_\infty$. If $a \neq 0$, then L is a *vertical line*, i.e. for a suitable $\lambda \in K$ $L = L_\lambda := V(x - \lambda z) = \{[\lambda, t, 1] : t \in K\} \cup \{P\}$. In other words, L_λ is the vertical line $x = 1$ in the affine xy-plane with an added point P on the line at infinity.

2. Consider an arbitrary collection of points $P_1, \dots, P_m \in \mathbb{P}^n$. Then since there is a one-to-one correspondence between $\mathbb{P}^n$ and the hyperplanes in $\mathbb{P}^n$ via the map $P \mapsto$

$P^\star$. Then pick a point P not in $P_i^\star$ for any i. We may do this since the intersection of proper subsets (of $\mathbb{P}^n$) itself is a proper subset (of $\mathbb{P}^n$). Then $P_i = P_i^{\star\star} \notin P^\star$. Here we use Corollary4.1.58 a few times together with the fact that $\mathbb{P}^n$ is an infinite set for $n \geq 1$. It should be noted that one can deduce the fact from more simple principles.

4.2 Multiprojective Algebraic Sets & Multiprojective Varieties

4.2.1 Basic Definitions

Definition 4.2.1. Let $S \subset K[x_{ij} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i + 1\}, n_i \geq 1]$. We define the *multiprojective vanishing set* S to be

$$V^{\text{mul}\mathbb{P}}(S) := V(S) := \{([v_1], \ldots, [v_m]) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} : f([v_1], \ldots, [v_m]) = 0 \text{ for all } f \in S\}.$$

A set $X \subset \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_m}$ is called a *(multiprojective) algebraic set* if it the multiprojective vanishing set of some subset of $K[\mathbf{x}]$. X is a *(multiprojective) hypersurface* if $X = V(F)$ for some m-homogeneous $F \in K[\mathbf{x}]$. If deg $F = 1$ it is called a *(multiprojective) hyperplane*.

Remark 4.2.2. By Lemma 1.8.129 we may assume that any multiprojective algebraic set may be written as the multiprojective vanishing set of some m-homogeneous ideal, i.e. as the multiprojective vanishing set of a finite set of m-forms. The system of multiprojective algebraic sets form the *Zariski topology* on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ with $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} = V^{\text{mul}\mathbb{P}}(0)$ and $\emptyset = V^{\text{mul}\mathbb{P}}(K[\mathbf{x}])$. Vanishing sets interact with $\subset$ as expected.

Definition 4.2.3. Let $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. We define *the ideal of* X to be

$$I^{\text{mul}\mathbb{P}}(X) := I(X) := \{f \in K[\mathbf{x_1}, \ldots, \mathbf{x_m}] : f([v_1], \ldots, [v_m]) = 0 \text{ for every } ([v_1], \ldots, [v_m]) \in X\}.$$

Remark 4.2.4. $I^{\text{mul}\mathbb{P}}(X)$ is an m-homogeneous ideal. The interactions of $I^{\text{mul}\mathbb{P}}(\bullet)$ with $\subset$ and $V^{\text{mul}\mathbb{P}}(\bullet)$ are as expected.

Definition 4.2.5. An algebraic set $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is called *reducible* if there a proper algebraic subsets of X, Y, Z say, such that $X = Y \cup Z$. If an algebraic set $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is not reducible, it is called *irreducible* or is said to be a *multiprojective variety*.

Remark 4.2.6. As expected any multiprojective set has a unique decomposition into multiprojective varieties. An algebraic set V is a variety if and only if $I^{\text{mul}\mathbb{P}}(V)$ is prime.

Definition 4.2.7. For a variety $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ we define the cone of V to be

$$C(V) := \left\{(v_1, \ldots, v_m) \in \mathbb{A}^{n_1+1} \times \cdots \times \mathbb{A}^{n_m+1} : ([v_1], \ldots, [v_m]) \in V \text{ or } v_i = 0\right\}$$

Remark 4.2.8. Lemmas 4.1.20 and 4.1.21 are easily generalizable to the multiprojective case.

There is a natural way of "embedding" a projective algebraic set $V \subset \mathbb{P}^{n_i}$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ by considering

$$\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times V \times \mathbb{P}^{n_{i+1}} \times \cdots \mathbb{P}^{n_m}.$$

We do find that if $V = V^{\mathbb{P}}(I)$, then $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times V \times \mathbb{P}^{n_{i+1}} \times \cdots \mathbb{P}^{n_m} = V^{\text{mul}\mathbb{P}}(I)$. In general, given $V_i = V(I_i) \subset \mathbb{P}^{n_i}$ algebraic sets, we have that

$$V^{\text{mul}\mathbb{P}}(\langle \bigcup I_i \rangle) = V_1 \times \cdots \times V_m \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}.$$

Lemma 4.2.9. *Let I be an m-homogeneous ideal in $K[\mathbf{x_1}, \ldots, \mathbf{x_m}]$. The following are equivalent.*

1. $V^{\text{mul}\mathbb{P}}(I) = \emptyset$.
2. $V^{\mathbb{A}}(I) \subset \{0\}$.
3. $\text{rad}(I) = I^{\mathbb{A}}(V^{\mathbb{A}}(I)) \supset \langle \mathbf{x_1}, \ldots \mathbf{x_m} \rangle$.
4. $\langle \mathbf{x_1}, \ldots \mathbf{x_m} \rangle^D \supset I$ *for a suitable large* $D \geq 0$.

Proof. The proof is the same as the singly projective version using proper generalizations of the results involved □

Theorem 4.2.10. *Let $I \subset K[\mathbf{x_1}, \ldots, \mathbf{x_m}]$ be an m-homogeneous ideal. Then*

1. $V^{\text{mul}\mathbb{P}}(I) = \emptyset$ *if and only if* $\langle \mathbf{x_1}, \ldots, \mathbf{x_m} \rangle^D \subset I$ *for some* $D \geq 0$.
2. *If* $V^{\text{mul}\mathbb{P}}(I) \neq \emptyset$, *then* $I^{\text{mul}\mathbb{P}}(V^{\text{mul}\mathbb{P}}(I)) = \text{rad}(I)$.

Proof. 1. follows from the prior lemma. 2. follows from the generalizations of the results about cones. □

Definition 4.2.11. We define the *multiprojective coordinate ring of a variety* V to be the integral domain.

$$\Gamma^m(V) := K[\mathbf{x_1}, \ldots, \mathbf{x_m}]/I^{\text{mul}\mathbb{P}}(V)$$

We set $K^b(V) := Q(\Gamma^m(V))$. We define *the function field of* V to be

$$K(V) := \left\{ z \in K^b(V) : z = \frac{f}{g} \text{ for suitable } m\text{-forms } f, g \in \Gamma^m(V) \right\}.$$

We say for an element $z \in K(V)$ and a point $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ that z *is defined at* P if there are m-forms of the same m-degree, $\alpha, \beta \in \Gamma^m(V)$, such that $z = \frac{\alpha}{\beta}$ and P is not a zero of β. We define *the local ring of* V *at* P to be

$$\mathcal{O}_P(V) = \{z \in K(V) : z \text{ defined at } P\}$$

Remark 4.2.12. Suppose $z = \frac{\alpha}{\beta} \in K(V)$. Then for a $P = ([v_1], \ldots, [v_m]) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ with $\beta(v_1, \ldots, v_n) \neq 0$. We have for each $\lambda_1, \ldots, \lambda m \in K \setminus 0$ that

$$\frac{\alpha(\lambda_1 v_1, \ldots, \lambda_m v_m)}{\beta(\lambda_1 v_1, \ldots, \lambda_m v_m)} = \frac{\prod_1^m \lambda_i^{d_i}}{\prod_1^m \lambda_i^{d_i}} \frac{\alpha(v_1, \ldots, v_m)}{\beta(v_1, \ldots, v_m)} = \frac{\alpha(v_1, \ldots, v_m)}{\beta(v_1, \ldots, v_m)}.$$

Hence we get a well-defined evaluation at P of multiprojective rational functions defined at P, given by $\mathcal{O}_P(V) \ni \frac{\alpha}{\beta} \mapsto \frac{\alpha(v_1,\ldots,v_m)}{\beta(v_1,\ldots,v_m)}$.

Definition 4.2.13. For $(i_1, \ldots, i_m)$, define

$$\varphi_{i_1,\ldots,i_m} : \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m} \to U_{i_1} \times \cdots \times U_{i_m} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$$
$$(v_1, \ldots, v_m) \mapsto ([v_{11}, \ldots, v_{1,i_1-1}, 1, v_{1,i_1+1}, \ldots, v_{1n_1}], \ldots, [v_{m1}, \ldots, v_{m,i_m-1}, 1, v_{m,i_m+1}, \ldots, v_{mn_m}])$$

Remark 4.2.14. In this way we identify $U_{i_1} \times \cdots \times U_{i_m}$ with $\mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$.

Definition 4.2.15. The *multiprojective closure* of an affine algebraic set $V = V(I) \subset \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$ is the set

$$V^* := V(I^*)$$

where for an $f \in K[x_{ij} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i\}]$, f^* denotes polynomial obtained from the following recursive process: set $f_0 = f$ and $f_{i+1} = f_i^*$ where $f_i \in K[\mathbf{x_1}, \ldots, \widehat{\mathbf{x_i}}, \ldots, \mathbf{x_m}, x_{n_1+1}, \ldots, x_{n_{i-1}+1}][\mathbf{x_i}]$ for $i \in \{1, \ldots, m\}$. We therefor define $I^* := \langle \{f^* : f \in I\} \rangle$.
For a polynomial $f \in K[x_{ij} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i + 1\}]$ we define f_* to be the image of f under the evaluation map taking x_{i,n_i+1} to 1 and x_{ij} to x_{ij} for every other pair of indices.

Lemma 4.2.16. *Let* $V = V(I), W = V(J) \subset \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$, $X, Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ *be algebraic sets. We have the following results:*

1. $\varphi_{n_1+1,\ldots,n_m+1}(V) = V^* \cap U_{n_1+1} \times \cdots \times U_{n_m+1}$ *and* $(V^*)_* = V$, $(X_*)^* \subset X$.

2. *If $V \subset W$, then $V^* \subset W^*$. If $X \subset Y \subset$, then $X_* \subset Y_*$.*

3. *If V is a variety so is V^*.*

4. *$V^* = \overline{\varphi_{n_1+1,\ldots,n_m+1}(V)}$, where $\overline{\bullet}$ is the Zariski-closure in $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_m}$.*

5. *If $\bigcup_1^k V_i$ is the decomposition of V into affine varieties, then $\bigcup_1^k V_i^*$ is the decomposition of V^* into projective varieties.*

6. *If $\emptyset \neq V \subsetneq \mathbb{A}^n$, then no component is contained in or contains $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H_\infty^i \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$ for each i where $H_\infty^i := V^{\mathbb{P}}(x_{i,n_i+1})$.*

7. *If $X \subset \mathbb{P}^n$ is an algebraic set lies not in or is contained in any $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H_\infty^i \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$ for each i, then $X_* \subsetneq \mathbb{A}^n$ and $(X_*)^* = X$.*

Proof. 1. If $f \in I^a st$, then $f(\varphi_{n_1+1,\ldots,n_m+1}(v_1,\ldots,v_m)) = f([v_1,1],\ldots,[v_m,1]) = f_*(v_1,\ldots,v_m) = 0$ for every $\varphi(v_1,\ldots,v_m) \in \varphi_{n_1+1,\ldots,n_m+1}(V)$. Conversely if $v \in V^* \cap U_{n_1+1} \times \cdots \times U_{n_m+1}$. Then for a $f \in I$, $f(v_1,\ldots,v_m) = f^*([v_1,1],\ldots,[v_m,1]) = 0$, hence $v = \varphi(v_1,\ldots,v_m)$. It is fairly obvious that $(I^*)_*$, hence the second equality follows. Similarly $f = \left[\prod_1^m x_{n_i+1}^{r_i}\right](f_*)^*$, hence we get the inclusion of $(X_*)^*$ in X.
2. - 5. follow from the exact same arguments using the proper generalizations.
6. Let C be a component of V. Then $\emptyset \neq C \subsetneq \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$. Then $C^* \cap U_{n_1+1} \times \cdots \times U_{n_m+1} = \varphi_{n_1+1,\ldots,n_m+1}(C) \neq \emptyset$. Suppose $\emptyset \neq B \subset \mathbb{A}^n$ is algebraic such that $B^* \supset \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H_\infty^i \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$ for some i. Then $I(B)^* \subset I(B^*) \subset \langle x_{n_i+1} \rangle$ by the generalization of PNS. Let $f \in I(B) \subset K[\mathbf{y_j} : j \neq i][\mathbf{y_i}]$. Then $f^*([v_i,0]) = 0$ for every $v_i \in \mathbb{A}^{n_i}$. Suppose d_i was the $\mathbf{y_i}$-degree of f. Writing $f = f_{d_i} + \sum_{j \leq d_i} f_j$, we see that

$$f^* = f_{d_i} + \sum_{j \leq d_i} x_{n_i+1}^{d_i - j} f_j.$$

The vanishing condition on f^*, hence corresponds to $f_{d_i} = 0$, implying $I(B) = 0$, meaning $B = \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$. We therefor conclude that $V \not\supset \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H_\infty^i \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$. 7. The case $X_* = \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$ is trivially not possible under the assumptions. We thus have that $X_* \subsetneq \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$. Let $Z = V^{\mathrm{mul}^{\mathbb{P}}}(a)$ be a component of X. Then Z lies not in $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H_\infty^i \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$ for any i. It is sufficient to prove that $Z \subset (Z_*)^*$ and therefor sufficient to check that $I(Z) \supset I(Z_*)^*$ in $K[\mathbf{x_1},\ldots,\widehat{\mathbf{x_i}},\ldots,\mathbf{x_m}][\mathbf{x_i}]$. For an $f \in I(Z_*)$ we utilize the generalization of PNS to get that for suitably large $N \geq 0$, and for suitable r, $x_{n_i+1}^r (f^N)^* \in I(Z)$, using the fact that $x_{n_i+1} \notin I(Z)$ and that Z is irreducible we get that $f^* \in I(Z)$. □

Lemma 4.2.17. *Let $V \subset W \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be multiprojective varieties such that $V = V^{\mathrm{mul}\mathbb{P}}(f)$ for some m-form f. Then $V = W$ or $W = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. It follows that if $\mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H^i_\infty \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m} \subset W \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, then $W = \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_{i-1}} \times H^i_\infty \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$ or $W = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$*

Proof. The proof of the first statement is identical to the projective case, the second statement immediately follows from the first due to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times H^i_\infty \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m} = V(x_{i,n_i+1})$. □

4.2.2 Algebraic Geometry in Multispaces

We extend the theory to so-called *multispaces* which are defined to be $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ for $n, n_1, \ldots, n_m \geq 1$ with the convention that $\mathbb{A}^0$ is a point.

Definition 4.2.18. We say that a point $P = (v, [v_1], \ldots, [v_m]) \in \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is a *zero* of a polynomial $f \in K[x_1, \ldots, x_n, \mathbf{y_1}, \ldots, \mathbf{y_m}]$. If for every $\lambda_1, \ldots, \lambda_m \in K \setminus 0$

$$f(v, \lambda_1 v_1, \ldots, \lambda_m v_m) = 0$$

and we write $f(P) = 0$.

Remark 4.2.19. If $f \in K[x_1, \ldots, x_n][\mathbf{y_1}, \ldots, \mathbf{y_m}]$ is an m-form, then if $(v, v_1, \ldots, v_m) \in \mathbb{A}^n \times \mathbb{A}^{n_1+1} \times \cdots \mathbb{A}^{n_m+1}$ is a zero of f so is $(v, [v_1], \ldots, [v_m]) \in \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.

Definition 4.2.20. Let $S \subset K[x_1, \ldots, x_n, \mathbf{y_1}, \ldots, \mathbf{y_m}]$. We define the *vanishing set of S* in multispace to be

$$V(S) := \{P \in \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} : \forall f \in S, f(P) = 0\}.$$

A subset X in a multispace is said to be *algebraic* if it is the vanishing set of such an S

Remark 4.2.21. Again it is readily verifiable $V(S) = V(\langle S \rangle)$. By Lemma 1.8.129 we may therefor assume that any algebraic set is the vanishing set of some ideal $I \subset K[\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m}]$ such that $I \subset K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]$ is m-homogeneous. Hence we may assume that an algebraic set arises as the vanishing set of $f_1, \ldots, f_l \in K[\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m}]$ such that when seen as elements of $K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]$ these are m-forms. The algebraic sets define closed sets in a *Zariski topology* on multispaces and interacts with $\subset$ as expected. This therefor lays the ground for a natural generalization of the affine and multiprojective (therefor also projective) theory we have already developed.

To completely reconcile the above claim of generalization we also introduce the ideal of subsets of multispace.

Definition 4.2.22. Let $X \subset \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. The *ideal of* X is the ideal

$$I(X) = \{f \in K[\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m}] : \forall P \in X, f(P) = 0\}.$$

Remark 4.2.23. If X is algebraic, then $I(X) \subset K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]$ is m-homogeneous. $I(\bullet)$ interacts with $V(\bullet)$ and $\subset$ in the way one expects.

Definition 4.2.24. An algebraic set in multispace is *reducible* if it is the union of two proper algebraic subsets. An algebraic that is not reducible is called *irreducible* and is a *variety*.

Remark 4.2.25. An algebraic subset of multispace is a variety if and only if the ideal of the subset is prime. There is a unique decomposition of an algebraic set in multispace into a union of varieties that are not subsets of each other.

We could define the cone of a algebraic set in multispace and write generalizations of proper lemmas to prove. Things to check!.

Theorem 4.2.26. *Let* $I \subset K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]$ *be* m*-homogeneous. Then* $I(V(I)) = \text{rad}(I)$.

Definition 4.2.27. For $(i_1, \ldots, i_m)$ we define

$$\varphi_{i_1,\ldots,i_m} : \mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \mathbb{A}^{n_m} \to \mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$$
$$(v, v_1, \ldots, v_m) \mapsto (v, \varphi_{i_1,\ldots,i_m}(v_1, \ldots, v_m))$$

We homogenize an element f in $K[\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m}]$ by homogenizing f with respect to each set of the y-variables which we denote f^*. We homogenize an ideal in the obvious way and take the projective closure in the obvious way.

Remark 4.2.28. A generalization of Proposition 4.2.16 goes through. The addition of an affine coordinates changes nothing in the approach of the proof. The conditions in 6. and 7. of this result should be given in terms of $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times H^i_\infty \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_m}$. What we obtain from this result is, given a variety $V \subset \mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$, V^* defines a variety in $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ which is neither contained or contains $V(x_{n_i+1})$ for any i. If $V = \mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$ then $V^* = \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. Conversely given a variety $Z \subsetneq \mathbb{A}^n \times \mathbb{P}^{n_1} \times \mathbb{P}^{n_m}$ satisfying the conditions of 7. we get that X_* is a non-empty variety, not equal to $\mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$, and if $X = \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, then $X_* = \mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$. The maps $\bullet^*$ and $\bullet_*$ are mutual inverses under the restrictions of 6. and 7. Hence we get a one-to-one correspondence between non-empty varieties in $\mathbb{A}^n \times \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$ and varieties in $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ intersecting $\mathbb{A}^n \times U_{n_1+1} \times \cdots \times U_{n_m+1}$.

Definition 4.2.29. Let V be a variety in multispace. We define the *coordinate ring of* V to be

$$\Gamma(V) := K[\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_m}]/I(V).$$

Define the *ring of ration functions on* V to be

$$K(V) := \left\{ z = \frac{\alpha}{\beta} \in Q(\Gamma(V)) : \alpha, \beta \text{ are } m\text{-forms of the same degree} \right\}.$$

In the above we view $I(V) \subset K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]$, hence we identify $Q(\Gamma(V))$ with $Q(K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}]/I(V))$.

The coordinate ring is an integral domain an $K(V)$ is a field. These are generalizations of the affine/projective/multiprojective cases. Given a point $P = (v, [v_1], \ldots, [v_m]) \in V \subset \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ where V is a variety and a rational function $\frac{\alpha}{\beta} \in K(V)$, where P is not a zero of β. Then for every $\lambda_1, \ldots, \lambda_m \in K \setminus 0$

$$\frac{\alpha(v, \lambda_1 v_1, \ldots, \lambda_m v_m)}{\beta(v, \lambda_1 v_1, \ldots, \lambda_m v_m)} = \frac{\alpha(v, v_1, \ldots, v_m)}{\beta(v, v_1, \ldots, v_m)}$$

A rational function on V is defined at P if it has a representation where the denominator does not vanish on P. The local ring of such functions is denoted $\mathcal{O}_P(V)$. The pole set of a rational function on V, ϕ; i.e. the set of points in V on which ϕ is not defined is denoted $\mathcal{P}(\phi)$.

Remark 4.2.30. (cf. Remark 4.1.66) Consider $V = \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. Then

$$\Gamma(V)/\langle y_{n_1+1} - 1, \ldots, y_{n_m+1} - 1\rangle \to \Gamma(V_*), f + \langle y_{n_1+1} - 1, \ldots, y_{n_m+1} - 1\rangle \mapsto f_*.$$

One sees that

$$K(V) \simeq Q(\Gamma(V)/\langle y_{n_1+1} - 1, \ldots, y_{n_m+1} - 1\rangle) \simeq K(V_*)$$

defines an isomorphism. When V does not contain or is not contained in $V_{y_{n_i+1}}$ for some i, then

$$\Gamma(V) \to \Gamma(V_*), f + I(V) \mapsto f_* + I(V_*),$$

defines an isomorphism. From this we get that $x_{n+1} + I(V) = 1 + I(V)$, and hence that

$$K(V) = Q(\Gamma(V)) \simeq Q(\Gamma(V_*)) = K(V_*).$$

In any case it is clear to see that if $\alpha(P) \neq 0$ for some $P = (v, [v_1, 1], \ldots, [v_m, 1]) \in V \subset \mathbb{A}^n \times U_{n_1+1} \times \cdots \times U_{n_m+1}$, then $0 \neq \alpha(P) = \alpha_*(v, v_1, \ldots, v_m)$, hence $\mathcal{O}_P(V) \simeq \mathcal{O}_{(v,v_1,\ldots,v_m)}(V)$.

Remark 4.2.31. The pole set of a rational function $\phi \in K(V)$ is an algebraic set. Indeed, $J_\phi := \{g \in K[\mathbf{x}][\mathbf{y_1}, \ldots, \mathbf{y_m}] : (g + I(V))\phi \in \Gamma(V)\}$ is a m-homogeneous such that $\mathcal{P}(\phi) = V(J_\phi)$.

Definition 4.2.32. We define the *Segre embedding* of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ to be the map

$$S_{n_1 n_2} := S : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_1+n_2+n_1 n_2}$$
$$([v_1], [v_2]) \mapsto [v_{11}v_{21}, \ldots, v_{11}v_{2,n_2+1}, \ldots, v_{1,n_1+1}v_{21}, \ldots, v_{1,n_1+1}v_{2,n_2+1}]$$

Remark 4.2.33. This map is well-defined. First of all $\#\{(i,j) : 1 \leq i \leq n_1 + 1, 1 \leq j \leq n_2 + 1\} = (n_1+1)(n_2+1) = n_1 n_2 + n_1 + n_2 + 1$ and for some pair (i,j), $v_{1i} \neq 0 \neq v_{2j}$, hence the image of S is within the assigned co-domain. Secondly, if $(v_1, v_2) = (\lambda_1 v_1, \lambda_2 v_2)$ for some $\lambda_1, \lambda_2 \in K \setminus 0$, then for each $\lambda \in K \setminus 0$,

$$(\lambda\lambda_1\lambda_2 v_{1i} v_{2j}) = \lambda\lambda_1\lambda_2(v_{1i}v_{2j}),$$

hence S is well-defined. Note also that if $v_{1i}v_{2i} = \lambda v_{1i}v_{2j}$ for a pair (i,j) with $v_{1i}, v_{2j} \neq 0$, then $v_{1i} = \lambda v_{1i}$ and $v_{2j} = \lambda v_{2j}$ which implies the map is injective.

Lemma 4.2.34. *Let $n_1, n_2 \geq 1$, and $W = V(f_1, \ldots, f_m) \subset \mathbb{P}^{n_1+n_2+n_1 n_2}$ be a variety. Define*

$$g_i := f_i(x_{11}x_{21}, \ldots, x_{11}x_{2,n_2+1}, \ldots, x_{1,n_1+1}x_{21}, \ldots, x_{1,n_1+1}x_{2,n_2+1}),$$

which are polynomials in $K[\mathbf{x_1}, \mathbf{x_2}]$. Then

$$S^{-1}(W) = V(g_1, g_2, \ldots, g_m).$$

Proof. Indeed, $g(v) = f(S(v))$, hence

$$v \in S^{-1}(W) \iff g_i(v) = f_i(S(v)) = 0 \iff v \in V(g_1, \ldots, g_m).$$

□

Lemma 4.2.35. *Set $V := V(\{y_{ij}y_{kl} - y_{il}y_{kj} : i,k \in \{1, \ldots, n_1+1\}, j,l \in \{1, \ldots, n_2+1\}\})$. Then $S(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}) = V$, $S(U_i \times U_j) = V \cap U_{ij}$ and V is a variety.*

Proof. Let $S(v) \in S(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$. Then

$$\mathrm{ev}_{S(v)}(y_{ij}y_{kl} - y_{il}y_{kj}) = v_{1i}v_{2j}v_{1k}v_{2l} - v_{1i}v_{2l}v_{1k}v_{2j} = 0.$$

Conversely, if $v \in V$, then $v_{ij}v_{kl} = v_{il}v_{kj}$. For some pair k,l, $v_{kl} \neq 0$. Then for any $(i,j) \neq (k,l)$,

$$v_{ij} = (v_{il}v_{kj})/v_{kl}.$$

Define $w_{1i} = v_{il}/v_{kl}$ and $w_{2j} = v_{kj}/v_{kl}$. Define $w_1 = (w_{11}, \ldots, w_{1,n_1+1})$ and $w_2 = (w_{21}, \ldots, w_{2,n_2+1})$. Consider $\overline{S} : \mathbb{A}^{n_1} \setminus 0 \times \mathbb{A}^{n_2} \setminus 0 \to \mathbb{A}^{n_1+n_2+n_1n_2} \setminus 0, (u_1, u_2) \mapsto (u_{1i}u_{2j})$. Then

$$\overline{S}(w_1, w_2)_{ij} = w_{1i}w_{2j} = v_{il}v_{kj}/v_{kl} = v_{ij} \implies S([w_1], [w_2]) = [\overline{S}(w_1, w_2)] = [v].$$

In particular if $([v_1], [v_2]) \in U_i \times U_j$, then

$$\overline{S}(v_1, v_2)_{ij} = v_{1i}v_{2j} \neq 0 \implies S([v_1], [v_2]) = [\overline{S}(v_1, v_2)] \in U_{ij}.$$

Suppose $V = V_1 \cup V_2$ is a decomposition of V. Then

$$S^{-1}(V_1) \cup S^{-1}(V_2) = S^{-1}(V_1 \cup V_2) = S^{-1}(V) = S^{-1}(S(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})) = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$$

is a decomposition of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, meaning $S^{-1}(V_i) = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ for some i, hence $V_i = V$ for some i. □

4.3 Projective Plane Curves

4.3.1 Definitions and Basic Results

Definition 4.3.1. We say that two non-constant forms $F, G \in K[x, y, z]$ are equivalent if there is a $\lambda \in K \setminus 0$ such that $G = \lambda F$. In this case we write $F \sim G$. An element of $\{F \in V_K(d, 3) : d \geq 1\}/\sim$ is called a *projective plane curve*. The *degree* of a curve is just the degree of a representative of curve.

Remark 4.3.2. Every non-trivial algebraic set is characterized by such curves. Consider the factors of F, $F_1, \ldots, F_m$, Then $I(V(F)) = F_1 \cdots F_m$, we thus (as in the affine case) lose information about multiplicities. We call these factors *components*. For an irreducible f, $V(f)$ is irreducible. There is a one-to-one correspondence between projective varieties in $\mathbb{P}^2$ and irreducible projective plane curves f. We define $\Gamma(F) := \Gamma^h(V(F))$, $K(F) := K(V(F))$, for a point $P \in F$, $\mathcal{O}_P(F) := \mathcal{O}_P(V(F)$, when F is irreducible. If $P = [v, w, 1] \in U_3$ then $V(F)$ does not contain, and is not contained in L_∞, hence $\mathcal{O}_P(F) \simeq \mathcal{O}_{(v,w)}(F_*)$, since if $\frac{\beta}{\alpha} \in K(F)$, then P is a zero of α if and only if $\alpha_*(v, w) = 0$. We thus find that the isomorphism, $\Gamma(F) \simeq \Gamma(F_*)$ extends to an isomorphism $\mathcal{O}_P(F) \simeq \mathcal{O}_{(v,w)}(F_*)$. This result is obviously also true for U_1, U_2 with $F_{*,1}, F_{*,2}$. The upshot of this is that the results of 3.4.2 carries over to projective curves. Thus upon, for an arbitrary curve F, $P \in U_i$, $i = 1, 2, 3$ we define $m_P(F) := m_{(v_k,v_l)}(F_{*,i})$, $k, l \neq i$ which will only depend the local ring of each component of F (with multiplicity) by Corollary 1.8.121. This definition is independent of choice of U_i. Indeed for a component C passing through

$P = [v_1, v_2, v_3] \in U_i \cap U_j$. Then $L_{\infty,k} = \{[w_1, w_2, w_3] \in \mathbb{P}^3 : w_k = 0\}$ is not contained in and does not contain C. Then $\mathcal{O}_{(v_k,v_h)}(C_{*,i}) \simeq \mathcal{O}_P(C) \simeq \mathcal{O}_{(v_l,v_m)}(C_{*,j})$ for $k, h \neq i$, $l, m \neq j$. From this point we therefor let $\bullet_*$ denote dehomogenization with respect to the appropriate variable. The multiplicity is also invariant under a projective change of coordinates. Let $\varphi = (l_1, l_2, l_3)$ be a projective change of coordinates. Then

$$\mathcal{O}_{\varphi^{-1}(P)}(C(\varphi_1, \varphi_2, \varphi_3)) \simeq \mathcal{O}_P(C).$$

We say that a point on F is *simple* if $m_P(F) = 1$ and *multiple* if $m_P(F) > 1$.

Lemma 4.3.3. *Let F be a projective plane curve and $P \in F$. Then P is a multiple point of F if and only if $F_x(P) = F_y(P) = F_z(P) = 0$*

Proof. WLOG $P = [v_1, v_2, 1] \in U_3$, since multiplying with z commutes with $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$,

$$(F_x)_* = (F_*)_x \text{ and } (F_y)_* = (F_*)_y.$$

Using Euler's formula we get (in any characteristic)

$$dF = xF_x + yF_y + zF_z.$$

" $\Longrightarrow$ ": If P is a multiple point of F. Then $F_x(P) = (F_x)_*(v_1, v_2) = (F_*)_x(v_1, v_2) = 0$ and $F_y(P) = (F_y)_*(v_1, v_2) = (F_*)_y(v_1, v_2) = 0$. Then

$$0 = dF(P) = v_1F_x(P) + v_2F_y(P) + F_z(P) = F_z(P).$$

(Note that morally in the above, we take evaluation in P to evaluation in an arbitrary representative of P).
" $\Longleftarrow$ ": We get that $(F_*)_x(v_1, v_2) = F_x(P) = 0$ and $(F_*)_y(v_1, v_2) = F_y(P) = 0$. It follows that $m_P(F) = m_{(v_1,v_2)}(F_*) > 1$. □

Definition 4.3.4. Let F be a projective curve and $P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points. Let L be a line that does pass through these points (cf. Example 4.1.68 2.). We define $F_* = \frac{F}{L^{\deg F}} \in K(\mathbb{P}^2)$.

Remark 4.3.5. 1. Suppose Λ is another line that does not pass through $P_1, \ldots, P_n$. Then

$$\frac{F}{\Lambda^{\deg F}} = \left(\frac{L}{\Lambda}\right)^d \frac{F}{L^d},$$

and since $L/\Lambda \in K(\mathbb{P}^2)^*$, F_* is unique up to multiplication by a unit.

2. After an appropriate projective change of coordinates, φ say. We may arrange that $\varphi(L) = L_\infty = z$ is line passing through no $Q_i = \varphi(P_i)$. Hence $Q_1, \ldots, Q_n \in U_3$ are distinct point that L_∞ does not pass through and $G = F \circ \varphi$ is curve of degree deg F. Therefor, under the identification $K(\mathbb{P}^2) \simeq K(\mathbb{A}^2), \alpha/\beta \mapsto \alpha_*/\beta_*$, $G_* = G/z^{\deg\ F} \mapsto G(x, y, 1)/1^{\deg\ F}$, hence in this case $\bullet_*$ is in fact just dehomogenization under the aforementioned identification.

3. If F is an irreducible curve, then F_* is irreducible by Corollary 1.8.121. Therefor if P is simple, then $\mathcal{O}_P(F)$ is a DVR. Then we have an order function ord_P^F on $Q(\mathcal{O}_P(F)) = K(F)$.

Definition 4.3.6. Let F be an irreducible projective curve with $P \in F$ a simple point. Consider a form $G \in K[x, y, z]$. We then define $\text{ord}_P^F(G) := \text{ord}_P^F(G_*)$.

Remark 4.3.7. Let H be any form of degree deg G such that $H(P) \neq 0$. Then since $\frac{1}{H+I(F)}$ is unit in $\mathcal{O}_P(F)$, $\text{ord}_P^F(G/H) = \text{ord}_P^F(G)$.

Definition 4.3.8. Let F, G be projective plane curves and $P \in \mathbb{P}^2$. We define the *intersection number of F and G at P* to be

$$I(P, F \cap G) := \dim\ \mathcal{O}_P(\mathbb{P}^2)/\langle F_*, G_* \rangle$$

Remark 4.3.9. One should have in mind that we dehomogenize with respect to a non-zero coordinate of P and that the definition is independent of which one we choose (if there is more than one). This has the properties of the affine version (cf. 3.4.3), in 3. however φ should be a projective change of coordinates and in 7. h should be a form of degree deg $G -$ deg F (simply to ensure that $g + hf$ is a form).

Definition 4.3.10. A projective line L is a *tangent* of a curve projective plane F if $I(P, L \cap F) > m_P(F)$. A multiple point $P \in F$ is called *ordinary* if F has $m_P(f)$ distinct tangents at P.

Remark 4.3.11. Let the Q be appropriate affine coordinates that are identified with P under some φ_i. We see that $I(P, L \cap F) > m_P(F)$ if and only if $I(Q, L_* \cap F_*) > m_Q(F_*)$ if and only if L_* is tangent to F_* at Q (cf. Corollary 3.4.57).

Proposition 4.3.12. *Let F be a curve at $P \in U_i$ and Q be the identification with P in $\mathbb{A}^2$. Then the tangents of F at P are the tangents of F_* at Q.*

Proof. Let $F_* = \prod_1^n l_i^{r_i} + \ldots$, where $l_1, \ldots, l_m$ are the tangents of F.

$$I(P, l_i \cap F) = I(Q, l_i \cap F_*) > m_Q(F_*) = m_P(F),$$

hence l_i is a tangent of F. Suppose conversely that L is a tangent of F. Then

$$I(Q, L_* \cap F_*) = I(P, L \cap F) > m_P(F) = m_Q(F_*).$$

Note that L_* is a 1-degree polynomial, for otherwise $I(P, L \cap F) \in \{0, \infty\}$. Therefor L_* is a tangent for F_* at Q, hence L_* is a form, meaning $L_* = L$. □

Example 4.3.13. 1. Consider the curve $F = xy^4 + yz^4 + xz^4$ over K with characteristic 0. Consider the system of polynomial equations

$$\begin{cases} F = 0 \\ F_x = y^4 + z^4 = 0 \\ F_y = 4xy^3 + z^4 = 0 \\ F_z = 4yz^3 + 4xz^3 = 0 \end{cases}$$

Suppose $z = 1$, then y needs to be a solution to $y^4 + 1$, which we denote by s. Then x has to be a common zero of $4s^3x + 1$ and $4s + 4x$, hence there is no solution in U_3. If $y = 1$, we find that $z = s$ and again x has to be a common zero of $4x+1$ and $4s^3+4xs^3$, which is not possible in characteristic 0. Suppose $x = 1$. Then $4yz^3+4z^3 = 0$, hence $y = -1$ or $z = 0$. In the first case $z^4 - 4 = 0$, hence $z = \pm 2$, but then $y^2 + z^2 = 5 \neq 0$, hence no such solution can exist. If $z = 0$, then $4y^3 = 0$, hence $y = 0$. Hence in U_1 the only solution is $[1, 0, 0]$. Note that $F(1, y, z) = y^4 + z^4 + yz^4$. Then the lowest degree form of $F(1, x, y)$ at $(0, 0)$ is

$$y^4 + z^4 = (y^2 + s^2z^2)(y^2 - s^2z^2) = \underbrace{(y + s^3z)}_{l_1}\underbrace{(y - s^3z)}_{l_2}\underbrace{(y + sz)}_{l_3}\underbrace{(y - sz)},$$

hence by the prior proposition these are the tangents of F at $[1, 0, 0]$. Consider also $F(x - s, y + s, 1) = (x - s)(y + s)^4 + y + s + x - s = (x - s)(y^4 + 4sy^3 + 6s^2y^2$.

2. Set $F = x^2y^3 + x^2z^3 + y^2z^3$. The multiple points of F are the solution to the system of polynomial equations

$$\begin{cases} F = 0 \\ F_x = 2xy^3 + 2xz^3 = 0 \\ F_y = 3x^2y^2 + 2yz^3 = 0 \\ F_z = 3x^2z^2 + 3y^2z^2 = 0 \end{cases}$$

We first aim to find a solution in U_3. We identify that $(0, 0)$ is a valid solution. Note that $x = 0$ if and only if $y = 0$. We investigate whether there can exist others.

Suppose $x \neq 0$. Then $2xy^3+2x = 0 \iff y^3 = -1$ call any solution to this equation α. Then x must satisfy $3\alpha^2x^2 + 2\alpha = 0$, hence $x^2 = \frac{-2}{3\alpha}$. Call this value β. Note however that

$$\alpha(3\beta^2 + 3\alpha^2) = -2 - 3 = -5 \neq 0.$$

So the only solution is $[0, 0, 1]$. In U_2, $[0, 1, 0]$ is a solution. again $x = 0$ if and only if $z = 0$. Suppose $x \neq 0$. Then $2x + 2xz^3 = 0$, hence we must have that $z = \alpha$. Then we must have that $3x^2\alpha^2 + 3\alpha^2 = 0 \iff x^2 = -1$. Call any solution of this equation i. Then $-3\alpha^2 + 2\alpha = 0 \iff \alpha(-3\alpha + 2) = 0$, hence $[0, 1, 0]$ is the only solution. In U_1 we get that $[1, 0, 0]$ is a solution with $y = 0 \iff z = 0$. We must have that $y^3 = -z^3$ and $0 = 3z^2 + 3y^2z^2 = 3z^2(y^2 + 1)$, hence $y = i$ for a solution with $y \neq 0$ to exist. But then $z^3 = -i^3$, hence $0 = -3 - 2ii^3 = -3 - 2 = -5 \neq 0$, which implies $[1, 0, 0]$ is the only possible solution.
Note that $F(x, y, 1) = x^2y^3 + x^2 + y^2$, hence the tangents of F at $[0, 0, 1]$ are $(x + iy)(x - iy)$. Note secondly that $F(x, 1, z) = x^2 + x^2z^3 + z^3$, hence F has x as a double tangent at $[0, 1, 0]$. Note lastly $F(1, y, z) = y^3 + z^3 + y^2z^3$, hence the tangents of F at $[1, 0, 0]$ are $x + z, 2x + (1 + i\sqrt{3})z, 2x + (1 - i\sqrt{3})z$.

3. Consider $F = y^2z - x(x - z)(x - \lambda z) = y^2z - (x^2 - xz)(x - \lambda z) = y^2z - x^3 + (z + \lambda z)x^2 - \lambda xz^2$ where $\lambda \in K$. To find multiple points of F in $\mathbb{P}^2$ we solve the system of polynomial equations

$$\begin{cases} F = 0 \\ F_x = 3x^2 - 2(1 + \lambda)zx + \lambda z^2 = 0 \\ F_y = 2yz = 0 \\ F_z = y^2 + (1 + \lambda)x^2 + 2\lambda xz = 0 \end{cases}$$

Note that either $y = 0$ or $z = 0$. If $z = 0$, then $3x^2 = 0$, hence $x = 0$, but then $y^2 = 0$. Assume $z = 1$, then $y = 0$ and x must be a common zero of $3x^2 - 2(1 + \lambda)x + \lambda$ and $F(x, 0, 1) = x(x - 1)(x - \lambda)$. $[0, 0, 1]$ is a multiple point if and only if $\lambda = 0$, since $F_x(0, 0, 1) = \lambda$. If $x = 1$

$$F_x(1, 0, 1) = 1 - \lambda,$$

then $[1, 0, 1]$ is a multiple point if and only if $\lambda = 1$. Similarly, if $x = \lambda$,

$$F(\lambda, 0, 1) = 3\lambda^2 - 2\lambda - 2\lambda^2 + \lambda = \lambda^2 - \lambda = \lambda(\lambda - 1),$$

hence $[\lambda, 0, 1]$ is a multiple point if and only $\lambda = 0$ or $\lambda = 1$. In conclusion, when $\lambda = 0$, $[0, 0, 1]$ is a multiple point, in which case $F(x, y, 1) = y^2 + x^2 - x^3$, hence

the tangents at this point are $x + iy$ and $x - iy$. When $\lambda = 1$, $[1, 0, 1]$ is a multiple point and $F(x + 1, y, 1) = y^2 - x^2(x + 1) = y^2 + x^2 - x^3$, hence the tangent at this point are $x + iy$ and $x - iy$ as well. For every other value of λ, F is non-singular.

4. Consider $F = x + y + z$. Then $F_x = F_y = F_z = 1$, hence F has no multiple points

5. Consider $F = x^n + y^n + z^n$, $n > 1$, Then $F_x = nx^{n-1}$, $F_y = ny^{n-1}$ and $F_z = nz^{n-1}$, we see that the only solution to $F_x = F_y = F_z = 0$ is $x = y = z = 0$, hence F has no multiple points.

Example 4.3.14. 1. Let $F = y^2z - x(x - 2z)(x + z)$ and $G = y^2 + x^2 - 2xz$. Note that

$$\begin{cases} F_* = y^2 - x^3 + x^2 + 2x \\ G_* = y^2 + x^2 - 2x \end{cases}$$

We check whether there are common zeros of F and G in U_3. (a, b) is a common zero of $F(x, y, 1)$ and $G(x, y, 1)$ if and only if a is a zero of $x(x-2)(x+1) - x(x-2) = x(x - 2)(x + 1 - 1) = x^2(x - 2)$, hence $a = 0$ or $a = 2$ and $b = 0$. The common zeros of F and G in U_3 therefor are $P_1 = [0, 0, 1]$ and $P_2 = [2, 0, 1]$.

$$\begin{aligned} I(P_1, F \cap G) &= I((0,0), y^2 - x^3 + x^2 + 2x \cap y^2 + x^2 - 2x) \\ &= I((0,0), y^2 - x^3 + x^2 + 2x \cap x^3 - 4x) \\ &= I((0,0), y^2 - x^3 + x^2 + 2x, x) \\ &+ I((0,0), y^2 - x^3 + x^2 + 2x \cap x^3 - 4) \\ &= I((0,0), y^2, x) = 2. \end{aligned}$$

Secondly, note that

$$\begin{cases} F_*(x + 2, y) = y^2 - x(x + 2)(x + 3) \\ G_*(x + 2, y) = y^2 + x(x + 2) \end{cases}$$

$$\begin{aligned} I(P_2, F \cap G) &= I((0,2), F_* \cap G_*) = I((0,0), F_*(x, y + 2, 1) \cap G(x, y + 2, 1)) \\ &= I((0,0), y^2 - x(x + 2)(x + 3) \cap y^2 + x(x + 2)) \\ &= I((0,0), y^2 \cap y^2 + x^2 + 2x) \\ &= 2I((0,0), y \cap y^2 + x^2 + 2x) = 2. \end{aligned}$$

Note that $F(x, y, 0) = x^3$ and $G(x, y, 0) = y^2 + x^2$. It thus follows that F and G have no other common zeros.

Definition 4.3.15. Two curves F and G are *projectively equivalent* if there is a projective change of coordinates φ such that $\varphi^{-1}(F) = G$

Remark 4.3.16. Any statement that only depends on local rings of curves will thus be true for any curve in an equivalence class under this equivalence relation. This is useful since we may prove a general statement on projective curves by reducing it to a statement to nicer equivalent curves

Lemma 4.3.17. *Let P be a simple point of a projective curve F. Then the tangent of F at P is given by*

$$F_x(P)x + F_y(P)y + F_z(P)z$$

Proof. WLOG $P = [0, 0, 1] \in U_3$. Then then tangent of F at P is equal to the tangent of F_* at $(0, 0)$ which is given by $L = (F_x)_*(0, 0)x + (F_y)_*(0, 0)y$. Set $d := \deg F$. Using Euler's formula

$$0 = dF(P) = F_z(P),$$

hence we find that

$$L = F_x(P)x + F_y(P)y + F_z(P)z.$$

□

Lemma 4.3.18. *Let P be a point on a projective curve F. Then*

$$m_P(F_x) \geq m_P(F) - 1.$$

Proof. WLOG $P = [0, 0, 1]$. Set $m := m_P(F) = m_0(F_*)$. We consider $F_* = F(x, y, 1) = \sum_m^d f_i$. Then

$$(F_*)_x = \sum_m^d (f_i)_x.$$

If $(f_i)_x = 0$ for every i, then trivially $\infty = m_P(F_x) = m_0((F_*)_x) > m_0(F_*) = m_P(F)$. So let $n \geq m$ be the smallest index such that $(f_n)_x \neq 0$. Then $(f_n)_x$ is a form degree $n - 1$, hence

$$m_P(F_x) = m_0((F_*)_x) = n - 1 \geq m_P(F) - 1$$

□

Proposition 4.3.19. *Any two curves F and G with no common components, have only finitely many points of intersection*

Proof. If F and G have no common zeroes, then $\gcd(F_{*,i}, G_{*,i}) = 1$, $i = 1, 2, 3$ by Corollary 1.8.121. Hence

$$\#V(F,G) \leq \sum_1^3 \#(V(F,G) \cap U_i) < \infty.$$

It can also be seen from the fact that

$$\sum_P I(P, F \cap G) \leq \sum_1^3 \sum_P I(P, F_{*,i} \cap G_{*_i}) < \infty,$$

by Corollary 3.4.56. □

Proposition 4.3.20. *Let F be an irreducible projective curve. F has only finitely many multiple points.*

Proof. By proposition 3.4.22, F has only finitely many multiple points in each U_i (note that this proof works in any characteristic).
(**Note to self:** In Fulton this is marked as an obligatory exercise while the proposition is not. The (I think) intended proof uses the fact that WLOG $F_x \neq 0$ (since F is non-constant), implying $(F_x)_* \neq 0$ hence deg $(F_x)_* <$ deg F_*, implying $F_* \nmid (F_x)_*$, hence $\gcd(F_*, (F_x)_*) = 1$, hence $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$ for only finitely many P in each U_i. This ONLY works in characteristic 0. To prove the characteristic $p > 0$ case, we would apply Lemma 3.4.21 to F in each U_i, before using the char $K = 0$-argument. This is just to say, I haven't strayed from the books approach in going to the simple two line proof.) □

Example 4.3.21. 1. Let C be an irreducible *conic*; i.e. a projective curve of degree 2. In this example we aim to show that up to projective equivalence $C = yz - x^2$. Suppose $P = [0, 0, 1]$ is a simple point of C with y as tangent line of C at P. Write $C = ax^2 + by^2 + cz^2 + dxy + exz + fyz$. Then $C_* = ax^2 + by^2 + c + dxy + ex + fy$. Then $c = e = 0$ and $f \neq 0$, hence $C = ax^2 + by^2 + dxy + fyz$. If $a = 0$, C is reducible. If $a \neq 0$, it follows from Eisenstein's criterion using y that $C \in K[z, y][x]$ is irreducible. WLOG $C = fyz - (ax^2 + by^2 + dxy)$ with $a, f \neq 0$. Set φ equal to the projective change of coordinates induced by $\mathrm{diag}(a^{-1}, f^{-1}, 1)$. Then $C^\varphi = yz - x^2 - \alpha y^2 - \beta xy$. Consider then the projective change of coordinates ϕ induced by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & \alpha & 1 \end{pmatrix}$$

Then

$$\begin{aligned}(C^{\varphi})^{\phi} &= y(\beta x + \alpha y + z) - x^2 - \alpha y^2 - \beta xy \\ &= \beta xy + \alpha y^2 + yz - x^2 - \alpha y^2 - \beta xy = yz - x^2.\end{aligned}$$

Given an arbitrary irreducible conic we apply a projective change of coordinates such that it has P as a simple point and y as a tangent line at P to see that it will be projectively equivalent to $F := yz - x^2$. Note that

$$\begin{cases} F_x = 2x \\ F_y = z \\ F_z = y \end{cases}$$

hence any irreducible conic will be non-singular when char $K \neq 2$ and the only common zero of the partial derivatives when char $K = 2$, is $[1, 0, 0]$ which is not a zero of F.

2. Consider an irreducible cubic C with a cusp $P := [0, 0, 1]$ and tangent y at P. We aim to show that up to projective equivalence an irreducible cubic with a cusp is $y^2z - x^3$. Write

$$\begin{aligned}C = a_{300}x^3 + a_{030}y^3 + a_{003}z^3 + a_{210}x^2y + a_{201}x^2z \\ + a_{120}xy^2 + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 + a_{111}xyz\end{aligned}$$

Then

$$\begin{aligned}C_* = a_{300}x^3 + a_{030}y^3 + a_{003} + a_{210}x^2y + a_{201}x^2 \\ + a_{120}xy^2 + a_{021}y^2 + a_{102}x + a_{012}y + a_{111}xy.\end{aligned}$$

Then since P is a double point and y is the only tangent of C, we necessarily have that $a_{003} = a_{102} = a_{012} = a_{111} = a_{201} = 0$ and $a_{021} \neq 0$ hence

$$C = ay^2z - bx^3 - cy^3 - dx^2y - exy^2,$$

where $a \neq 0$. If $b = 0$, then $y \mid C$, hence $b \neq 0$. Take φ to be the projective change of coordinates induced by

$$\begin{pmatrix} b^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $C^\varphi = y^2z - x^3 - \alpha y^3 - \beta x^2 y - \gamma x y^2$ Let ϕ be the projective change of coordinates induced by

$$\begin{pmatrix} 1 & -\beta/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We thus get that

$$\begin{aligned}(C^\varphi)^\phi &= y^2z - (x - \beta/3y)^3 - \alpha y^3 - \beta(x - \beta/3y)^2 y - \gamma(x - \beta/3y)y^2 \\ &= y^2z - x^3 + \beta x^2 y - \beta^2/3xy^2 + \beta^3/27y^3 - \alpha y^3 - \beta x^2 y - \beta^2/9y^3 + 2\beta/3xy^2 - \gamma xy^2 + \gamma\beta/3 \\ &= y^2z - x^3 - \underbrace{(\beta^2/3 - 2\beta/3 + \gamma)}_{A}xy^2 - \underbrace{(-\beta^3/27 + \alpha^3 + \beta^2/9 - \gamma\beta/3)}_{B}y^3\end{aligned}$$

Let ψ be the projective change of coordinates induced by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & 1 \end{pmatrix}$$

Then

$$((C^\varphi)^\phi)^\psi = y^2(Ax + By + z) - x^3 - Axy^2 - By^3 = y^2z - x^3 =: F.$$

Then

$$\begin{cases} F_x = 2x \\ F_y = 2yz \\ F_z = y^2 \end{cases}$$

Then we see that $[0, 0, 1]$ is a singularity. It is clear that it's the only singularity for char $K \neq 2$. In characteristic 2 another candidate would be $[\alpha, 0, \beta]$, $(\alpha, \beta) \neq 0$, but for such a point to be a zero of F, $\beta = 0$. Therefor any irreducible cubic with a cusp has only one multiple point, namely the cusp.

3. Up to projective equivalence there is only one irreducible cubic with a node. Consider an irreducible cubic C with a node (a double point at which there are distinct tangents of the curve) at $P := [0, 0, 1]$ and tangents x and y Write

$$\begin{aligned}C = a_{300}x^3 + a_{030}y^3 + a_{003}z^3 + a_{210}x^2y + a_{201}x^2z \\ + a_{120}xy^2 + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 + a_{111}xyz\end{aligned}$$

Then

$$C_* = a_{300}x^3 + a_{030}y^3 + a_{003} + a_{210}x^2y + a_{201}x^2 + a_{120}xy^2 + a_{021}y^2 + a_{102}x + a_{012}y + a_{111}xy.$$

Then $a_{003} = a_{201} = a_{021} = a_{102} = a_{012} = 0$ and $a_{111} \neq 1$. Hence

$$C = axyz - bx^3 - cy^3 - dx^2y - exy^2$$

with $a \neq 0$. If b or c are 0, then $x \mid C$ respectively $y \mid C$, hence $b, c \neq 0$. Changing coordinates by φ induced by $\mathrm{diag}(b^{-1}, c^{-1}, a^{-1})$, we get

$$C^\varphi = xyz - x^3 - y^3 - \alpha x^2y - \beta xy^2.$$

Let ϕ be the projective change of coordinates induced by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{pmatrix}$$

Then we find that

$$(C^\varphi)^\phi = xy(z + \alpha x + \beta y) - x^3 - y^3 - \alpha x^2y - \beta xy^2 = xyz - x^3 - y^3 =: F$$

See that

$$\begin{cases} F_x = yz - 3x^2 \\ F_y = xz - 3y^2 \\ F_z = xy \end{cases}$$

In any characteristic it is true that $x = 0$ or $y = 0$, plugging this into F, we see that both x and y have to be 0, hence any irreducible cubic with a node has only the node as a multiple point.

Lemma 4.3.22. *Let F be a curve of degree $d > 0$ passing through $Q := [0, 0, 1]$. Then*

$$\sum_P I(P, F \cap x) = d \text{ or } \sum_P I(P, F \cap y) = 0$$

Proof. Since $F(0, 0, 1) = 0$, $F_* = ax^d + by^d + \ldots$ where $a \neq 0$ or $b \neq 0$, then $x \nmid F_*$ or $y \nmid F_*$. Note that F does not pass through $[\alpha, \beta, 0]$. Suppose $x \nmid F_*$. It follows from Proposition 3.4.56 that

$$\sum_P I(P, F \cap x) = \sum_{P \in \mathbb{A}^2} I(P, F_* \cap x) = \dim\ K[x, y]/\langle F_*, x\rangle = \dim\ K[y]/\langle F_*(0, y)\rangle = d,$$

where the last equality follows from Lemma 1.8.55. □

Remark 4.3.23. Given an arbitrary curve F of degree d and line L not contained in F that intersect at a point Q, we may take a projective change of coordinates taking Q to $[0,0,1]$,L to x and F to some degree n curve G. Then

$$\sum_P I(P, F \cap L) = \sum_P I(P, G, x) = d.$$

Proposition 4.3.24. *An irreducible cubic is either non-singular or has exactly one double point.*

Proof. Let F be an irreducible curve of degree n with two multiple points $P_1, P_2 \in F$. Let L be the line through these two points. Note that since F is irreducible, $L \not\subset F$. Then by prior lemma

$$n = \sum_P I(P, F \cap L) = I(P_1, F \cap L) + I(P_2, F \cap L) + ... \geq m_{P_1}(F) + m_{P_2}(F) \geq 4,$$

hence any irreducible curve of degree < 4 has at most one multiple point. Could it be a triple point? No, since any cubic with a triple is projectively equivalent to $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. Note that if an irreducible cubic admits a double point, the tangents at that point are either distinct or not, hence up to projective equivalence the curve is either $y^2z - x^3$ (the irreducible cubic with a cusp) or $xyz - x^3 - y^3$ (the irreducible cubic with a node).

This hints at another proof: An irreducible cubic cannot have $m_P(F) > 3$ for any point. We have also seen that it cannot have $m_P(F) = 3$ at any point. So if it has a multiple point it is either a node or a cusp in which case the curve is either projectively equivalent to $y^2z - x^3$ resp. $xyz - x^3 - y^3$, meaning such a curve has only one double point. □

Lemma 4.3.25. *Let $P, P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points. Then there are infinitely many lines passing through P not passing through P_1.*

Proof. the set of lines through P is in one-to-one correspondence with points distinct from P. Indeed, set $P = [v]$ and consider $Q = [w]$,$Q' = [w']$, $L(Q, P) = L(Q', P)$. Then for suitable $\alpha, \beta, \gamma \in K$, we have that $w = \alpha v + \beta w' = \alpha v + \gamma w$, implying $\alpha = 0$ and $w = \beta w'$ with $\beta \neq 0$. This means that $\mathbb{P}^2 \setminus \{P_1, \ldots, P_n, P\}$ is in bijection with the lines passing through P and not passing through $P_1, \ldots, P_n$, implying that there are infinitely many such lines. □

Remark 4.3.26. Let $P \in F$ be a simple point on a curve. Let $P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points distinct from P. Then there are infinitely many lines not passing through $P_1, \ldots, P_n$ that intersect F transversely at P.

Lemma 4.3.27. *Let C be an irreducible projective plane curve. Consider $P_1, \ldots, P_n \in C$ all distinct and simple with integers $m_1, \ldots, m_n$. Then there is a $z \in K(C)$ with $\operatorname{ord}_{P_i}^C(z) = m_i$ for each i.*

Proof. By the prior lemma and remark we can for each i pick a line L_i passing through P_i and not passing through P_j for $j \neq i$ that intersects C transversally at P_i. We may also pick a line L_0 that passes through no P_i. Pick

$$z := \prod_1^n L_i^{m_i} L_0^{-\sum_{j \neq i} m_j}.$$

Indeed notice first that by Proposition 3.4.54

$$\operatorname{ord}_{Q_i}^{C_*}\left(\prod_1^n (L_i)_*^{m_i}\right) = I\left(P_i, C \cap \prod_1^n L_i^{m_i}\right) = m_i,$$

where Q_i is the appropriate affine coordinate corresponding to P_i. Since the element $\prod_1^n (L_i)_*^{m_i} \in \mathcal{O}_{Q_i}(C_*)$ is identified with $z \in \mathcal{O}_{P_i}(C)$, we find that $\operatorname{ord}_{P_i}(z) = \operatorname{ord}_{Q_i}(\prod_1^n (L_i)_*^{m_i}) = m_i$. □

Lemma 4.3.28. *Let $P \in F$ be a point on an irreducible curve. Suppose $I(P, F \cap z) = 1$ and $P \neq [1,0,0]$. Then $F_x(P) \neq 0$.*

Proof. We prove the contrapositive: If G is a curve that intersect with z at a point P and $G_x(P) = 0$, then $P = [\alpha, \beta, 0]$ and by Euler's formula

$$0 = (\deg\ G)G(P) = \beta G_y(P).$$

Then $P = [1,0,0]$ or $G_y(P) = 0$, which implies $G_z(P) = 0$, meaning $I(P, F \cap z) > 1$. □

4.3.2 Linear Systems Of Curves

The projective plane curves of $V(d,3)$, $(d > 1)$ has $\{\mathbf{x}^v : |v| = d\}$ as a basis over K. Set $N = \binom{d+2}{2} = \frac{(d+1)(d+2)}{2}$, we then get a commutative diagram

$$\begin{array}{ccc} V(d,3) & \xrightarrow{\sim} & \mathbb{A}^N \\ \downarrow & & \downarrow \\ V(d,3) \setminus 0 & \xrightarrow{\sim} & \mathbb{A}^N \setminus 0 \\ \downarrow & & \downarrow \\ (V(d,3) \setminus 0)/\sim & \xrightarrow{\sim} & \mathbb{P}^{N-1} \end{array}$$

which is a natural motivation to study the behavior of curves of fixed degree via the identification of these with points in $\mathbb{P}^{N-1} = \mathbb{P}^{\frac{d(d+3)}{2}}$. We denote this space by $\mathcal{L}_d$.

Definition 4.3.29. Let d be some fixed positive integer. Let S be a subset of $\mathcal{L}_d$. We then have a natural identification of S with a subset of $\mathbb{P}^{\frac{d(d+3)}{2}}$. If $S \subset \mathbb{P}^{\frac{d(d+3)}{2}}$ is a linear subvariety, we say that it is *linear system of plane curves*.

Remark 4.3.30. Throughout this subsection we fix a positive integer d; and make the choice to denote $(d+1)(d+2)/2$ by N and the standard basis elements of $V(d,3)$ be $M_1, \ldots, M_d$.

Proposition 4.3.31. *Let d be some fixed positive integer and Fix a point $P \in \mathbb{P}^2$. The set S of curves passing through P is a linear system of plane curves (in particular a hyperplane).*

Proof. Let $F = \sum_1^N a_i M_i$, $[a_1, \ldots, a_N] \in \mathbb{P}^{N-1}$ be degree d curve. Then $F(P) = 0$ if and only if $\sum_1^d a_i M_i(P) = 0$. There is at least one monomial that does not vanish on P. This means that $\sum_1^N M_i(P) y_i \in K[y_1, \ldots, y_N] \setminus 0$ and that $S = V(\sum_1^N M_i(P) y_i)$, meaning S is a hyperplane. □

Corollary 4.3.32. *Let $P_1, \ldots, P_n \in \mathbb{P}^2$ be an arbitrary set of points. If $n \leq N-1$ The set of curves passing through these points is a linear system of plane curve.*

Corollary 4.3.33. *Let $P_1, \ldots, P_{N-1} \in \mathbb{P}^2$ be distinct points. There is exactly one curve of degree d passing through each of these points.*

Proof. This follows from Lemma 4.1.54. □

Lemma 4.3.34. *Let φ = be a projective change of coordinates on $\mathbb{P}^2$. Then the map $\mathcal{L}_d \to \mathcal{L}_d, F \mapsto F^\varphi$ induces a projective change of coordinates on $\mathbb{P}^{N-1}$.*

Proof. We define $\phi : \mathbb{P}^{N-1} \simeq \mathcal{L}_d \xrightarrow{\varphi} \mathcal{L}_d \simeq \mathbb{P}^{N-1}$. Let $F\mathcal{L}_d$. Note that since φ is invertible F^φ is a degree d curve, since otherwise $F = (F^\varphi)^{\varphi^{-1}} = 0$, hence $\phi([v_1, \ldots, v_N])$ is an element of $\mathbb{P}^{N-1}$. For each M_i we get that

$$M_i \circ \varphi = \sum_1^d b_{ij} M_j,$$

hence

$$F^\varphi = \sum_1^d a_i M_i \circ \varphi = \sum_1^d a_i \sum_1^d b_{ij} M_j = \sum_{j=1}^d \left[\sum_{i=1}^d b_{ij} a_i \right] M_j.$$

Therefor

$$\phi([v_1, \ldots, v_N]) = [\sum_1^d b_{i1} v_i, \ldots, \sum_1^d b_{iN} v_i]$$

hence ϕ is induced by the linear transformation on $\mathbb{A}^N$, $(b_{ij})^T$. This is invertible since it has mutual inverse induced by $F \mapsto F^{\varphi^{-1}}$, and is thus a projective change of coordinates. □

Definition 4.3.35. Let $P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points and $1 \leq r_i \leq d+1$ for each i. We define

$$V(d; r_1P_1, \ldots, r_nP_n) := \{F \in \mathcal{L}_d : m_{P_i}(F) \geq r_i, 1 \leq i \leq n\}$$

Lemma 4.3.36. *Let $P \in \mathbb{P}^2$ and $1 \leq r \leq d+1$. $V(d; rP) \subset \mathbb{P}^{N-1}$ is a linear subvariety of dimension*

$$N - 1 - \frac{r(r+1)}{2}.$$

In other words codim $V(d; rP) = \frac{r(r+1)}{2}$

Proof. By the prior lemma we may assume that $P = [0, 0, 1]$. Note that $F \in \mathcal{L}_d$ has multiplicity greater than r if and only if the coefficient of F at $x^i y^j z^k$ is 0 for every (i, j) with $i + j < r$, which is readily seen by writing $F = \sum_0^d F_i z^{d-i}$ for forms F_i. There are $D := \#\{(i, j) : i + j < r\} = \binom{r-1+2}{r} = \binom{r+1}{r} = \frac{r(r+1)}{2}$ It follows that $V(d; rP)$ can be identified with

$$\left\{[v_1, \ldots, v_{N-D}, 0 \ldots, 0] \in \mathbb{P}^{N-1}\right\} = V(y_{N-D+1}, \ldots, y_N)$$

It thus follows that dim $V(d; rP) = N - 1 - D$. □

Lemma 4.3.37. *Let $P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points and let $r_1, \ldots, r_n$ be positive integers $\leq d+1$. Then $V(d; r_1P_1, \ldots, r_nP_n)$ is linear subvariety of dimension*

$$\geq N - 1 - \sum_1^n \frac{r_i(r_i+1)}{2}.$$

Proof. Note that

$$V(d; r_1P_1, \ldots, r_nP_n) = \bigcap_1^n V(d; r_iP_i),$$

hence by Lemma 1.7.19 and the prior lemma, it follows that

$$\text{dim } V(d; r_1P_1, \ldots, r_nP_n) = N - 1 - \text{codim } V(d; r_1P_1, \ldots, r_nP_n) \geq N - 1 - \sum_1^n \text{codim } V(d; r_iP_i)$$

$$= N - 1 - \sum_1^n \frac{r_i(r_i+1)}{2}$$

□

Theorem 4.3.38. *Let $P_1, \ldots, P_n \in \mathbb{P}^2$ be distinct points and let $r_1, \ldots, r_n$ be positive integers. Suppose $d \geq [\sum_1^n r_i] - 1$. Then*

$$\text{dim } V(d; r_1P_1, \ldots, r_nP_n) = N - 1 - \sum_1^n r_i.$$

Proof. We prove statement by induction in value $m := [\sum_1^n r_i] - 1$. In the case $m = 0$ we get that $r_i = 1$ for some i and $r_j = 0$ for $j \neq i$. Then

$$V(d; r_1P_1, \ldots, r_nP_n) = V(d; r_iP_i) \implies \dim\, V(d; r_1P_1, \ldots, r_nP_n) = N - 1 - \frac{r_i(r_i+1)}{2}$$
$$= N - q - \sum_1^n \frac{r_j(r_j+1)}{2}.$$

So assume $m > 1$, $d > 1$.

Case 1: Suppose first that $r_i = 1$ for each i. Then $V(d; P_1, \ldots, P_n)$ is the intersection of n hyperplanes. By induction $\dim\, V(d; P_1, \ldots, P_{n-1}) = N - n$. Choose lines through L_i through P_i not passing through P_j for $j \neq i$ for $i \in \{1, \ldots, n-1\}$ and L_0 a line not passing through $P_1, \ldots, P_{n-1}$. Then $F = L_0^{d-n+1} \prod_1^{n-1} L_i \in V(d; P_1, \ldots, P_{n-1})$ but not in $V(d; P_1, \ldots, P_n)$, hence the hyperplanes generating $V(d; P_1, \ldots, P_n)$ are linearly independent (since $V(d; P_1, \ldots, P_n) \subsetneq V(d; P_1, \ldots, P_{n-1})$, meaning $\dim\, V(d; P_1, \ldots, P_n) = N - n - 1 = N - 1 - \sum_1^n \frac{r_i(r_i+1)}{2}$.

Case 2: Suppose some $r_i > 1$; WLOG $r := r_1 > 1$ and $P := P_1 = [0, 0, 1]$. Set

$$V_0 := V(d; (r-1)P, r_2P_2, \ldots, r_nP_n),$$

and define

$$V_i := \left\{ F \in V_0 : F_* = \sum_{j=i}^{r-1} a_j x^j y^{r-1-j} + \text{ higher order terms} \right\}, \quad (i = 1, \ldots, r).$$

Note that $V_r = V(d; rP, \ldots, r_nP_n)$ and in particular that

$$V_0 \supset V_1 \supset \cdots \supset V_{r-1} \supset V_r = V(d; rP, \ldots, r_nP_n).$$

If we can prove that non of these inclusions hold with equality, then by the induction hypothesis

$$N - 1 - \frac{(r_1 - 1)(r_1)}{2} - \sum_2^n \frac{r_i(r_i+1)}{2} = \dim\, V_0 > \dim\, V_1 > \ldots > \dim\, V_{r-1} > \dim V_r,$$

hence

$$\dim\, V \leq N - 1 - \sum_2^n \frac{r_i(r_i+1)}{2} - \frac{(r_1-1)r_1}{2} + r_1 = N - 1 - \sum_2^n \frac{r_i(r_i+1)}{2} - \frac{(r_1-1)r_1 + 2r_1}{2}$$
$$= N - 1 - \sum_1^n \frac{r_i(r_i+1)}{2} \leq \dim\, V,$$

where the last bound is due to Lemma 4.3.37. We would then conclude that dim V = $N-1-\sum_1^n \frac{r_i(r_i+1)}{2}$.

We proceed to prove the sufficient claim: Set

$$W_0 := V(d-1;(r-2)P, r_2P_2, \ldots, r_nP_n)$$

and

$$W_i := \left\{ F \in W_0 : F_* = \sum_{j=i}^{r-2} a_j x^j y^{r-2-j} + \text{ higher order terms} \right\} \quad (i = 1, \ldots, r-1),$$

by induction

$$W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_{r-2} \supsetneq W_{r-1} = V(d-1;(r-1)P, r_2P_2, \ldots, r_nP_n).$$

Fix an $i \in \{0, \ldots, r-2\}$ and pick an $F \in W_i \setminus W_{i+1}$. Then $F_* = \sum_{j=i}^{r-2} a_j x^j y^{r-2-j} + \ldots$ with $a_i \neq 0$, hence

$$(yF)_* = yF_* = \sum_{j=i}^{r-2} a_j x^j y^{r-1-j} + \ldots,$$

implying $yF \in V_i \setminus V_{i+1}$. Pick $F \in W_{r-2} \setminus W_{r-1}$. Then

$$F_* = a_{r-2}x^{r-2} + \cdots,$$

where $a_{r-2} \neq 0$. Then

$$(xF)_* = a_{r-2}x^{r-1} + \cdots,$$

hence $xF \in V_{r-1} \setminus V_r$ establishing the desired claim. □

Example 4.3.39. 1. Let $P_1, \ldots, P_4 \in \mathbb{P}^2$ be distinct and set $V := V(2; P_1, \ldots, P_4)$. Suppose all 4 points lie on a line. WLOG $P_1 = [1,0,0]$,$P_2 = [0,1,0]$,$P_3 = [\alpha, \beta, 0]$ and $P_4 = [\gamma, \beta, 0]$ with $\alpha, \beta, \gamma, \delta \neq 0$ Let

$$F = ax^2 + by^2 + cxy + dz^2 + exz + fyz \in V(2; P_1, \ldots, P_4)$$

Then we need have that

$$\begin{cases} 0 = F(P_1) = a \\ 0 = F(P_2) = b \\ 0 = F(P_3) = a\alpha^2 + b\beta^2 + c\alpha\beta \\ 0 = F(P_4) = a\gamma^2 + b\delta^2 + c\gamma\delta \end{cases}$$

Hence $a = b = c = 0$, hence $V \simeq \text{Span}([e_4], [e_5], [e_6])$ hence dim $V = 2$. If one of $P_1, \ldots, P_4$ does not sit on a common line with the other points, we may WLOG

assume that $P_1 = [1,0,0]$, $P_2 = [0,1,0]$, $P_3 = [0,0,1]$ and $P_4 = [\alpha,\beta,0]$ with $\alpha,\beta \neq 0$. Let $F = ax^2 + by^2 + cz^2 + dxy + exz + fyz \in V(2; P_1,\ldots,P_4)$. Then

$$\begin{cases} 0 = F(P_1) = a \\ 0 = F(P_2) = b \\ 0 = F(P_3) = c 0 = F(P_4) = \alpha^2 a + \beta^2 b + \alpha\beta d \end{cases}$$

Then we see that $a = b = c = d = 0$, hence $V \simeq \text{Span}([e_4],[e_5])$, hence dim $V = 1$.

2. Consider the points $[1,0,0],[0,1,0],[0,0,1],[1,1,1],[1,2,3] \in \mathbb{P}^2$. Denote them P_i. A curve

$$F = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

is in $V(2; P_1,\ldots,P_5)$ if and only if

$$\begin{cases} 0 = F(P_1) = a \\ 0 = F(P_2) = b \\ 0 = F(P_3) = c \\ 0 = F(P_4) = a + b + c + d + e + f \\ 0 = F(P_5) = a + 4b + 9c + 2d + 3e + 6f \end{cases}$$

hence F is in $V(2; P_1,\ldots,P_5)$ if and only if (a,b,c,d,e,f) is in the null-space of

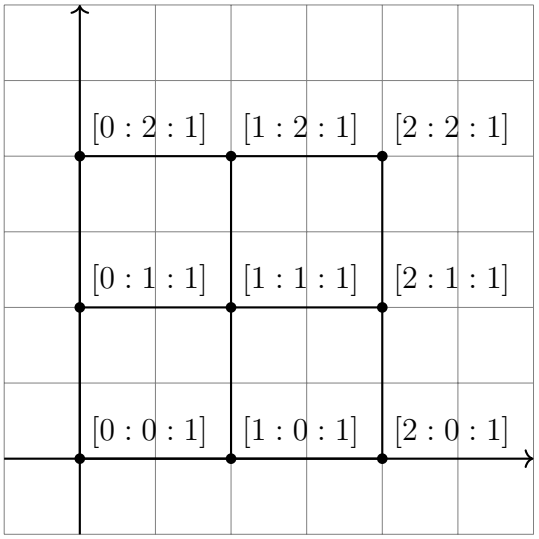
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 2 & 3 & 6 \end{pmatrix}$$

We thus see that $a = b = c = 0$ and $d = 3f$ and $e = -4f$, hence there is only one curve passing through these five points, namely $3xy - 4xz + yz$. Note that by Eisenstein in $K[x,y][z]$ using x this is irreducible. By Example 4.3.21 1. there (up to projective equivalence) is only one such curve, which is non-singular.

3. Consider the nine points

$$[0,0,1],[0,1,1],[1,0,1],[1,1,1],[0,2,1],[2,0,1],[1,2,1],[2,1,1],[2,2,1] \in \mathbb{P}^2,$$

which we denote $P_1,\ldots,P_9$. By a result seen in this subsection, we know only that dim $V(3; P_1,\ldots,P_9) \geq 0$, so it may be that there are only one. We aim to show that there are in fact infinitely many such. Below is a sketch of the positions of the points in U_3 (in $\mathbb{P}^2(\mathbb{R})$):



We see that any of the points lie on either $x, x-z$ or $x-2z$ and on either $y, y-z$ or $y-2z$. It follows that $f := x(x-z)(x-2z)$ and $g := y(y-z)(y-2z)$ are two distinct cubics passing through each of the 9 points. It follows that $L(f,g) \subset V(3; P_1, \ldots, P_9)$, hence $\#V(3; P_1, \ldots, P_9) \geq \#L(f,g) = \infty$.

4.3.3 Bézout's Theorem

Theorem 4.3.40. *(Bezout's Theorem)*
Let F and G be projective plane curves with no common components. Then

$$\sum_P I(P, F \cap G) = (\deg\ F)(\deg\ G)$$

Proof. By Proposition 4.3.19 F and G have only finitely many points of intersection. WLOG non of these lie on L_∞. Then

$$\sum_P I(P, F \cap G) = \sum_P I(P, F_* \cap G_*) = \dim\ K[x,y]/\langle F_*, G_* \rangle.$$

We define

$$\Gamma_* := K[x,y]/\langle F_*, G_* \rangle, \qquad \Gamma := K[x,y,z]/\langle F, G \rangle, \qquad R = K[x,y,z].$$

We let Γ_d denote the space of forms of degree d, and $R_d := V(d,3)$ for $d \geq 1$. Set $n := \deg\ F$ and $m := \deg\ G$. It is sufficient to prove that $\dim\ \Gamma_* = \dim\ \Gamma_d$ and that $\dim\ \Gamma_d = mn$ for some (large enough) d.

Claim 1: $\dim\ \Gamma_d = mn$ for every $d \geq m + n$. Consider the sequence

$$0 \longrightarrow R \xrightarrow{\tau} R \times R \xrightarrow{\sigma} R \xrightarrow{\pi} \Gamma \longrightarrow 0$$

where $\tau : C \mapsto (GC, -FC)$, $\sigma : (A, B) \mapsto AF + BG$ and π is the canonical surjection. It is clear that $\sigma\tau = 0 = \pi\sigma$ hence the sequence is exact. Note that τ restricts to a linear map from R_{d-m-n} to $R_{d-m} \times R_{d-n}$, that σ restricts to a linear map from $R_{d-m} \times R_{d-n}$ to R_d and that π restricts to linear map from R_d to Γ_d. We thus get by exactness (due to Lemma 1.7.12) that

$$\begin{aligned}
&\dim\ \Gamma_d = \dim\ R_d - \dim\ R_{d-m} \times R_{d-n} + \dim\ R_{d-m-n} \\
&= \frac{(d+1)(d+2) + (-d+m-1)(d-m+2) + (-d+n-1)(d-n+1)}{2} \\
&+ \frac{(d-m-n+1)(d-m-n+2)}{2} \\
&= mn.
\end{aligned}$$

Let $\overline{\bullet}$ denote $\bullet + \langle F, G\rangle$.

Claim: The map $\alpha : \Gamma \to \Gamma, \overline{H} \mapsto \overline{zH}$ is injective. Suppose $zH = AF + BG$. Note that since $z \cap F = \emptyset, z \cap G = \emptyset$, one sees that $F_0 := F(x, y, 0)$, $G_0 := G(x, y, 0)$ are non-zero forms in $K[x, y]$ that are coprime.

Indeed, suppose $J, I \in K[x_1, \ldots, x_{n+1}]$ such that $J(x_1, \ldots, x_n, 0) \neq 0 \neq I(x_1, \ldots, x_n, 0)$ have a non-trivial common factor D, then $\emptyset \neq H \subset J(x_1, \ldots, x_n, 0), H(x_1, \ldots, x_n, 0)$, meaning $x_{n+1} \subset J(x_1, \ldots, x_{n+1}), H(x_1, \ldots, x_{n+1})$.

Set $A_0 := A(x, y, 0)$ and $B_0 := B(x, y, 0)$. Then

$$A_0F_0 = -B_0G_0.$$

Hence for some $C \in K[x, y]$, $A_0 = -G_0C$ and $B_0 = F_0C$. Set $A_1 := A + CG$ and $B_1 := B - CF$. Note that

$$A_1(x, y, 0) = 0, \quad B_1(x, y, 0) = 0,$$

hence $A_1 = zA'$ and $B_1 = zB'$ for some $A', B' \in K[x, y, z]$, hence

$$zH = AF+BG+CFG-CFG = A_1F+B_1F = z(A'F+B'G) \implies H = A'F+B'G \in \langle F, G\rangle$$

proving that ker $\alpha = 0$.

Combining results: Let $d \geq m+n$ be given. Pick a basis $\{\overline{A_1}, \ldots, \overline{A_{nm}}\} \subset \Gamma_d$ for suitable $A_i \in R_d$. Consider the restriction of α to $\Gamma_d \to \Gamma_{d+1}, \overline{H} \mapsto \overline{zH}$, since ker $\alpha|_{\Gamma_d} = 0$. It follows by rank-nullity that the restriction of α is an isomorphism. Therefor $\{\overline{zA_i} : 1 \leq i \leq mn\}$ constitutes a basis of Γ_{d+1}, and by induction $\{\overline{z^r A_i} : 1 \leq i \leq mn\}$ form a basis for Γ_{d+r} for every $r \geq 0$. We claim that setting $a_i := (A_i)_* + \langle F_*, G_*\rangle$, $\{a_1, \ldots, a_{mn}\} \subset \Gamma_*$

constitutes a basis. Let $h = H + \langle F_*, G_* \rangle \in \Gamma_*$, then $\overline{z^N H^*} \in \Gamma_{d+r}$ for some sufficiently large $N \geq 0$ and $r \geq 0$, hence

$$z^N H^* = \sum_1^{mn} \lambda z^r A_i + BF + CG$$

for some $\lambda \in K$, $B, C \in K[x, y, z]$. One then sees that

$$H = (z^N H^*)_* = \sum_1^{nm} \lambda_i (A_i)_* + B_* F_* + C_* G_* \implies h = \sum_1^{nm} \lambda_i a_i.$$

Suppose, $\sum_1^{nm} \lambda_i a_i = 0$. Then $\sum_1^{nm} \lambda (A_i)_* = BF_* + CG_*$ for some $B, C \in K[x, y]$. We then for suitably large $r, s, t \geq 0$ that

$$z^s B^* G + z^t C^* = z^r (BF_* + CG_*)^* = z^r \left(\sum_1^{mn} \lambda_i A_i \right)^* = \sum_1^{mn} \lambda_i z^{r_i} A_i \sum R_{d+r_i},$$

implying that $0 = \sum_1^{mn} \overline{\lambda_i z^{r_i} A_i} \in \sum \Gamma_{d+r_i}$. Since $\Gamma_{d+l} \cap \Gamma_{d+k} = 0$ (this is seen readily), it follows that $\{z^{r_i} A_i\}$ is a basis of $\sum \Gamma_{d+r_i}$, hence in particular $\{z^{r_i} A_i\}$ are algebraically independent, meaning $\lambda_i = 0$. It follows that dim Γ_* = dim $\Gamma_d = mn$. □

Corollary 4.3.41. *Let F and G be curves with no common components. Then*

$$\sum_P m_P(F) m_P(G) \leq (\deg\ F)(\deg\ G).$$

Proof. This follows from property 5 of intersection numbers in conjunction with Bezout's theorem. □

Corollary 4.3.42. *If F and G (still have no common components) meets in $(\deg\ F)(\deg\ G)$ distinct points, then these points are simple.*

Proof. Under this extra assumption, $m_P(F) m_P(G) \geq 1$ for each point of intersection, meaning $\sum_P m_P(F) m_P(G) \geq (\deg\ F)(\deg\ G)$, hence $\sum_P m_P(F) m_P(G) = (\deg\ F)(\deg\ G)$. Then $m_P(G) m_P(F) = 1$ at each point of intersection, for otherwise we would have a strict inequality. We thus conclude that $m_P(F) = m_P(G) = 1$ at every point of intersection. □

Corollary 4.3.43. *If F and G have exactly $(\deg\ F)(\deg\ G)$ common points, then they have no common components.*

Proof. This just follows from Bezout and the fact that $\sum_P I(P, F \cap G) < \infty$ if and only if F and G intersect properly at every point. □

Proposition 4.3.44. *Every non-singular projective curve F is irreducible.*

Proof. Suppose F is reducible with two components G, H. G and H have at least one point in common, since if they have a component in common, then they intersect at infinitely many points and if they have no components in common, then by Bezout

$$\sum_P I(P, G \cap H) = (\deg\ G)(\deg\ H) > 1,$$

meaning they have at least one point in common. Let $P \in G \cap H \subset F$. Then

$$m_P(F) = m_P(GH) = m_P(G) + m_P(H) \geq 2,$$

hence F is singular. □

Remark 4.3.45. It is not the case that the above is true in the affine case. Consider for example $f = y(y+1)$. The zeroes of this curve are $(\alpha, 0)$ and $(\alpha, -1)$ where $\alpha \in K$. Since $f(x + \alpha, y + 0) = f(x, y) = y^2 + y$ and $f(x + \alpha, y - 1) = y(y - 1) = y^2 - y$, hence f is non-singular.

4.3.4 Bounds on the Number of Multiple Points of a Curve

Proposition 4.3.46. *Let F be an irreducible projective plane curve of degree d with $F_x \neq 0$. Then*

$$\sum_P m_P(F)(m_P(F) - 1) \leq d(d - 1),$$

hence F has at most $\frac{d(d-1)}{2}$ multiple points.

Proof. Since $F_x \neq 0$, we have that deg $F_x = d - 1$, hence the first bound follows from Corollary 4.3.41 and Lemma 4.3.18,

$$\sum_P m_P(F)(m_P(F) - 1) \leq \sum_P m_P(F) m_P(F_x) \leq d(d - 1)$$

. Note that the number of multiple points is given by $\sum_P (m_P(F) - 1)$. It then follows that

$$2\sum_P (m_P(F) - 1) \leq \sum_{Q \text{ multiple}} \sum_P m_Q(F)(m_P(F) - 1) \leq \sum_P m_P(F)(m_P(F) - 1) \leq d(d - 1).$$

It thus follows that

$$\sum_P (m_P(F) - 1) \leq \frac{d(d - 1)}{2}.$$

Another way to derive the bound:

$$\sum_P (m_P(F) - 1) \leq \sum_P \frac{m_P(F)(m_P(F) - 1)}{2} \leq \frac{d(d - 1)}{2}.$$

□

In Example 4.3.21 we saw that the optimal bound on the number of multiple points for an irreducible conic is 0 and for an irreducible cubic 1, indicating that there is a better bound in the general case.

Theorem 4.3.47. *Let F be an irreducible curve of degree $d \geq 1$. Then*

$$\sum_P (m_P(F) - 1) \leq \frac{(d-1)(d-2)}{2}.$$

Proof. Set

$$r := \frac{(d-1)(d-1+3)}{2} - \sum_P \frac{m_P(F)(m_P(F)-1)}{2} \geq \frac{d(d-1)}{2} - \sum_P \frac{m_P(F)(m_P(F)-1)}{2} \geq 0.$$

Pick $Q_1, \ldots, Q_r \in F$ simple. Let $P_1, \ldots, P_l \in F$ be the multiple points. Then

$$V := V(d-1; Q_1, \ldots, Q_r, (m_{P_1}(F)-1)P_1, \ldots, (m_{P_l}(F)-1)P_l)$$

is a linear subvariety of dimension greater then $\frac{(d-1)(d-1+3}{2} - r - \sum_P \frac{m_P(F)(m_P(F)-1)}{2} = 0$. It follows that can pick a curve $G \in V$. Note that if $P \neq Q_i$ is simple, then $m_P(G) \geq 0 = m_P(F) - 1$. so $m_P(G) \geq m_P(F) - 1$ for each $P \in F$ Since deg $G = d - 1 < d =$ deg F and F is irreducible, gcd$(G, F) = 1$, implying

$$\begin{aligned} r + \sum_{P \neq Q_i} m_P(F)(m_P(F)-1) &\leq \sum_1^r m_{Q_i}(G) m_{Q_i}(F) + \sum_{P \neq Q_i} m_P(G) m_P(F) \\ &= \sum_P m_P(G) m_P(F) \leq d(d-1). \end{aligned}$$

Here we use Corollary 4.3.41 for the last upper bound. Inserting the value of r into the left-hand side, we see that

$$\frac{(d-1)(d+2)}{2} + \sum_P \frac{m_P(F)(m_P(F)-1)}{2} \leq \frac{2d(d-1)}{2} \implies \sum_P \frac{m_P(F)(m_P(F)-1)}{2} \leq \frac{(d-1)(2d-d-2)}{2} = \frac{(d-1)(d-2)}{2}.$$

We therefor immediately get that $\sum_P (m_P(F) - 1) \leq \frac{(d-1)(d-2)}{2}$ □

Remark 4.3.48. This bound is sharp in the cases $d = 1, 2, 3$.

Proposition 4.3.49. *Let F be a projective plane curve of degree $d \geq 1$ with c components, each of which is not multiple. Then*

$$\sum_P \frac{m_P(F)(m_P(F)-1)}{2} \leq \frac{(d-1)(d-2)}{2} + c - 1 \leq \frac{d(d-1)}{2}.$$

Proof. write $F = F_1F_2$ where F_1 is a component and F_2 is a product of $c-1$ remaining simple components with $c > 1$. Set $d_1 := \deg\ F_1$ and $d_2 := \deg\ F_2$. By induction and Bezout, it follows that

$$\begin{aligned}
\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)+m_P(F_2)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_1)+m_P(F_2)-1)}{2} \\
&= \sum_1^2 \sum_P \frac{m_P(F_i)(m_P(F_i)-1)}{2} + \sum_P m_P(F_1)m_P(F_2) \\
&\leq \frac{(d_1-1)(d_1-2)+(d_2-1)(d_2-2)+2d_1d_2}{2} + c - 1 - 1 \\
&= \frac{d_1^2+2-d_1-2d_1+d_2^2+2-d_2-2d_2+2d_1d_2}{2} + c - 1 - 1 \\
&= \frac{(d_1+d_2)^2-(d_1+d_2)-2(d_1+d_2)+2}{2} + c - 1 \\
&= \frac{(d-1)(d-2)}{2} + c - 1.
\end{aligned}$$

Note that in the last step we use $d_1 + d_2 = d$. □

Proposition 4.3.50. *Assume* char $K = 0$. *Let F be an irreducible curve of degree $d \geq 1$. Let $P \in \mathbb{P}^2$ and set $r = m_P(F)$. For all but finitely many lines L through P, L intersects F in $d - r$ distinct points.*

Proof. WLOG $P = [0, 0, 1]$. The lines through P are

$$L_\lambda := V(x - \lambda y) = \{[\lambda, 1, t] : t \in K\} \cup \{P\} \quad (\lambda \in K)$$

(cf. Example 4.1.68 1.) together with $L = V(y)$. It is therefor sufficient to prove the statement for the L_λ. Write $F = \sum_r^d H_i z^{d-i}$ with $H_i \in K[x, y]$ a form degree i and $H_r \neq 0$. Consider for each $\lambda \in K$ the polynomial

$$G_\lambda := F(\lambda, 1, T) \in K[T]$$

whose roots are in one-to-one correspondence with $L_\lambda \cap F$. It is therefor sufficient to prove that G_λ has $n - r$ distinct roots for all but finitely many $\lambda \in K$. Suppose $\lambda \in K$ is given such that $H_r(\lambda, 1) \neq 0$ (making G_λ a $d-r$-degree polynomial) and $F \cap F_z \cap L_\lambda = \{P\}$, or equivalently that the common roots of G_λ and $(G_\lambda)_T = F_z(\lambda, 1, T)$ is the empty set. Then by the contrapositive of result the $d-r$ roots of G_λ are all distinct. Clearly $H_r(\lambda, 1) \neq 0$ for all but finitely many λ. Since $H_i(0, 0) = 0$ for all $i \geq 1$, it follows that $P \in F_z$. Note that since F is irreducible so is $F(x, 1, z)$ by Corollary 1.8.121, hence $F(x, 1, z)$ and $F_z(x, 1, z)$ are co-prime. It then follows that $F(x, 1, z) \cap F_z(x, 1, z)$ is finite. In particular there are only finitely $\lambda \in K$ such that $\emptyset \neq F(\lambda, 1, T) \cap F_z(\lambda, 1, T) = G_\lambda \cap (G_\lambda)_T$. It follows that G_λ has $d - r$ distinct roots for all but finitely many λ. □

Proposition 4.3.51. *The above proposition extends to curves F with no multiple components.*

Proof. Write $F = \prod_1^n F_i$. Set $r_i := m_P(F_i)$ and $d_i :=$ deg F_i. There are infinitely many lines passing through P that do not pass through $\bigcap_1^n F_i$. Denote this set $\mathcal{L}$. Then for each i, all but finitely many lines in $\mathcal{L}$, intersect F_i in $d_i - r_i$ points. Then L intersect each F_i in $d_i - r_i$ points for all but finitely many i. Let such an L be given. Then

$$L \cap F = L \cap \bigcup_1^n F_i = \bigsqcup_1^n L \cap F_i,$$

hence

$$\#(L \cap F) = \# \left(\bigsqcup_1^n L \cap F_i \right) = \sum_1^n d_i - r_i = d - \sum_1^n m_P(F_i) = d - m_P(F) = d - r.$$

□

Example 4.3.52. Suppose char $K = p > 0$. Set $F := x^{p+1} - y^p z$ and $P := [0, 1, 0]$. $L_\infty \cap F = \{P\}$. Let $\lambda \in K$. Then $L_\lambda \cap F = \{t : \lambda^{p+1} - t^p = 0\} \cup \{P\} = \left\{ \left[\lambda, 1, \lambda^{\frac{1}{p}} \lambda \right] \right\} \cup \{P\}$ More explicit description?. Note that

$$\begin{cases} F_x = x^p \\ F_y = 0 \\ F_z = y^p \end{cases}$$

4.3.5 Max Noether's Fundamental Theorem

Definition 4.3.53. A *zero-cycle* on $\mathbb{P}^2$ is an element of the free abelian group generated by $\mathbb{P}^2$, i.e. $\mathbb{Z}[\mathbb{P}^2]$.

Definition 4.3.54. We define the *degree* of a zero-cycle $s = \sum_P n_P P \in \mathbb{Z}[\mathbb{P}^2]$ is the quantity

$$\text{deg } s := \sum_P n_P.$$

For a $t = \sum_P m_P \in \mathbb{Z}[\mathbb{P}^2]$, we write $s \geq t$ if $n_P \geq m_P$ for each $P \in \mathbb{P}^2$.

Definition 4.3.55. Let F and G be curves with no common components. We define the *intersection cycle of F and G* to be

$$F \bullet G := \sum_P I(P, F \cap G) P \in \mathbb{Z}[\mathbb{P}^2]$$

Remark 4.3.56. By Bezout $\deg\ F \bullet G = (\deg\ F)(\deg\ G)$

The following properties of intersection cycles are trivial consequences of properties of intersection numbers:

Lemma 4.3.57. *Let F, G, H be curves and A a form of degree* $\deg\ G - \deg\ F$. *Then*

1. $F \bullet G = G \bullet F$.
2. $F \bullet GH = F \bullet G + F \bullet H$.
3. $F \bullet G + AF = F \bullet G$.

Definition 4.3.58. Consider projective plane curves F, G, H where F and G have not common components. Let $P \in \mathbb{P}^2$. We say that *Noether's condition (with respect to F, G and H) is satisfied at P* if $H_* \in \langle F_*, G_* \rangle \subset \mathcal{O}_P(\mathbb{P}^2)$.

Theorem 4.3.59. *(Max Noether's Fundamental Theorem/MNFT)*
Let F, G, H be projective plane curves where F and G have no common components. Then there are curves A and B with $\deg\ A = \deg\ H - \deg\ F$ *and* $\deg\ B = \deg\ H - \deg\ G$ *such that*

$$H = AF + BG$$

if and only if Noether's conditions are satisfied at every $P \in F \cap G$.

Proof. " $\Longrightarrow$ ": Is trivial
" $\Longleftarrow$ ": WLOG $F \cap G \cap z = \emptyset$. Since

$$K[x,y]/(K[x,y]F_* + K[x,y]G_*) \simeq \prod_{P=[v,1]\in F\cap G} \mathcal{O}_v/(\mathcal{O}F_* + \mathcal{O}_v G_*)$$

and $H_* \in \mathcal{O}F_* + \mathcal{O}_v G_*$ for each $P = [v,1] \in F \cap G$, we get that $H_* \in K[x,y]F_* + K[x,y]G_*$, hence for suitable $a, b \in K[x,y]$, $H_* = aF_* + bG_*$. For a suitably large $N \geq 0$,

$$z^r H = z^N (H_*)^* = z^N (aF_* + bG_*)^* = a^* z^s (F_*)^* + b^* z^t (G_*)^* = a^* z^{s'} F + b^* z^{t'} F,$$

hence $z^r H = AF + BG$ for suitable forms $A, B \in K[x,y,z]$. By the proof of Bezout, $K[x,y,z]/\langle F, G\rangle \to K[x,y,z]/\langle F, G\rangle, \Lambda \mapsto z^r \Lambda$ is injective, hence

$$H = A'F + B'G$$

for some $A', B' \in K[x,y,z]$, hence writing $A' = A'_{\deg\ H-\deg\ F} + \ldots$ and $B' = B'_{\deg\ H-\deg\ G} + \ldots$, as a result of cancellations

$$H = A'_{\deg\ H-\deg\ F} F + B'_{\deg\ H-\deg\ G} G.$$

□

Proposition 4.3.60. *Let F, G, H be projective plane curves and $P \in F \cap G$. Noether's condition at P are satisfied if:*

1. *F and G intersect transversally at P and $P \in H$.*
2. *P is simple on F and $I(P, H \cap F) \geq I(P, G \cap F)$.*
3. *F and G have distinct tangents at P and $m_P(H) \geq m_P(F) + m_P(G) - 1$.*

Proof. 1. Note that 1. implies 2. since then $I(P, H \cap F) \geq 1 = I(P, F \cap G)$, so it is sufficient to prove 2.
2. We get that $\text{ord}_P^F(H) = I(P, H \cap F) \geq I(P, G \cap G) = \text{ord}_P^F(F)$, hence $\mathcal{O}_P(F) \ni \overline{H_*} = uL^k$ and $\mathcal{O}_P(F) \ni \overline{G_*} = uL^h$ with $h \leq k$, hence $\overline{H_*} \in \langle \overline{G_*} \rangle \in \mathcal{O}_P(F)$. Then using $\mathcal{O}_P(F)/\langle \overline{G_*} \rangle \simeq \mathcal{O}_P(\mathbb{P}^2)/\langle F_*, G_* \rangle$, we find that $H_* + \langle F_*, G_* \rangle = 0$.
3. WLOG $P = [0, 0, 1]$. Note that then $m_P(H_*) \geq m_P(F_*) + m_P(G_*) - 1$. This implies that $H_* \in \langle x, y \rangle^{m_P(F_*)+m_P(G_*)-1}$, hence by Lemma 3.4.48, $0 = H_* + \langle F_*, G_* \rangle \in \mathcal{O}_{(0,0)}(\mathbb{A}^2)/\langle F_*, G_* \rangle \simeq \mathcal{O}_P(\mathbb{P}^2)/\langle F_*, G_* \rangle$ □

Corollary 4.3.61. *Let F and G be projective plane curves with no common components. Then there is a curve B where $B \cdot F = H \cdot F - G \cdot F$ if one of following two conditions are satisfied:*

1. *F and G intersect in $(\deg F)(\deg G)$ points and H passes through each of these points.*
2. *All points of $F \cap G$ are simple points of F and $H \cdot F \geq G \cdot F$.*

Proof. If 1. is satisfied, then F and G intersect transversally at every point of intersection, hence by 1. of the prior proposition Noether's condition is satisfied at every point of intersection.
If 2. is satisfied then $I(P, H \cap F) \geq I(P, G \cap F)$, hence 2. of the prior propositions shows that Noether's condition is satisfied at every $P \in F \cap G$.
In either case, this means $H = AF + BG$ for suitable forms A, B by Max Noether's Fundamental Theorem. □

Proposition 4.3.62. *Let F, G, H be plane curves where $\gcd(F, G) = 1$, $P \in F \cap G$.*

1. *When P is simple, then Noether's condition at P is satisfied at P if and only if $I(P, F \cap H) \geq I(P, F \cap G)$.*
2. *When F and G meet transversally at P, then Noether's condition at P is satisfied at P if and only if $P \in H$.*

Proof. WLOG $P = [0,0,1]$. 1" $\Longrightarrow$ ": Write $H_* = \frac{\alpha}{\beta}F_* + \frac{\lambda}{\mu}G_*$ for some $\frac{\alpha}{\beta}, \frac{\lambda}{\mu} \in \mathcal{O}_{0,0}(\mathbb{A}^2)$. Then

$$\zeta H_* = \alpha F_* + \lambda G_*, \quad (\zeta := \beta\mu).$$

Then

$$\begin{aligned} I(P, H \cap G) &= I((0,0), H_* \cap G_*) = I((0,0), \zeta H_* \cap G_*) = I((0,0), \alpha F_* \cap G_*) \\ &= I((0,0), \alpha \cap G_*) + I((0,0), F_* \cap G_*) \geq I((0,0), F_* \cap G_*) = I(P, F \cap G) \end{aligned}$$

" $\Longleftarrow$ ": Follows from Proposition 4.3.60.
2.Follows from 1. □

Remark 4.3.63. The above shows that in Proposition 4.3.60 in case 1. resp. case 2. if we presuppose the condition on F and G, then the condition on H is equivalent to Noether's condition being satisfied for F, G and P. The next example shows that the same augmentation can not be made in case 3.

Example 4.3.64. Consider $F = x^2 + x + y$, $G = x^2$ and $H = x + y$ and $P = [0,0,1]$. Note that F and G have distinct tangents, namely $x + y$ resp. x. One sees that $H_* = H = F - G = F_* - G_*$, so Noether's condition is satisfied at P wrt. F and G. Note that $1 = m_P(H) < 2 = m_P(F) + m_P(G) - 1$. So the condition on H in case 3. in Proposition 4.3.60 is not equivalent to Noether's condition on P for curves F, G, H with $\gcd(F,G) = 1$, $P \in F \cap G$ and F and G having distinct tangents.

Proposition 4.3.65. *Let F be an irreducible projective plane curve. Suppose $z \in K(F)$ is given such that $z \in \mathcal{O}_P(F)$ for every $P \in F$. Then $z \in K$.*

Proof. write $z = \frac{H+\langle F\rangle}{G+\langle F\rangle}$, for some equidegree forms $H, G \in K[\mathbf{x}]$, where $G(P) \neq 0$ for every $P \in F$. Then $F \cap G = \emptyset$ hence Noether's condition is vacuously satisfied for every $P \in F \cap G$. Then by MNFT there are forms $A, B \in K[\mathbf{x}]$ such that deg AF = deg H, deg BG = deg H = deg G. Then $B \in K$ and

$$z = \frac{H + I(F)}{G + I(F)} = \frac{AF + BG + I(F)}{G + I(F)} = \frac{BG + I(F)}{G + I(F)} = B + I(F) \in K.$$

□

4.3.6 Applications of Noether's Theorem

Proposition 4.3.66. *Let C, C' be cubics such that $C \bullet C' = \sum_1^9 P_i$. Let Q be a conic such that $Q \bullet C = \sum_1^6 P_i$. Assume $P_1, \ldots, P_6$ are simple on C. Then P_7, P_8 and P_9 lie on a line.*

Proof. Since $P_1, \ldots, P_6$ are distinct, $I(P_i, C' \cap C) \geq 1 = I(P_i, Q \cap C)$ for $i = 1, \ldots, 6$. For $i = 7, 8, 9$ (7 8 9?! I guess that's why 6 is afraid of 7), $I(P_i, C' \cap C) \geq 1 \geq I(P_i, Q \cap C)$, since $I(P_i, Q \cap C) = 1$ if $P_i \in Q \cap C$ and 0 otherwise. It follows that $C' \bullet C \geq Q \bullet C$. By Corollary 4.3.61 there is an L such that $L \bullet C = C' \bullet C - Q \bullet C = P_7 + P_8 + P_9$. Since deg $C = 3$ and $(\deg L)(\deg C) = \deg L \bullet C = 3$, we get that deg $L = 1$, hence L is a line passing through P_7, P_8, P_9. □

Definition 4.3.67. Given $2n$ points in the projective plane $P_1, \ldots, P_{2n}$, we form a $2n$-gon by connecting these points via $2n$ lines $L_i := L(P_i, P_{i+1})$ for $i \in \{1, \ldots, 2n-1\}$ and $L_{2n} := L(P_{2n}, P_1)$. For the first n lines, we define the *opposite side of* L_i, denoted $\text{op}(L_i)$ to be L_{i+n}. For the remaining lines, the opposite to L_i is L_{i-n}.

Remark 4.3.68. Thus we get bijection of any collection of n line segments, LS satisfying that the opposite segment of any $L \in LS$ is not LS, to the opposite line segments to those in LS, via op.

Corollary 4.3.69. *(Pascal's Theorem) Let Q be an irreducible conic (an ellipse, parabola or hyperbola). Pick 6 distinct points on Q, in some ordering $P_1, \ldots, P_6$ and form the hexagon containing these points. Then the points $P_i := L_i \cap \text{op}(L_i)$, $i := 1, 2, 3$ lie on a line. More generally given a set $LS := \{L_{i_1}, L_{i_2}, L_{i_3}\}$, satisfying the conditions of the prior remark, the points $Q_j := L_{i_j} \cap \text{op}(L_{i_j})$ lie on a line.*

Proof. Indeed, consider the cubics $C := L_{i_1} L_{i_2} L_{i_3}$ and $C' := \text{op}(L_{i_1})\text{op}(L_{i_2})\text{op}(L_{i_3})$ and apply the prior proposition. □

Corollary 4.3.70. *(Pappus' Theorem) Let L_1, L_2 be two lines, $P_1, P_2, P_3 \in L_1$ and $Q_1, Q_2, Q_3 \in L_2$ such that $P_i, Q_j \notin L_1 \cap L_2$. Set $L_{ij} := L(P_i, Q_j)$. For each i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, set $R_k := L_{ij} \bullet L_{ji}$. Then R_1, R_2, R_3 lie on a line.*

Proof. Indeed, Consider $Q := L_1 L_2$, $C := L_{12} L_{13} L_{23}$ and $C' := L_{21} L_{31} L_{32}$. By construction non of the L_{ij}'s are components of Q. We see that $C \bullet C' = \sum_1^3 P_i + Q_i + R_i$ and $Q \bullet C = \sum_1^3 P_i + Q_i$ and also note that we chose P_i, Q_i such that they are simple. We are thus in position where we can apply the proposition to see that R_1, R_2, R_3 are on a line. □

Proposition 4.3.71. *Let C, C', C'' be cubics with C irreducible, and $Q \in C$. Suppose $C' \bullet C = \sum_1^9 P_i$, where the P_i are simple but not necessarily distinct points on C and that $C'' \bullet C = Q + \sum_1^8 P_i$, then $Q = P_9$.*

Proof. Take any line L passing through P_9. Note that $L \bullet C = P_9 + R + S$ for some $R, S \in C$. Then $LC'' \bullet C = C' \bullet C + Q + R + S$. By Noether's Theorem there is a line L' passing through Q, R, S. But then $L = L'$, hence $P_9 + R + S = L \bullet C = L' \bullet C = Q + R + S$, meaning $P_9 = Q$. □

Definition 4.3.72. Let C be a non-singular cubic. Take any two points P, Q on C. Take the unique line L such that $L \bullet C = P + Q + R_{P,Q}$ for some unique $R_{P,Q} \in C$. When $P \neq Q$, $L = L(P, Q)$ and when $P = Q$, L is the tangent at P. Define a binary operation on C

$$\boxplus : C \times C \to C$$
$$(P, Q) \mapsto R_{P,Q}$$

Remark 4.3.73. With this operation C becomes a commutative magma I.e. $\boxplus$ is a commutative operation.

Definition 4.3.74. Let C be a non-singular cubic. Choose any point O on C. Define a binary operation

$$\oplus_O := \oplus : C \times C \to C$$
$$(P, Q) \mapsto O^{\boxplus}(P^{\boxplus}Q)$$

Proposition 4.3.75. *Let C be a non-singular cubic. With $\oplus$, C becomes an additive group with O being the identity.*

Proof. We first show that the operation is associative. Let $P, Q, R \in C$. We pick unique lines $L_1, L_2, L_3, M_1, M_2, M_3$ such that

$$L_1 \bullet C = P + Q + P^{\boxplus}Q$$
$$L_2 \bullet C = P \oplus Q + R + (P \oplus Q)^{\boxplus}R$$
$$L_3 \bullet C = O + Q^{\boxplus}R + \underbrace{O^{\boxplus}(Q^{\boxplus}S)}_{Q \oplus R}$$
$$M_1 \bullet C = O + P^{\boxplus}Q + \underbrace{O^{\boxplus}(P^{\boxplus}Q)}_{P \oplus Q}$$
$$M_2 \bullet C = Q + R + Q^{\boxplus}R$$
$$M_3 \bullet C = P + Q \oplus R + P^{\boxplus}(Q \oplus R)$$

Set $C' := L_1L_2L_3$ and $C'' := M_1M_2M_3$. Then

$$C' \bullet C = O + P + Q + R + P^{\boxplus}Q + Q^{\boxplus}R + P \oplus Q + Q \oplus S + (P \oplus Q)^{\boxplus}R,$$
$$C'' \bullet C = O + P + Q + R + P^{\boxplus}Q + Q^{\boxplus}R + P \oplus Q + Q \oplus S + P^{\boxplus}(Q \oplus R).$$

Applying the prior proposition, it follows that $(P \oplus Q)^{\boxplus} R = P^{\boxplus}(Q \oplus R)$. Then $O + (P \oplus Q)^{\boxplus} R (P \oplus Q) \oplus R = O + P^{\boxplus}(Q \oplus R) + P \oplus (Q \oplus R)$, hence $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$. Moreover, the line through O and $O \oplus P$ is the line through O and P, hence $O \oplus P = P$. Define $-P := P^{\boxplus}(O^{\boxplus} O)$. Then the third intersection of C with the line through P and $-P$ is $O^{\boxplus} O$. Moreover the third point of intersection of C with the line through O and $O^{\boxplus} O$ is O, hence $P \oplus -P = O^{\boxplus}(P^{\boxplus} - P) = O^{\boxplus}(O^{\boxplus} O) = O$. Commutativity, follows from commutativity of $\boxplus$. □

Definition 4.3.76. Let C be a cubic with no multiple components. Define

$$C^o := \{P \in C : P \text{ simple}\}.$$

Remark 4.3.77. When C is irreducible, we may define $\boxplus$ as in the non-singular case, since $L(P, Q)$ is not a component of C in this case there is a unique point $R_{P,Q}$ such that $L(P, Q) \cdot C = P + Q + R_{P,Q}$. We see that $I(R_{P,Q}, L(P, Q) \cap C) = 1$, hence $R_{P,Q}$ is simple, meaning $\boxplus$ is indeed well-defined. Pick a point $O \in C^o$. Upon for $P, Q \in C^o$ defining, $P \oplus Q := O^{\boxplus}(P^{\boxplus} Q)$, the same computations as in the nonsingular case shows that $(C^o, \oplus)$ is an additive group.
If C has a non-trivial component Then I don't know.

Proposition 4.3.78. *Consider an irreducible cubic C. Let $O, O' \in C^\circ$ be given. Set $Q := O^{\boxplus} O'$. Then*

$$\alpha : (C^\circ, \oplus_O) \to (C^\circ, \oplus_{O'})$$
$$P \mapsto Q^{\boxplus} P$$

defines a group isomorphism.

Proof. □

Proposition 4.3.79. *Let P_1, P_2, P_3 be distinct points on an irreducible conic Q. Let L_1, L_3, L_5 be the tangents at each of the respective points. Let $L_2 := L(P_1, P_2)$, $L_4 := L(P_2, P_3)$ and $L_6 := L(P_3, P_1)$. Then $Q_i := L_i \cap \text{op}(L_i) = L_i \cap L_{i+3}$ for $i = 1, 2, 3$ are collinear*

Proof. Set $C = L_1 L_3 L_5$, $C' := L_2 L_4 L_6$. For each i note that there are exactly two lines in C' intersecting P_i. Since neither of these points are tangents to Q, they intersect the tangent to P_i transversally. It follows that $I(P_i, C \cap C') = 2$, hence

$$C \cdot C' = \sum_1^3 2P_i + Q_i.$$

Additionally

$$Q \bullet C' = \sum_1^3 2P_i,$$

it follows from a Corollary 4.3.61 2. that there is a line L for which $L \bullet C' = \sum_1^3 Q_i$, hence Q_1, Q_2, Q_3 are collinear. □

Proposition 4.3.80. *Let $P_1, \ldots, P_5$ be distinct points on an irreducible conic Q. Let L_1 be the tangent at P_1 and connect the points with 5 lines L_i as before. Defining Q_1, Q_2, Q_3 as before, we get that these are collinear.*

Proof. Define C and C' as before. Then

$$C \bullet C' = 2P_1 + \sum_2^5 P_i + Q_1 + Q_2 + Q_3$$

and

$$C \bullet Q = 2P_1 + \sum_2^5 P_i.$$

We get a line through Q_1, Q_2, Q_3 in the usual way. □

Remark 4.3.81. The above result gives a way to construct a tangent to a point on an irreducible conic using only a straightedge: Call the point P. Draw 4 other distinct points also distinct from P. Call these points $P_1, \ldots, P_4$. Find the intersection of the line $L(P_4, P)$ and $L(P_1, P_2)$. Call that point Q_1. Do the same with $L(P_1, P)$ and $L(P_3, P_4)$. Call that point Q_2. Find the intersection of $L(P_2, P_3)$ and $L(Q_1, Q_2)$. Call that point Q_3. Then $L(P, Q_3)$ is the tangent of the conic at P.

Proposition 4.3.82. *Consider the hexagon formed by points $P_1, \ldots, P_6$. Suppose the intersections of the sides with their opposite side lie on a line. Then $P_1, \ldots, P_6$ lie on a conic.*

Proof. □

Proposition 4.3.83. *Let C be an irreducible cubic and L a line such that $L \bullet C = P_1 + P_2 + P_3$ for P_1, P_2, P_3 distinct. Let L_i be the tangent to C at P_i. Then $L_i \bullet C = 2P_i + Q_i$ for some Q_i. Then Q_1, Q_2, Q_3 are collinear.*

Proof. □

Proposition 4.3.84. *On a cubic, a line through two flexes on the cubic, passes through a third flex on the cubic.*

Proof. □

5 Algebraic Geometry with Abstract Varieties

We have thus far been studying algebraic geometry by studying systems of polynomial equations in some space in which this makes sense. We want to study properties of such objects independently of the space in which they are embedded. This will be accomplished by relying the fact that on the spaces of study there is a topology - namely the Zariski topology. We thus aim to study algebraic geometry through the lens of this topology. We thus aim to define varieties in terms of this topology and have these definitions be consistent with our prior studies and also adding some more substance to the theory. One upshot of what we will develop is that we get a notion of a ring of regular functions/coordinate ring on any variety that on the nose will correspond to some ring of K-valued functions on that variety.

5.1 Some Topology

Lemma 5.1.1. *The Zariski topology on a variety V is never Hausdorff (It is however Frechét (T_1) since points are closed) and every non-empty open subset of V is dense in V.*

Proof. Let $U_1 \cap V$ and $U_2 \cap V$ be open such that $U_1 \cap U_2 \cap V = \emptyset$. Then $V \setminus U_1 \cup V \setminus U_2 = V$, hence WLOG $V \setminus U_1 = V$, implying $U_1 = \emptyset$. Hence for any two open subsets of V, their intersection will never be empty. So consider distinct point P and Q. Any open neighborhood of P will then intersect any open neighborhood of Q, meaning V is not Hausdorff.
Any open neighborhood of a point P would intersect U, hence any point in V is a point of closure in U, hence cl$(U) = V$. □

We denote the subspace topology of a subset Y of topological space (X, τ) by τ_Y^τ.

Lemma 5.1.2. *Let (X, τ) be a topological space and $Z \subset Y \subset X$. Then $\tau_Z^{\tau_Y^\tau} = \tau_Z^\tau$.*

Proof. "$\subset$": Let $U \cap Z \in \tau_Z^{\tau_Y^\tau}$. Then $U = U' \cap Y$ for some $U' \in \tau$, hence

$$U \cap Z = U' \cap Z.$$

"$\supset$": Conversely, given $U \cap Z \in \tau_Z^\tau$, set $U' := U \cap Y \in \tau_Y^\tau$. Then

$$U \cap Z = U' \cap Z \in\in \tau_Z^{\tau_Y^\tau}$$

□

Lemma 5.1.3. *Let (X, τ) be topological space with an open covering $\{U_\alpha\}_{\alpha \in A}$.*

1. *$V \subset X$ closed if and only if $V \cap U_\alpha$ is closed in U_α with respect to $\tau^\tau_{U_\alpha}$ for every $\alpha \in A$.*

2. *Consider an additional topological space (Y, τ') with an open covering $\{V_\alpha\}$ and a function $f : X \to Y$ with $f(U_\alpha) \subset V_\alpha$ for each α. f is then continuous if and only if $f_\alpha := f|_{U_\alpha} : U_\alpha \to V_\alpha$ is continuous for each $\alpha \in A$.*

Proof. 1. Only if follows from

$$X \setminus V = \bigcup_\alpha (U_\alpha \setminus V) = \bigcup_\alpha (U_\alpha \setminus (V \cap U_\alpha))$$

and if $X \setminus V$ is open then so is $U_\alpha \setminus (V \cap U_\alpha) = U_\alpha \setminus V = X \setminus V \cap U_\alpha$ in the subspace topology for each α.
2. For only if, let $V \subset Y$ be open. Then $V = \bigcup_\alpha (V \cap V_\alpha)$ with $V'_\alpha := V \cap V_\alpha$ being open in τ' for each α. Then $f_\alpha^{-1}(V'_\alpha)$ is open in $\tau^\tau_{U_\alpha}$, for each α by continuity. Upon suitably writing $f_\alpha^{-1}(V'_\alpha) = O_\alpha \cap U_\alpha$, we find that $f_\alpha^{-1}(V'_\alpha)$ is open in τ. We thus get that

$$f^{-1}(V) = f^{-1}\left(\bigcup_\alpha V'_\alpha\right) = \bigcup f_\alpha^{-1}(V'_\alpha) \in \tau.$$

For the if part, let $V_\alpha \cap V \in \tau^{\tau'}_{V_\alpha}$, then $V_\alpha \cap V \in \tau'$, hence $f_\alpha^{-1}(V_\alpha \cap V) = f^{-1}(V_\alpha \cap V) \cap U_\alpha \in \tau^\tau_{U_\alpha}$. □

Example 5.1.4. The map $\varphi_i : \mathbb{A}^n \to U_i \subset \mathbb{P}^n$ is a homeomorphism when U_i is induced with Zariski subspace topology. Indeed, this is a consequence of Lemma 4.1.62. 6. and 7. Note that each U_i is open since $U_i = \mathbb{P}^n \setminus V(x_i)$. It thus follows from 1. of the prior lemma that $W \subset \mathbb{P}^n$ is closed if and only if $W \cap U_i$ is closed in the Zariski subspace topology for each i if and only if $\varphi_i^{-1}(W) = \varphi_i^{-1}(W \cap U_i)$ is closed in $\mathbb{A}^n$ for each i.

Proposition 5.1.5. *Let S be an infinite subset of an (irreducible!) plane curve $V(f) \subset \mathbb{A}^2$. Then S is dense in $V(f)$.* ***I don't see that the statement could be true as stated in Fulton. Take for instance f to be the product of two distinct lines l_1 and l_2. Then $V(f) \setminus V(l_1) = (\mathbb{A}^2 \setminus V(l_1)) \cap V(f)$, implying that $V(l_1)$ is closed in $V(f)$, but then*** $\mathrm{cl}(V(l_1)) = V(l_1) \subsetneq V(f)$

Proof. Note that $\mathrm{cl}(S)$ is infinite and therefor must contain a plane curve g. This g is thus component of f and since f is irreducible, $f = g$. □

Proposition 5.1.6. *Any bijection from an irreducible curve f to an irreducible curve g is a homeomorphism.*

Proof. Call such a map ϕ. To prove that ϕ is continuous, we prove that $\phi(\text{cl}(S)) \subset \text{cl}(\phi(S))$ for any $S \subset f$. Note that if S is finite, then S is closed in f, hence $\phi(\text{cl}(S)) = \phi(S)$ which is a finite subset of g, and therefor also closed in g, implying $\phi(\text{cl}(S)) = \phi(S) = \text{cl}(\phi(S))$. If S is infinite, then by the prior proposition $\phi(\text{cl}(S)) = \phi(f) = g$. We also have that $\phi(S)$ is infinite since ϕ is injective, hence $\text{cl}(\phi(S)) = g$. Applying the same argument to ϕ^{-1}, we find that ϕ is bicontinuous. □

Lemma 5.1.7. *Let X be a topological space and $\phi : X \to \mathbb{A}^n$ some map. Then ϕ is continuous if and only if $\phi^{-1}(V(f))$ is closed for every non-constant polynomial $f \in K[x_1, \ldots, x_n]$. In particular for $n = 1$, ϕ is continuous if and only if $\phi^{-1}(\lambda)$ is closed for any $\lambda \in K$.*

Proof. This follows from the fact that $\{V(f) : f \in K[\mathbf{x}], f \text{ non-constant}\}$ forms a basis for the Zariski closed sets in $\mathbb{A}^n$ except for the trivially closed sets $\mathbb{A}^n$ and $\emptyset$, but under the preimage of ϕ these will always be closed. Moreover for $n = 1$, $\{V(f) : f \in K[\mathbf{x}], f \text{ non-constant}\}$ correspond to finite unions of points in $\mathbb{A}^1$. □

Lemma 5.1.8. *Let V be an affine variety and $f \in \Gamma(V)$. Define $V(f) := \{P \in V : f(P) = 0\}$. $V(f) \subset V$ is closed and*

$$V \neq V(f) \iff f \neq 0.$$

Proof. Write $f = F + I(V)$ for some F. Then $V \setminus V(f) = V \setminus V(F) = (\mathbb{A}^n \setminus V(F)) \cap V$. So $V \setminus V(f)$ is open, hence $V(f)$ is closed. If $f = 0$, then clearly every point on V is mapped to 0, hence $V = V(f)$. If $f \neq 0$, then there is a point on V that is not mapped to 0, hence $V(f) \subsetneq V$. □

Lemma 5.1.9. *Let V be an affine variety, $f \in \Gamma(V)$. Suppose $U \subset V$ is dense and $f(P) = 0$ for every $P \in U$. Then $f = 0$.*

Proof. Since f is a polynomial function, it is Zariski-continuous from V to K. We thus find that $f(V) = f(\text{cl}(U)) \subset \text{cl}(f(U)) = \text{cl}(\{0\}) = \{0\}$, where the last equality is due to points being Zariski-closed. It follows that $f(V) = \{0\}$. □

Lemma 5.1.10. *Let U be an open subset of a variety V. Consider some rational function $z \in K(V)$. Then $U_z := \{P \in U : z(P) \neq 0\}$ is open in V. Furthermore, $U \to K, P \mapsto z(P)$ is continuous.*

Proof. Write $z = f/g$. Note that $U_z = \{P \in U : f(P) \neq 0\}$, hence

$$U \setminus U_z = \{P \in U : f(P) = 0\} = (\{P \in V : f(P) = 0\} \cup \mathcal{P}_z) \cap U.$$

By a similar argument as in the prior lemma $V(f) := \{P \in V : f(P) = 0\}$ is closed in V, and we already know that the pole set of a rational function is closed in V, hence $U \setminus U_z$ is closed, meaning U_z is open. Let $\lambda \in K$. To check continuity of evaluation of z on U, let $\lambda \in K$. Then $z^{-1}(\lambda) = \{P \in U : z - \lambda = 0\}$, and since $z - \lambda \in \mathcal{O}_P(V)$ for every $P \in U$, it follows that by the first statement that $z^{-1}(\lambda)$ is closed. By Lemma 5.1.7 it follows that $\text{ev}_\bullet(z)$ is continuous as a function from U to K. □

5.2 Redefining notions

5.2.1 Varieties and Regular Functions on Varieties

Definition 5.2.1. Let $\emptyset \neq V \subset \mathbb{A}^m \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_l}$ be an irreducible algebraic set. A *variety of* V is a Zariski open subset X. We endow this with the *Zariski topology*, which will just be be subspace topology of X in V. We define $K(X) := K(V)$ and $\mathcal{O}_P(X) := \mathcal{O}_P(V)$ for $P \in X$.

Remark 5.2.2. By Lemma 5.1.2 if $U \subset X$ is open, then $U \subset V$ is also open, hence it is also variety. Such a set is called an *open subvariety of* X.

Definition 5.2.3. A closed subset Y of a variety $X \subset V$ is called *reducible* if it is the union of two proper subsets that are closed in X. Otherwise, it is called *irreducible*.

Lemma 5.2.4. *If Y is an irreducible closed subset of a variety $X \subset V$, then $\text{cl}_V(Y)$ is irreducible in the underlying multispace and hence a variety.*

Proof. Suppose $\text{cl}_V(Y) = A \cup B$ where A and B are closed sets in multispace. WLOG $A, B \subset V$. Then $Y \subset A \cup B$, hence $Y = (Y \cap A) \cup (Y \cap B)$. Then if $Y \cap A = Y$,

$$Y = Y \cap A \subset \text{cl}_V(Y \cap A) \subset \text{cl}_V(Y) \cap A = A,$$

hence $\text{cl}_V(Y) = A$. It thus follows that $\text{cl}_V(Y)$ is irreducible. □

Definition 5.2.5. A an set $Y \subset X \subset V$ that is irreducible and closed in X is called *a closed subvariety of* X

Lemma 5.2.6. *A closed subvariety $Y \subset X \subset V$ is a variety in $\text{cl}_V(Y)$.*

Proof. Let Y be a closed subvariety of $X \subset V$. From topology we know that

$$Y = \text{cl}_X(Y) = X \cap \text{cl}_V(Y),$$

hence Y is a variety in $\text{cl}_V(Y)$. □

More generally,

Lemma 5.2.7. *if U is an open subvariety of a variety X and Y is a closed subvariety of U, setting $Z := \mathrm{cl}_X(Y)$,*

1. *Z is a closed subvariety of X.*
2. *Y is an open subvariety of Z.*

Definition 5.2.8. Let $X \subset V$ be a variety and $U \subset X$ an open subvariety. Define

$$\Gamma(U, \mathcal{O}_X) := \Gamma(U) := \bigcap_{P \in U} \mathcal{O}_P(X),$$

which is a subring of $K(X)$.

Remark 5.2.9. Note that if $U' \subset U$ are open subvarieties of X, then

$$\Gamma(U') \supset \Gamma(U),$$

and if $X = U$ is an affine variety, then $\Gamma(X)$ is the coordinate ring on X (cf. Proposition 3.3.53).

Proposition 5.2.10. *Let $U \subset X$ be an open subvariety of a variety $X \subset V$. Consider $z \in \Gamma(U)$ such that $z(P) = 0$ for every $P \in U$. Then $z = 0$.*

Proof. Since $K(X) = K(V)$, we may WLOG assume X is an irreducible algebraic set. In general we are situated in some multispace $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_m}$. Let $i_1, \ldots, i_m$ be given such that $X \cap \mathbb{A}^n \times U_{i_1} \times \cdots U_{i_m} \neq \emptyset$. We then apply Remark 4.2.28 to see it is sufficient to consider the affine variety $\varphi^{-1}_{i_1,\ldots,i_m,n}(X) \subset \mathbb{A}^{n+n_1+\cdots+n_m}$. In other words X is an affine variety and U is an open subvariety of X. Write $z = \frac{f}{g}$ and set

$$U' := \{P \in U : g(P) \neq 0\} \subset U,$$

which is open by Lemma 5.1.10. By Remark 5.2.9, $\Gamma(U') \supset \Gamma(U)$, hence $z \in \Gamma(U')$. Then since $f(P) = 0$ for every $P \in U'$ and due the fact that open subsets of X are dense in X it follows Lemma 5.1.9 that $f = 0$, hence $z = 0$. □

Remark 5.2.11. The above proposition shows that the K-algebra homomorphism,

$$\Gamma(U) \to \mathrm{Set}(U, K)$$
$$z \mapsto \mathrm{ev}_\bullet(z)$$

is injective. This means that we may identify $\Gamma(U)$ with some ring of K-valued functions on U. Explicitly, we identify $\Gamma(U)$ with the set

$$\left\{ f \in \mathrm{Set}(U, K) : \text{ there is a } z \in K(X) \text{ for every } P \in U, z \text{ is defined at } P \text{ and } f(P) = z(P) \right\}$$

Remark 5.2.12. We get an alternative way of thinking about the fact

$$U' \subset U \implies \Gamma(U') \supset \Gamma(U)$$

A function $f \in \Gamma(U)$ can be thought of as a function of $\Gamma(U')$ via restriction to U'. Indeed, since there is a $z \in K(X)$ defined at each point in U such that $z(P) = f(P)$ for each $P \in U$, in particular z is defined at each point in U' and $z(P) = f|_{U'}(P)$. This can also be seen from the fact that $\text{Set}(U, K)/I$ can be identified with $\text{Set}(U', K)$, where I is the ideal

$$\{f \in \text{Set}(U, K) : f(P) = 0 \text{ for all } P \in U'\}$$

An equivalence class in $\text{Set}(U, K)/I$ is the set of functions that agree on U'. So we may think of functions in $\Gamma(U')$ as functions of $\Gamma(U)$ where equality is up to agreement on U', in the same way we do in the affine case.

Example 5.2.13. Note that

$$\Gamma(\mathbb{A}^2 \setminus \{(0,0)\}) \supset \Gamma(\mathbb{A}^2) = K[x, y].$$

Conversely, if $z \in \Gamma(\mathbb{A}^2 \setminus \{(0,0)\})$. Writing $z = \frac{f}{g}$ for $f, g \in K[x, y]$ with $g \neq 0$ and $\gcd(f, g) = 1$, we have that $g(P) \neq 0$ for every $P \neq 0$. Then $V(g)$ is finite, hence g must be constant (cf. Proposition 3.1.34). It follows that $z \in K[x, y] = \Gamma(\mathbb{A}^2)$.

Proposition 5.2.14. *Let $\{X_\alpha\}$ be a family of closed subset of some variety $X \subset V$. Such a family has a minimal element.*

Proof. Set $W := \text{cl}(X)$. Set $V_\alpha := \text{cl}_V(X_\alpha)$ for each α. The family of algebraic sets $\{V_\alpha\}$ has a minimal element V_β by Lemma 3.2.8. We claim that X_β is minimal for $\{X_\alpha\}$. Indeed, suppose $X_\alpha \subset X_\beta$. Then $V_\alpha \subset V_\beta$ meaning $V_\alpha = V_\beta$. In general,

$$X_\gamma = \text{cl}_X(X_\gamma) = X \cap \text{cl}_V(X_\gamma) = X \cap V_\gamma,$$

hence $X_\alpha = X \cap V_\alpha = X \cap V_\beta = X_\beta$. □

Proposition 5.2.15. *Let X be a variety with an open cover $\{U_\alpha\}_{\alpha \in A}$. Then X has a finite subcover $\{U_{\alpha_i}\}_1^n$.*

Proof. If A is finite we are done, so suppose this is not the case. By the prior proposition, the family $\{X \setminus \bigcup_1^n U_{\alpha_i} : n \geq 1, \alpha_1, \ldots, \alpha_n \in A\}$ has a minimal element $X \setminus \bigcup_1^m U_{\beta_i}$. This means that $\bigcup_1^m U_{\beta_i}$ is a maximal element of the family

$$S := \left\{ \bigcup_1^n U_{\alpha_i} : n \geq 1, \alpha_1, \ldots, \alpha_n \in A \right\}.$$

Let $x \in X$. Then $x \in U_{\alpha_x}$ for some $\alpha_x \in A$, hence

$$\bigcup_1^m U_{\beta_i} \subset U_{\alpha_x} \cup \bigcup_1^m U_{\beta_i} \in S \implies x \in U_{\alpha_x} \cup \bigcup_1^m U_{\beta_i} = \bigcup_1^m U_{\beta_i} \implies X = \bigcup_1^m U_{\beta_i}.$$

□

Remark 5.2.16. A topological space for which every open cover has a finite subcover is called *quasi-compact*. If such a space is also Hausdorff it is called *compact*. A variety is thus always quasi-compact, but never compact.

Lemma 5.2.17. *Let $X \subset V$ be a variety, $z \in K(X) = K(V)$. The pole set,*

$$\mathcal{P}_X(z) := \{P \in X : z \text{ not defined at } P\}$$

is closed in X. If $z \in \mathcal{O}_P(X)$ for some $P \in X$. Then there is some open neighborhood of P in X, U such that $z \in \Gamma(U)$, hence

$$\mathcal{O}_P(X) = \bigcup_{U \text{ open neighborhood of } P} \Gamma(U).$$

Proof. Note first that $\mathcal{P}_V(z)$ is an algebraic set and hence closed, hence

$$\mathcal{P}_X(z) = \mathcal{P}_V(z) \cap X$$

is closed in X.
Set $U := X \setminus \mathcal{P}_X(z)$, which we have just seen is open in X. Since z is defined at P, $P \notin \mathcal{P}_X(z)$. In particular, if $Q \in U$, then z is defined at Q, hence $z \in \mathcal{O}_Q(X)$, meaning $z \in \bigcap_{Q \in U} \mathcal{O}_Q(X) = \Gamma(U)$. We already knew that

$$\mathcal{O}_P(X) \supset \bigcup_{U \text{ open neighborhood of } P} \Gamma(U)$$

and we have just established the converse inclusion. □

5.2.2 Rational Functions as Equivalence Classes of Regular Functions

Definition 5.2.18. Let $X \subset V \subset \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be a variety. Let $P \in X$. A function $f \in \text{Set}(X, K)$ is *regular at P* if there exists an open neighborhood U of P in X and a pair of same degree m-forms $a, b \in K[\mathbf{x}][\mathbf{y}_1, \ldots, \mathbf{y}_m]$ with $P \notin V(b)$ such that for $z := \frac{a}{b} \in K(\mathbf{x}\mathbf{y}_1, \ldots, \mathbf{y}_m)$,

$$f(P) = z(P).$$

Definition 5.2.19. Let $X \subset V \subset \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ be a variety. A function $f \in \text{Set}(X, K)$ is *regular*, if f is regular at every $P \in X$.

Definition 5.2.20. For a variety $X \subset V$, consider the set

$$\text{Reg}(X) := \{f \in \text{Set}(U, K) : U \subset V \text{is open }, f \text{ is regular}\}$$

Consider $f, g \in \text{Reg}(X)$. We define addition on $\text{Reg}(X)$ by

$$f + g := f|_{\text{dom } f \cap \text{dom } g} + g|_{\text{dom } f \cap \text{dom } g},$$

and multiplication by

$$fg := f|_{\text{dom } f \cap \text{dom } g}\, g|_{\text{dom } f \cap \text{dom } g}$$

Remark 5.2.21. We wish for a set of partial functions on K that are regular on its domain to somehow define a field. The above approach will fail. One readily verifies that the above defines commutative monoids with respect to both addition and multiplication with 0 and 1 being the constant functions on V induced by $0, 1 \in K$ and multiplication distributes over addition. It is in general, however, not a group with respect to addition, so it cannot define a ring. Moreover non-zero elements are not always units, so it cannot be a field. This is due to the fact that if U is a proper open subset of V, then a regular function f with domain U cannot have an additive multiplicative inverse, since for every regular function g on V,

$$\text{dom } (f + g) = U \cap \text{dom } g \subsetneq V \implies f + g \neq 0$$

and

$$\text{dom } fg = U \cap \text{dom } g \subsetneq V \implies fg \neq 1.$$

To mend this, we somehow need to replace $\text{Reg}(X)$ a set of something resembling regular functions, where restriction of such a "regular function" yields the same "regular function".
Note that $\text{Reg}(X)$ is a field when V is a point P. In this case it is the field $\text{Set}(P, K)$ which is isomorphic to K.

Lemma 5.2.22. *Consider a variety X and a rational function $z \in K(X)$. Note that*

$$U := X \setminus \mathcal{P}_X(z)$$

is an open set in X. The function

$$z : U \to K$$
$$P \mapsto z(P)$$

is regular.

Proof. This is easily seen from the definition of being regular. □

The above lemma leads one to consider whether $K(X)$ is somehow isomorphic to a field of regular functions. We have seen that the naive approach of defining $\mathrm{Reg}(X)$ does not even yield a field. We now aim to define an equivalence relation $\sim$ on $\mathrm{Reg}(X)$ such that $\mathrm{Reg}(X)/\sim$ has the obvious structure of a field and $K(X) \simeq \mathrm{Reg}(X)/\sim$.

Definition 5.2.23. Let X be a variety. For $f, g \in \mathrm{Reg}(X)$ we define

$$f \sim g \iff f = g \text{ on } \mathrm{dom}\, f \cap \mathrm{dom}\, g$$

Remark 5.2.24. This is easily seen to define an equivalence relation. Indeed, to verify the transitive property, if $f \sim g$ and $g \sim h$, then f and h agree on $\mathrm{dom}\, f \cap \mathrm{dom}\, g \cap \mathrm{dom}\, h$, which is a dense subset of $\mathrm{dom}\, f \cap \mathrm{dom}\, h$, hence f and h agree on $\mathrm{dom}\, f \cap \mathrm{dom}\, h$, which means $f \sim h$.

Definition 5.2.25. For a variety X, we define

$$K^{\mathrm{Reg}}(X) := \mathrm{Reg}(X)/\sim$$

For $[f], [g] \in K^{\mathrm{Reg}}(X)$, we define

$$[f] + [g] := [f + g]$$

and

$$[f][g] := [fg]$$

Remark 5.2.26. Suppose $f \sim f'$ and $g \sim g'$. Then for $P \in \mathrm{dom}\, f \cap \mathrm{dom}\, g \cap \mathrm{dom}\, f' \cap \mathrm{dom}\, g'$,

$$\begin{aligned}(f+g)(P) &= f|_{\mathrm{dom}\, f \cap \mathrm{dom}\, g}(P) + g|_{\mathrm{dom}\, f \cap \mathrm{dom} g}(P) = f(P) + g(P) = f'(P) + g'(P) \\ &= f'|_{\mathrm{dom}\, f' \cap \mathrm{dom}\, g'}(P) + g'|_{\mathrm{dom}\, f' \cap \mathrm{dom} g'}(P) = (f' + g')(P).\end{aligned}$$

So addition is well-defined. It is similarly easy to check that multiplication is well-defined. It follows easily that $K^{\mathrm{Reg}}(X)$ is a commutative monoid with respect to both addition and multiplication and that multiplication distributes over addition. Let $[f] \in K^{\mathrm{Reg}}(X)$. Then we define $-[f] := [-f]$. Then

$$[f] - [f] = [f - f] = [0].$$

Suppose $[f] \neq [0]$. Since f is regular on $\mathrm{dom}\, f$, there is a rational function $\frac{a}{b}$ for multiforms a, b such that $\mathrm{dom}\, f \cap V(b) = \emptyset$ and $f(P) = \frac{a(P)}{b(P)}$. Then $a \neq 0$, hence $\frac{a}{b}$ is a unit. Then

$$[f]\left[\frac{b}{a}\right] = \left[\frac{a}{b}\right]\left[\frac{b}{a}\right] = \left[\frac{a}{b}\frac{b}{a}\right] = [1].$$

where $\left[\frac{a}{b}\right]$ is the equivalence of the regular function

$$\frac{a}{b} : V \setminus \mathcal{P}_X\left(\frac{a}{b}\right) \to K,$$

$\frac{b}{a}$ is interpreted as a regular function in the same way. It thus follows that $K^{\text{Reg}}(X)$ is a field.

Lemma 5.2.27. *Let X be a variety. Then*

$$K(X) \to K^{\text{Reg}}(X)$$
$$z \mapsto [z]$$

defines an isomorphism of rings.

Proof. One readily verifies that this is a ring map and thus a field map, which is automatically injective. It thus remains to show surjectivity. Let f be a regular function and let $\frac{a}{b}$ be a rational function of multiforms defined on the domain of f that agrees with f. Let z denote the image of $\frac{a}{b}$ in $K(X)$. This is well-defined since $b \notin I(V)$. Then

$$[z] = f.$$

□

This leads to an alternative charaterisation of local rings on a variety

Lemma 5.2.28. *Let X be a variety and $P \in X$. Then $\mathcal{O}_P(X)$ is isomorphic to the subring of $K^{\text{Reg}}(X)$,*

$$\left\{[f] \in K^{\text{Reg}}(X) \middle| \, f \in \text{Reg}(X), P \in \text{dom } f\right\}$$

Proof. One readily verifies that this is a subring of $K^{\text{Reg}}(X)$. If $z \in \mathcal{O}_P(X)$, then since z is defined at P, $P \in \text{dom } z$, hence $[z]$ is an element of the subring. Let $[f] \in K^{\text{Reg}}(X)$ such that $f \in \text{Reg}(X)$ and $P \in \text{dom } f$. Then f agrees with a rational function z in $K(X)$ on the domain of f. It follows that $[z] = [f]$. □

Remark 5.2.29. For an open subset $U \subset X$, we thus have

$$\Gamma(U) = \left\{[f] \in K^{\text{Reg}}(X) : f \in \text{Reg}(X), f \text{ is regular on } U\right\}.$$

In the next section we introduce morphisms of varieties the above remark sets up the most straightforward way to verify that a map is a morphism.

5.2.3 Morphisms of Varieties

Consider the contravariant functor

$$\text{Set}(_, K) : \text{Set} \to \text{Alg}_K$$

For a map $\varphi : A \to B$, we define

$$\widetilde{\varphi} := \text{Set}(\varphi, K) : \text{Set}(B, K) \to \text{Set}(A, K)$$
$$f \mapsto f \circ \varphi$$

For affine varieties we have seen that this functor restricts to a fully faithful functor between the category of affine varieties over K to the category of integral domain finitely generated K-algebras. We want to generalise this notion such that we obtain a class of morphism such that taking the class of all abstract varieties as objects will form a category. This motivates the following definition

Definition 5.2.30. Let X and Y be varieties. A morphism from X to Y is a Zariski-continuous map $\varphi : X \to Y$ such that for every $U \subset Y$ open, $\widetilde{\varphi}$ restricts to a well-defined map

$$\Gamma(U) \to \Gamma(\varphi^{-1}(U)).$$

For $f \in \Gamma(U)$, when we write $\widetilde{\varphi}(f)$, we mean the restriction of φ to $\varphi^{-1}(U)$ precomposed with f.

Remark 5.2.31.

1. For any variety $\text{id}_X : X \to X, x \mapsto x$ is a morphism. Indeed it is continuous and $\widetilde{\text{id}} = \text{id}_{\Gamma(U)}$ for each open $U \subset X$.

2. For morphisms $\varphi : X \to Y$ and $\phi : Y \to Z$, $\phi \circ \varphi : X \to Z$ is a again a morphism. Indeed, the composition is continuous. Let $U \subset Z$ be open and $f \in \Gamma(U)$. Note that $\widetilde{\phi}(f) \in \Gamma\left(\phi^{-1}(U)\right)$. Note then that $\phi^{-1}(U) \subset Y$ is open by continuity, hence
$$\widetilde{\phi \circ \varphi}(f) = \widetilde{\varphi}(\widetilde{\phi}(f)) \in \Gamma\left(\varphi^{-1}\left(\phi^{-1}\left(U\right)\right)\right) = \Gamma\left(\left(\phi \circ \varphi\right)^{-1}\left(U\right)\right).$$

This shows that abstract varieties with morphisms is a category which we denote Var_k.

Definition 5.2.32. A variety that is isomorphic to a closed subvariety of $\mathbb{A}^n$ for some n is called *an affine variety*. A variety that is isomorphic to a closed subvariety of $\mathbb{P}^n$ is called *a projective variety*.

As to not confuse these notions with prior notions of affine and projective variety; "$X \subset \mathbb{A}^n$ is an affine variety", means that X is an affine variety in the sense developed in chapter 4, while "X is an affine variety" refers to the more general notion of being affine described in this definition. A similar clarification of notation also applies to the notion of projective varieties.

Remark 5.2.33. The subcategory of Var_K of affine varieties are denoted AffVar_K. Its skeleton is by definition the irreducible affine algebraic sets.

Lemma 5.2.34. *Given a closed or open subvariety $Y \subset X$ of a variety X*

$$\iota : Y \hookrightarrow X$$
$$y \mapsto y$$

is a morphism.

Proof. We already (from general topology) know that embeddings are continuous. Let $U \subset X$ be open. Let $f \in \Gamma(U)$. This is a rational funtion defined at each $P \in U$, and in particular it is defined at each $P \in \iota^{-1}(U) = U \cap Y$. In other words $f|_{\mathrm{cl}_X(Y)} \in K(Y)$ is regular on $Y \cap U$, and since for $P \in Y \cap U$

$$\widetilde{\iota}(f)(P) = f \circ \iota(P) = f(P) = f|_{\mathrm{cl}_X(Y)}(P)$$

we get $\widetilde{\iota}(f) \in \Gamma(\iota^{-1}(U))$. □

Lemma 5.2.35. *The functor*

$$\Gamma(_) : \mathrm{AffVar}_K \to \mathrm{Alg}_K$$
$$X \mapsto \Gamma(X)$$
$$\varphi : X \to Y \mapsto \widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$$

is a fully faithful functor. Moreover, a morphism between affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is a polynomial map.

Proof. Functoriality of $\Gamma(_) : \mathrm{Aff}_K \to \mathrm{Alg}_K$, together with the fact that we have a commutative square for each morphism of algebraic varieties $\varphi : X \to Y$ for suitable $V \subset \mathbb{A}^m$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow\wr & & \downarrow\wr \\ V & \dashrightarrow & W \end{array}$$

ensures that this functor is well-defined, since then $\widetilde{\varphi} = \widetilde{\nu}\widetilde{\nu^{-1}}\widetilde{\varphi}\widetilde{\mu}\widetilde{\mu^{-1}} : \Gamma(Y) \to \Gamma(X)$ is well-defined K-algebra map. Here $\mu : X \simeq V$ and $\nu : Y \simeq W$.
Let $\sigma : \Gamma(Y) \to \Gamma(X)$ be a K-algebra map. Then there is a unique polynomial map $\phi : V \to W$ such that

$$\widetilde{\mu\phi\nu^{-1}} = \sigma,$$

hence $\Gamma(_) : \mathrm{AffVar}_K \to \mathrm{Alg}_K$ is fully faithful.
Consider affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ a morphism $\varphi : V \to W$. Then there is a polynomial map $\phi : V \to W$ such that $\widetilde{\phi} = \widetilde{\varphi}$, since polynomial maps are morphisms (this is seen in an example below) and $\Gamma(_)$ is fully faithful, it follows that $\varphi = \phi$, hence φ is a polynomial map. □

Lemma 5.2.36. *Let X, Y be varieties and $\varphi : X \to Y$ be some function. Let $\{U_\alpha\}$ be a cover of X of open subvarieties of X and $\{V_\alpha\}$ be a cover of Y of open subvarieties of Y such that $\varphi(U_\alpha) \subset V_\alpha$ for each α.*

1. *Then φ is a morphism if and only if $\varphi_\alpha := \varphi|_{U_\alpha}$ is a morphism for each α.*
2. *If we furthermore assume that each U_α, V_α are affine, then φ is a morphism if and only if each $\widetilde{\varphi}(\Gamma(V_\alpha)) \subset \Gamma(U_\alpha)$.*

Proof. 1. " $\Longrightarrow$ ": Let α be given. Let $W \subset V_\alpha$ be open. Then W is also open in Y. Let $f \in \Gamma(W)$. Note that $\varphi_\alpha^{-1}(W) \subset \varphi^{-1}(W)$. There is a $z \in K(X)$ defined at every point in $\varphi^{-1}(W)$ such that $z(P) = \widetilde{\varphi}(f)(P)$. Then in particular z is defined at every point in $\varphi_\alpha^{-1}(W)$ and

$$z(P) = \widetilde{\varphi}(f)(P) = f(\varphi(P)) = f(\varphi_\alpha(P)) = \widetilde{\varphi_\alpha}(f)(P).$$

From this it follows that $\widetilde{\varphi_\alpha}(f) \in \Gamma(\varphi_\alpha^{-1}(W))$ and as a consequence we conclude that φ_α is a morphism for α. " $\Longleftarrow$ ": That φ is continuous follows from Lemma 5.1.3. Let $U \subset Y$ be open and $f \in \Gamma(U)$. Then we have in particular that $f \in \Gamma(U \cap V_\alpha)$ for every α. To prove that φ is a morphism, we need to prove that $\widetilde{\varphi}(f) \in \Gamma\left(\varphi^{-1}(U)\right)$. Note that $\varphi^{-1}(U) = \bigcup_\alpha \varphi_\alpha^{-1}(U \cap V_\alpha)$. Then

$$\Gamma\left(\varphi^{-1}(U)\right) = \bigcap_\alpha \Gamma\left(\varphi_\alpha^{-1}(U \cap V_\alpha)\right).$$

It is thus sufficient to prove that for every α, $\widetilde{\varphi}(f) \in \Gamma\left(\varphi_\alpha^{-1}(U \cap V_\alpha)\right)$.
We already know that $\widetilde{\varphi_\alpha}(f) \in \Gamma\left(\varphi_\alpha^{-1}(U \cap V_\alpha)\right)$. I.e. $\widetilde{\varphi_\alpha}(f)$ is an element in $K(X)$ defined at every point in $U \cap V_\alpha$. On every point Q in $\varphi_\alpha^{-1}(U \cap V_\alpha)$,

$$\left(\widetilde{\varphi_\alpha}(f)\right)(Q) = \left(\widetilde{\varphi}(f)\right)(Q),$$

hence $\widetilde{\varphi}(f) \in \Gamma\left(\varphi_\alpha^{-1}(U \cap V_\alpha)\right)$.
2. " $\Longrightarrow$ ": Since $\varphi(U_\alpha) \subset V_\alpha$, it follows that $U_\alpha \subset \varphi^{-1}(V_\alpha)$, hence if $f \in \Gamma(V_\alpha)$ using that φ is a morphism,

$$\widetilde{\varphi}(f) \in \Gamma\left(\varphi^{-1}(V_\alpha)\right) \subset \Gamma(U_\alpha) \implies \widetilde{\varphi}\left(\Gamma(V_\alpha)\right) \subset \Gamma(U_\alpha).$$

" $\Longleftarrow$ ":Let α be given. By assumption,

$$\widetilde{\varphi_\alpha} : \Gamma(V_\alpha) \to \Gamma(U_\alpha)$$
$$f \mapsto f \circ \varphi_\alpha$$

is a well-defined K-algebra homomorphism. Since U_α and V_α are affine, there is a unique morphism $\phi : U_\alpha \to V_\alpha$ inducing $\widetilde{\varphi_\alpha}$. Considering these two varieties as subsets of affine spaces, we may identify $\Gamma(V_\alpha)$ with some $K[y_1, \ldots, y_m]/I$ and transport the two maps via suitable isomorphisms to maps $\varphi'_\alpha, \phi : U'_\alpha \subset \mathbb{A}^n \to V'_\alpha \subset \mathbb{A}^m$ given by coordinate maps $(\varphi_\alpha)'_i$ and ϕ'_i respectively. Then

$$(\varphi_\alpha)'_i = \widetilde{\varphi'_\alpha}(y_i + I) = \widetilde{\phi'}(y_i + I) = \phi'_i \implies \varphi'_\alpha = \phi',$$

hence transporting back via the isomorphisms we get that $\varphi_\alpha = \phi$. Then φ_α is a morphism for each α, hence by 1. φ is a morphism. □

Lemma 5.2.37. *Consider a morphism $\varphi : X \to Y$ and open/closed subvarieties $X' \subset X$ and $Y' \subset Y$ such that $\varphi(X') \subset Y'$. Then $\varphi|_{X'} : X' \to Y'$ is a morphism*

Proof. Clearly $\varphi|_{X'} : X' \to Y'$ is continuous. Let $U \cap Y' \subset Y'$ be open. Let $f \in \Gamma(U \cap Y')$. To prove that $\widetilde{\varphi|_{X'}}(f) \in \Gamma(\varphi^{-1}(U) \cap X')$, it is sufficient to find a rational function $z \in K(X)$ such that z is defined on $U \cap Y'$ and $z(P) = \widetilde{\varphi|_{X'}}(f)(P)$ for every $P \in U \cap Y'$. Letting $\iota : X' \hookrightarrow X$ denote the embedding of X' in X, we find that $\widetilde{\varphi}\iota(f)$ is a such a rational function since it agrees with $\widetilde{\varphi|_{X'}}$ on $U \cap Y'$. It follows that $\varphi|_{X'}$ is a morphism. □

Example 5.2.38.

1. If $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are affine varieties, then a polynomial map $\varphi : V \to W$ is a morphism. The category of affine varieties over K is thus a subcategory of Var_K.

2. Set $A := \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $B := \mathbb{A}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$. Consider the map $\pi : A \times B \to A, (v, w) \mapsto v$. Note that $\pi^{-1}(V) = V \times B$ for every closed $V \subset A$ and this set is equal to $V(I(V)) \subset A \times B$, meaning π is continuous. Consider $K(A)$ canonically as a subring of $K(A \times B)$. Explicitly, a multiform in $(K[x_1, \ldots, x_n])[\mathbf{y}]$ is in particular an $(n_1, \ldots, n_k, m_1, \ldots, m_l)$-form in $(K[x_1, \ldots, x_n, z_1, \ldots, z_m])[\mathbf{y}, \mathbf{w}]$, hence a rational function a/b where $a, b \in K[\mathbf{x}, \mathbf{y}]$ are same degree $(n_1, \ldots, n_k)$-forms is canonically a rational function a/b in $Q(K[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}])$. Denote this identification by ι Let $U \subset A$ be open. Note that $\pi^{-1}(U) = U \times B$. Let $f \in \Gamma(U)$. Let

$Q \in U$. We may find suitable multiforms a, b with $b(Q) \neq 0$, and $\frac{a(Q)}{b(Q)} = f(Q)$. It follows that $\iota(\frac{a}{b})$ is a function regular on $U \times B$ and for each $P = (v, w) \in U \times B$,

$$\widetilde{\pi}(f)(P) = f(v) = \iota(f)(P),$$

hence $\widetilde{\pi}(f) \in \Gamma(\pi^{-1}(U))$.

3. Consider the quotient map $\pi : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$. Let $V = V^{\mathbb{P}}(F_1, \ldots, F_m)$ where F_i are homogeneous. We claim that $\pi^{-1}(V) = V^{\mathbb{A}}(F_1, \ldots, F_m)$. Indeed, for each i

$$v \in \pi^{-1}(V) \iff F_i([v]) = 0 \iff F_i(v) = 0 \iff v \in V^{\mathbb{A}}(F_1, \ldots, F_m).$$

It follows that ϕ is continuous. Let $U \subset \mathbb{P}^n$ be open. Consider the K-algebra map

$$\alpha : K(\mathbb{P}^n) = \left\{ \frac{f}{g} \in Q(K[\mathbf{x}]) : f, g \text{ forms}, \deg\ f = \deg\ g \right\} \to K(V) = K\left(\mathbb{A}^{n+1}\right) = Q(K[\mathbf{x}])$$
$$f/g \mapsto f/g$$

Let $P \in \pi^{-1}(U)$ and consider $z \in \Gamma(U)$. There is a rational function $\frac{a}{b} \in K(\mathbb{P}^n)$ such that $\lambda P \notin V(b)$ for any $\lambda \in K \setminus 0$. In particular $b(P) \neq 0$. It follows that α restrict to a K-algebra map

$$\alpha : \Gamma(U) \to \Gamma\left(\pi^{-1}(U)\right)$$
$$\frac{f}{g} \mapsto \frac{f}{g}$$

and for each $v \in \pi^{-1}(U)$ and $z := f/g \in \Gamma(U)$

$$\alpha(z)(v) = z(v) = z([v]) = (z \circ \pi)(v) = \widetilde{\pi}(z)(v) \implies \alpha = \widetilde{\pi},$$

hence π is a morphism.

Lemma 5.2.39. *$U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}(U) \subset \mathbb{A}^{n+1} \setminus 0$ is open, where $\pi : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is the quotient map.*

Proof. If $U \subset \mathbb{P}^n$ is open, then $\pi^{-1}(U)$ is open, since π is continuous.
Suppose $\pi^{-1}(U)$ is open. Then the complement of $\pi^{-1}(U)$ is $V^{\mathbb{A}}(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in K[x_1, \ldots, x_{n+1}]$. Let $[v] \in U$. Then $v \in \pi^{-1}(U)$, which means $f_i(v) \neq 0$ for some i. Then $[v]$ is not a zero of f_i, which means $[v] \notin V^{\mathbb{P}}(f_1, \ldots, f_m)$. Conversely, if $[v] \notin V^{\mathbb{P}}(f_1, \ldots, f_m)$, then there is a $\lambda \in K \setminus 0$ such that $f_i(\lambda v) \neq 0$ for some i. Then $\lambda v \in \pi^{-1}(U)$, hence $[v] = [\lambda v] \in U$. It follows that $U = \mathbb{P}^n \setminus V^{\mathbb{P}}(f_1, \ldots, f_m)$, hence U is open. □

Example 5.2.40. Consider $V := \mathbb{A}^2 \setminus \{(0,0)\}$ and the morphism $\iota : V \to \mathbb{A}^2$. By Example 5.2.13 the map

$$\widetilde{\iota} : \Gamma\left(\mathbb{A}^2\right) \to \Gamma(V) = \Gamma\left(\mathbb{A}^2\right)$$
$$f \to f$$

is an isomorphism.
In general, consider varieties X and Y and a morphism $\phi : X \to Y$. If X and Y are affine, then either both ϕ and $\widetilde{\phi}$ are isomorphisms or both of the maps are not isomorphisms.
So since since ι is not an isomorphism and $\widetilde{\iota}$ is an isomorphism, it follows that the domain or codomain is not affine. Since $\mathbb{A}^2$ is affine it follows that $\mathbb{A}^2 \setminus \{(0,0)\}$ is not affine.

Lemma 5.2.41. *Let X be a variety and $f \in \Gamma(X)$.*

1. *Set $z := \frac{1}{f-\lambda} \in K(X)$. For every $\lambda \in K$, $\mathcal{P}_X(z) = f^{-1}(\lambda)$.*
2. *f is a morphism. In other words a regular function $f : X \to K = \mathbb{A}^1$ is a morphism.*

Proof. 1. Let $P \in X$ and write $f = \frac{g}{h} \in K(X)$ with $h(P) \neq 0$. We then have $z = \frac{h}{g-\lambda h}$, hence

$$P \in \mathcal{P}_X(z) \iff g(P) = \lambda h(P) \iff \lambda = \frac{g(P)}{h(P)} = f(P) \iff P \in f^{-1}(\lambda).$$

2. f is continuous by Lemma 5.1.7 and 1. Let $U = \mathbb{A}^1 \setminus \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{A}^1$ (an arbitrary open subset). Let $\gamma = \frac{a}{b} \in \Gamma(U) \subset K(x)$ with $\gcd(a,b) = 1$. Then b is polynomial whose roots can only be among $\{\alpha_1, \ldots, \alpha_n\}$. Note $a(f), b(f) \in K(X)$ and that b vanishes on no point in $f^{-1}(U)$, which implies $b(f) \neq 0$, hence $\frac{a(f)}{b(f)} \in K(X)$. Note also that $b(f(P)) \neq 0$ for every $P \in U$. Let $P \in f^{-1}(U)$. Pick $\frac{g}{h} \in K(X)$ such that $h(P) \neq 0$. write $a = \sum_0^{d_1} a_i x^i$ and $b = \sum_0^{d_2} b_i x^i$. Then

$$\frac{a(f)}{b(f)} = \frac{\sum_0^{d_1} a_i \frac{g^i}{h^i}}{\sum_0^{d_2} b_i \frac{g^i}{h^i}} = \frac{\left(\frac{\sum_0^{d_1} a_i g^i h^{d_1-i}}{h^{d_1}}\right)}{\left(\frac{\sum_0^{d_2} b_i g^i h^{d_2-i}}{h^{d_2}}\right)} = \frac{\sum_0^{d_1} a_i g^i h^{d_1+d_2-i}}{\sum_0^{d_2} b_i g^i h^{d_1+d_2-i}}.$$

Note that

$$0 \neq b(f(P)) = \frac{\sum_0^{d_2} b_i g(P)^i h(P)^{d_2-i}}{h(P)^{d_2}} \implies \sum_0^{d_2} b_i g(P)^i h(P)^{d_1+d_2-i}(P) \neq 0,$$

so $\frac{a(f)}{b(f)}$ is defined at P, hence and since $\gamma \circ f$ agrees with $\frac{a(f)}{b(f)}$ on $f^{-1}(U)$ it follows that $\widetilde{f}(\gamma) \in \Gamma(f^{-1}(U))$. □

Lemma 5.2.42. *If $\varphi : X \to Y$ is a morphism and a homeomorphism, then φ is an isomorphism.*

Proof. It suffices to check that φ^{-1} is a morphism. Let $U \subset X$ be open. Then

$$\widetilde{\varphi^{-1}} : \Gamma(U) \to \text{Set}(\varphi(U), K)$$

is a well-defined K-algebra map, such that

$$\widetilde{\varphi^{-1}} \circ \varphi = \text{id}_{\Gamma(\varphi^{-1}(U))}$$

hence im $\widetilde{\varphi^{-1}} = \Gamma(\varphi(U))$. □

Lemma 5.2.43. *$\mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$ is an open subvariety of $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. Let V be a closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \mathbb{P}^{n_m}$. Then $V_{n,i_1,\ldots,i_m} := \varphi^{-1}_{n,i_1,\ldots,i_m}(V)$ is a closed subvariety in $\mathbb{A}^{n+n_1+\cdots+n_m}$. The map $\varphi_{n,i_1,\ldots,i_m} : \mathbb{A}^{n+n_1+\cdots+n_m} \to \mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$ is an isomorphism, hence this maps restricts to an isomorphism of $V_{n,i_1,\ldots,i_m}$ with $V \cap \mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$*

Proof. Indeed $\mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$ is the complement of $\bigcap_1^m V(x_{ki_k})$. We see that $V_{n,i_1,\ldots,i_m} = V_{*,n,i_1,\ldots,i_m}$. If V does not intersect $\mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$, then $V_{n,i_1,\ldots,i_m} = \emptyset$, which is a closed subvariety of $\mathbb{A}^{n+n_1+\cdots+n_m}$. Otherwise if $V_{*,n,i_1,\ldots,i_m} \cap \mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m} \neq \emptyset$, due to Remark 4.2.28, $V_{n,i_1,\ldots,i_m}$ is an affine variety in $\mathbb{A}^{n+n_1+\cdots+n_m}$, hence in particular it is a closed subvariety of $\mathbb{A}^{n+n_1+\cdots+n_m}$. This shows that $\varphi^{-1}_{n,i_1,\ldots,i_n}$ sends closed sets in $\mathbb{A}^n \times U_{i_1} \times \cdots \times U_{i_m}$ to closed sets in $\mathbb{A}^{n+n_1+\ldots+n_m}$ and $\varphi_{n,i_1,\ldots,i_m}$ maps closed sets in $\mathbb{A}^{n+n_1+\cdots+n_m}$ to closed sets in $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ by the proper generalization of Lemma 4.2.16, hence $\varphi_{n,i_1,\ldots,i_m}$ is a continuous map. To verify that it is a morphism it is sufficient to check that $\varphi := \varphi_{n,n_1+1,\ldots,n_m+1}$ is a morphism since permuation of coordinates is a morphism. Let

$$U = U' \cap \mathbb{A}^n \times U_{n_1+1} \times \cdots \times U_{n_m+1} \subset \mathbb{A}^n \times U_{n_1+1} \times \cdots \times U_{n_m+1}$$

be open. Let $f \in \Gamma(U)$. Let $P = (v, v_1, \ldots, v_m) \in \varphi^{-1}(U)$. Then

$$\widetilde{\varphi}(f)(P) = f(\varphi(P)) = f(v, [v_1, 1], \ldots, [v_m, 1]) = f_*(P)$$

and since $f_* \in \mathcal{O}_P(A)$ it follows that φ is a morphism.
Note that φ^{-1} is also continuous, since if $W \subset \mathbb{A}^{n+n_1+\ldots+n_m}$ is closed, then $\varphi(W) = (W)^* \cap U_{n,n_1+1,\ldots,n_m+1}$ is closed in $U_{n,n_1+1,\ldots,n_m+1}$. Then φ is a homeomorphism, hence, by the prior lemma, φ is an isomorphism. □

Example 5.2.44. Set $A := \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $B := \mathbb{A}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$. Let $v \in B$. Consider the map

$$\iota : A \to A \times B$$
$$a \mapsto (a, v)$$

Let $V = V(F_1, \ldots, F_t) \subset A \times B$ for $F_1, \ldots, F_t \in K[\mathbf{x}_0, \ldots, \mathbf{x}_k, \mathbf{y}_0, \ldots, \mathbf{y}_l]$. We readily verify that $\iota^{-1}(V) = V(F_1(\mathbf{x}, w), \ldots, F_t(\mathbf{x}, w))$, where $(w_0, [w_1], \ldots, [w_k]) = v$. Using this and the fact that $\iota^{-1}(V \times \{v\}) = V$, we readily verify that $V \times \{v\}$ is irreducible by the usual argument. Let $U \subset A \times B$ be open such that $\iota^{-1}(U) \neq \emptyset$. Consider a multiform $f \in \Gamma(A \times B) = K[\mathbf{x}, \mathbf{y}]$ that does not vanish on any $P \in U$. Then in particular, f does not vanish on any point $(u, v) \in U$. For any chosen representative of v, w say, we then have that $f(\mathbf{x}, w)$ does not vanish on any $u \in \iota^{-1}(U)$. Then for any $z \in \Gamma(U)$, $\widetilde{\iota}(z) \in \Gamma\left(\iota^{-1}(U)\right)$. It then follows that $\iota|_V$ is an isomorphism with inverse $\pi|_{V \times \{v\}} : V \times \{v\} \to V$. It follows that

$$V \times \{v\} \simeq V.$$

Lemma 5.2.45. *Let X, Y be varieties $\varphi : X \simeq Y$ an isomorphism, $X' \subset X$ a closed subvariety. Then $\varphi|_{X'}$ is an isomorphism onto its image.*

Proof. Indeed, by a prior lemma it suffices to show that $\varphi(X')$ is a closed subvariety in Y. Since φ is a homeomorphism it and its inverse are closed maps, hence $\varphi(X')$ is closed. Write $\varphi(X') = A \cup B$ for closed subsets A and B. Then $X' = \varphi^{-1}(A) \cup \varphi^{-1}(B)$, hence WLOG $X' = \varphi^{-1}(A) \implies \varphi(X') = A.$, □

Proposition 5.2.46. *Let V be a closed subvariety of $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. $\varphi_{n,i_1,\ldots,i_m}|_{V_i} : V_i \to V \cap \mathbb{A}^n \times U_{1i_1} \times \cdots \times U_{mi_m}$ is an isomorphism of varieties, hence $V \cap \mathbb{A}^n \times U_{1i_1} \times \cdots \times U_{mi_m}$ is an affine variety. A projective variety is therefor the union of a finite number of affine varieties.*

Proof. From Lemma 5.2.43 $\varphi_{n,i_1,\ldots,i_m}|_V$ defines a morphism with image $V \cap U_{n,i_1,\ldots,i_m}$, which is a closed subvariety of $U_{n,i_1,\ldots,i_m}$ by Lemma 5.2.45. Then

$$V = \bigcup_{i_1,\ldots,i_m} V \cap U_{1i_1} \times \cdots \times U_{mi_m}.$$

is a finite cover of open affine varieties. □

Lemma 5.2.47. *The Segre embedding is an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with*

$$V := V(\{z_{ij}z_{kl} - z_{il}z_{kj} : i, k \in \{1, \ldots, n_1 + 1\}\}).$$

Proof. By Lemma 4.2.35. $\mathbb{P}^n \times \mathbb{P}^m$ is in bijection with V. We have a covering of $\mathbb{P}^n \times \mathbb{P}^m$ by

$$\{U_i \times U_j : 1 \leq i \leq n+1, 1 \leq j \leq m+1\}$$

and a covering of V by

$$\{U_{ij} \cap V : 1 \leq i \leq n+1, 1 \leq j \leq m+1\}.$$

By Lemma 5.2.36, it suffices to show that $S_{ij} := S|_{U_i \times U_j}$ is an isomorphism for arbitrary i and j. We have the following commutative diagram

$$\begin{array}{ccc} U_i \times U_j & \xrightarrow{S_{ij}} & V \cap U_{ij} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbb{A}^{n+m} & \xrightarrow{S_*} & V_* \end{array}$$

where

$$S_*(v,w)_{pq} = \begin{cases} v_p & \text{if } q = i \\ w_q & \text{if } p = j \\ v_p w_q & \text{otherwise} \end{cases}$$

and the isomorphisms are given by $\varphi_{i,j}$ and the restriction of φ_{ij} to V_*. Since $U_i \times U_j$ is affine and $V \cap U_{ij}$ is affine it suffices to show that the induced map of S_{ij} is an isomorphism or indeed that the induced map of S_* is an isomorphism. Define

$$\sigma : K[\{z_{pi}\} \cup \{z_{jq}\}][z_{pq} : p \neq j, q \neq i] \to K[\{z_{pi}\} \cup \{z_{jq}\}]$$

$z_{pq} \mapsto z_{pi} z_{jq}$ for $q \neq j$, $p \neq i$. This is clearly surjective, so by Corollary 1.8.48,

$$K[z_{pq} : (p,q) \neq (i,j)]/\langle \{z_{pq} - z_{pi} z_{jq} : p \neq i, q \neq j\} \rangle \simeq K[\{z_{pi}\} \cup \{z_{jq}\}] \simeq K[\mathbf{x}, \mathbf{y}].$$

Note that $\langle \{z_{pq} - z_{pi} z_{jq} : p \neq i, q \neq j\} \rangle = I(V_*)$ and that $\widetilde{S_*}$ is equal to the map given by $f + I(V_*) \mapsto \sigma(f)$, hence S_* is an isomorphism. □

Proposition 5.2.48. *A closed subvariety Y of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is a projective variety. A variety is isomorphic to an open subvariety of a projective variety.*

Proof. The result is obviously true for $m = 1$. For $m \geq 2$, define $S_2 = S$ and

$$S_{m+1} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{m+1}} \to \mathbb{P}^{(n_1+1)\cdots(n_{m+1}+1)-1}$$
$$(v,w) \mapsto S(S_m(v), w)$$

By induction $W := \operatorname{im} S_m$ is a projective variety. We claim that $W \times \mathbb{P}^{n_{m+1}} \subset \mathbb{P}^{(n_1+1)\cdots(n_m+1)-1} \times \mathbb{P}^{n_{m+1}}$ is a variety. Consider the map

$$\varphi : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{m+1}} \to W \times \mathbb{P}^{n_{m+1}}$$
$$(v, w) \mapsto (S_m(v), w)$$

Let $Z = V(F_1, \ldots, F_k) \subset \mathbb{P}^{(n_1+1)\cdots(n_m+1)-1} \times \mathbb{P}^{n_{m+1}}$ for $F_i \in K[x_1, \ldots, x_{(n_1+1)\cdots(n_m+1)}, y_1, \ldots, y_{n_{m+1}+1}]$. Set

$$G_i := F_i(w_1, \ldots, w_{(n_1+1)\cdots(n_m+1)}, z_{m+1,1}, \ldots, z_{m+1,n_{m+1}})$$

sitting in the polynomial ring $K[\{z_{ij} : 1 \leq i \leq m+1, 1 \leq j \leq n_i+1\}]$ where w_i are products on the form

$$\prod_1^m z_{ki_k}$$

ordered in a suitable way. One readily verifies that

$$\varphi^{-1}(Z) = V(G_1, \ldots, G_k).$$

By a similar argument to the one given in the proof of Lemma 3.3.4, $W \times \mathbb{P}^{n_{m+1}}$ is a bi-projective variety. Then since we already know that S restricted to a closed varieties is isomorphic to a projective variety, it follows that S_{m+1} is an isomorphism to a projective variety in $\mathbb{P}^{(n_1+1)\cdots(n_{m+1}+1)-1}$, namely $S(W \times \mathbb{P}^{n_{m+1}})$. It follows that a closed subvariety of some multiprojective space is projective.
For the second statement note that

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} \times \mathbb{A}^n \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} \times U_1$$
$$([v_1], \ldots, [v_m], v) \mapsto ([v_1], \ldots, [v_m], [v])$$

defines an isomorphism, hence by the first statement it is isomorphic to an open subvariety of $\mathbb{P}^{(n_1+1)\cdots(n_m+1)(n+1)-1}$. □

Remark 5.2.49. If one knows that the cartesian product of two closed subvarieties is again a closed subvariety, then the above argument can be simplified.

Lemma 5.2.50. *Let R be an integral domain, with $K := Q(R)$. Consider $f \in R \setminus 0$. Consider also a ring map $\sigma : R \to S$ such that $\sigma(f)$ is a unit in S. Then σ uniquely extends to a ring map $R[1/f] \subset K \to S$. The map*

$$\gamma : R[x] \to R[1/f], x \mapsto 1/f$$

induces an isomorphism, $R[x]/\langle xf - 1\rangle \simeq R[1/f]$

Proof. Let $X = \{f^n : n \geq 0\}$. Then $X^{-1}R \simeq R\left[\frac{1}{f}\right]$ and $\sigma(X)^{-1}S \simeq S$ canonically, so by Lemma **??**, there is a unique ring homomorphism extending σ to a map $R\left[\frac{1}{f}\right] \to S$. It is clear that

$$\begin{aligned} \alpha : R[x] &\to R\left[\frac{1}{f}\right] \\ x &\mapsto \frac{1}{f} \end{aligned}$$

is surjective. Suppose $g(1/f) = 0$, then $g \in \langle x - \frac{1}{f}\rangle \subset R[1/f][x]$. Since f is a unit in $R[1/f]$, it follows that $g \in (R[1/f])[x](fx-1)$. Then $g = \frac{a}{f^k}(fx-1)$ implying $f^k g \in \langle fx-1\rangle \subset R[x]$ which is a prime ideal, and since deg $f = 0$, we have $f \notin \langle fx-1\rangle$, hence $g \in \langle fx-1\rangle$. One sees that ker $\alpha = \langle fx-1\rangle$, and therefore that α is an isomorphism. □

Proposition 5.2.51. *Let V be an affine variety and $f \in \Gamma(V) \setminus 0$. Set*

$$V_f := \{P \in V : f(P) \neq 0\},$$

which is an open subvariety of V. Then

1. $\Gamma(V_f) = \Gamma(V)\left[\frac{1}{f}\right] \subset K(V)$.
2. *V_f is affine*

Proof. WLOG $V \subset \mathbb{A}^n$. Set $I := I(V)$. Then $\Gamma(V) = K[x_1, \ldots, x_n]/I$ and we thus pick $F \in K[\mathbf{x}]$ such that $f = F + I$.
1. Let $z \in \Gamma(V_f)$. The pole set of z is equal to $V(J)$ where

$$J = \{G \in K[\mathbf{x}] : Gz \in \Gamma(V)\},$$

(cf. Lemma 3.3.50). Since $V(J) \subset V(F)$, by HNS there is an N such that $F^N \in J$. Then $zf^N \in \Gamma(V)$, hence $z = (zf^N)\frac{1}{f^N} \in \Gamma(V)\left[\frac{1}{f}\right]$. Since $V_f \subset V$, $\Gamma(V_f) \supset \Gamma(V)$, hence $\Gamma(V)\left[\frac{1}{f}\right] \subset \Gamma(V_f)$.
2. Set $I' := \langle I \cup \{x_{n+1}F - 1\}\rangle \subset K[x_1, \ldots, x_{n+1}]$. Set $V' := V(I')$. Consider

$$\alpha : K[x_1, \ldots, x_{n+1}] \to \Gamma(V_f)$$

defined by $x_i \mapsto x_i + I$ for $i = 1, \ldots, n$ and $x_{n+1} \mapsto \frac{1}{f}$. 1. shows that α is surjective. Lemma 5.2.50 shows that ker $\alpha = I'$. Then I' is prime since $\Gamma(V_f) = \Gamma(V)[1/f]$ is an integral domain, meaning V' is an affine variety. Denote the induced isomorphism

of $\Gamma(V')$ and $\Gamma(V_f)$ by β. Consider the morphism $\pi : \mathbb{A}^{n+1} \twoheadrightarrow \mathbb{A}^n$ which restricts to a morphism $\varphi : V' \to V_f$. We claim that this is a bijection inducing β^{-1}. Indeed define

$$\begin{aligned} \phi : V_f &\to V' \\ v &\mapsto (v, f(v)^{-1}) \end{aligned}$$

This is well-defined since $v \in V$ implies that $(v, f(v)^{-1})$ is a zero of every polynomial in $\langle I \rangle \subset K[x_1, \ldots, x_{n+1}]$. Moreover, $\mathrm{ev}_{(v,f(v)^{-1})}(fx_{n+1} - 1) = f(v)f(v)^{-1} - 1 = 0$. It is clear that $\pi\phi = \mathrm{id}$. Note that for $v \in V'$, $v_{n+1}f(v) = 1$, hence $v_{n+1} = f(v_1, \ldots, v_n)^{-1}$. It follows that

$$\phi\pi(v) = \phi(v_1, \ldots, v_n) = (v_1, \ldots, v_n, f(v)^{-1}) = v.$$

Let $v = (v_1, \ldots, v_n, f(v)^{-1}) \in V'$ and $\frac{g}{f^k} \in \Gamma(V_f)$. Then

$$\left(\widetilde{\varphi}\left(\frac{g}{f^k}\right)\right)(v) = \frac{g(v_1, \ldots, v_n)}{f(v)} = \mathrm{ev}_v\left(g(x_{n+1} + I(V))^k\right) = \left(\overline{\alpha}^{-1}\left(\frac{g}{f^k}\right)\right)(v) \implies \widetilde{\varphi} = \overline{\alpha}^{-1}.$$

Suppose $W := V(H_1, \ldots, H_k) \subset V'$ with $H_i \in K[x_1, \ldots, x_{n+1}]$. Then we claim that $\varphi(W) = V\left(F^N H_1(x_1, \ldots, x_n, 1/F), \ldots, F^{N_i} H_k(x_1, \ldots, x_n, 1/F)\right)$ for $N_i \geq \deg F_i$. Indeed,

$$w = (u, f(u)^{-1}) \in W \iff \forall i, 0 = H_i(u, F(u)^{-1}) \iff \forall i, 0 = F(u)^{N_i} H_i(u, F(u)^{-1}) \iff$$
$$u \in V(F^{N_1} H_1(x_1, \ldots, x_n, 1/F), \ldots, F^{N_k} H_k(x_1, \ldots, x_n, 1/F).$$

So ϕ is continuous. Lastly by functoriality,

$$\widetilde{\phi} = \widetilde{\varphi}^{-1} = \overline{\alpha},$$

hence ϕ is a morphism. It follows that $V_f \overset{\varphi}{\simeq} V'$, hence V_f is affine. □

Example 5.2.52. Consider that rational functions $x, y \in \Gamma(\mathbb{A}^2) = K[x, y]$. Then

$$V_x \cup V_y = \mathbb{A}^2 \setminus \{(0, v_2) : v_2 \in K\} \cup \mathbb{A}^2 \setminus \{(v_1, 0) : v_1 \in K\} = \mathbb{A}^2 \setminus \{(0, 0)\},$$

which shows (cf. Example 5.2.40) that the union of two open affine subvarieties need not be affine.

Corollary 5.2.53. *Let X be a variety, U an open neighborhood of some point $P \in X$. Then there is some open neighborhood V of P contained in U, which is affine.*

Proof. By Proposition 5.2.48 X is isomorphic to an open subvariety of a projective variety. I.e. WLOG $X \subset V \subset \mathbb{P}^n$. Suppose $P \in U_i$. WLOG we may replace X with V. In other

words we may assume that $X \subset \mathbb{P}^n$ is a projective variety. Then $U \cap U_i$ is an open neighborhood of P sitting in $X \cap U_i$. If we can find an affine open neighborhood of P contained in $U \cap U_i$, then we are done, since this will also be an open subset of U. I.e. WLOG we may assume that $X \subset \mathbb{A}^n$ is an affine variety. $X \setminus U$ is an algebraic set. By Proposition 3.1.43 1. we can find a polynomial $F \in K[x_1, \ldots, x_n]$ such that $F(P) \neq 0$ and $F(Q) = 0$ for every $Q \in X \setminus U$. Define $f := F + I(X) \in \Gamma(X)$. Then $P \in X_f$ and $X_f \subset U$, by the construction of F. The prior proposition shows that X_f is affine. □

Remark 5.2.54. This provides an alternative way of showing that a variety has a finite open affine cover: Let X be a variety. By Corollary 5.2.53 there is a covering of X, $\{U_P\}_{P \in X}$, where U_P is an affine open neighborhood of P. Since varieties are quasi-compact, we may then find finite set of points $P_1, \ldots, P_n$ such that $\{U_{P_i}\}_1^n$ is a covering of X.

Lemma 5.2.55. *Let $X \subset \mathbb{P}^n$ be a projective variety and H a hyperplane in $\mathbb{P}^n$ not containing X.*

1. *There is an $X_* \subset \mathbb{A}^n$ isomorphic to $X \setminus (H \cap X)$.*
2. *if L is a linear form defining H, then $\Gamma(X_*) \simeq K[x_1/L, \ldots, x_{n+1}/L] \subset K(x_1, \ldots, x_{n+1})$.*

Proof. 1. $X \cap H$ is closed in X, hence $Y := X \setminus (X \cap H)$ is an open subvariety of X. Write $H = V\left(\sum_1^{n+1} a_i x_i\right)$ and let A be the projective change of coordinates such that $H^A = H_\infty$. Then $Y \simeq Y' := X' \setminus (X' \cap H_\infty)$ for some $X' \subset \mathbb{P}^n$ projective variety not containing H_∞. Note that $Y' = X' \cap U_{n+1}$, hence $X_* := \varphi_{n+1}^{-1}(Y') \subset \mathbb{A}^n$ is an affine variety and φ_{n+1}, defines an isomorphism of Y' with X_*, hence $\phi := \varphi_{n+1} \circ A^{-1}$ is an isomorphism of Y with X_*.
2. Note that

$$\phi([v]) = (v_1/l(v), \ldots, v_n/l(v))$$

for every $[v] \in Y$. We thus get a K-algebra isomorphism

$$\begin{aligned} \widetilde{\varphi} : \Gamma(X_*) &\to \Gamma(Y) \\ z &\mapsto z\left(\frac{x_1}{l}, \ldots, \frac{x_n}{l}\right) \end{aligned}$$

where l is the image of L in $\gamma_h(X)$. It thus follows that $\Gamma(X_*) \simeq \Gamma(Y) = K\left[\frac{x_1}{l}, \ldots, \frac{x_n}{l}\right] = K\left[\frac{x_1}{l}, \ldots, \frac{x_{n+1}}{l}\right]$. □

Lemma 5.2.56. *Let X be a variety and P, Q points on X. There is an open affine subvariety in X containing P and Q.*

Proof. As in Corollary 5.2.53 assume WLOG $X \subset \mathbb{P}^n$. By a change of coordinates we may arrange that P, Q are in the same open affine patch in $\mathbb{P}^n$, so we may assume that $X \subset \mathbb{A}^n$ is affine. Let $U \ni P$, $U' \ni Q$ be open neighborhoods in X and set $U'' := U \cup U'$. Then $X \setminus U''$ is algebraic, hence by Corollary 3.1.43 3. we can pick $G_1, G_2 \in K[\mathbf{x}]$ such that $G_i(P), G_i(Q) = 1$ and G_i vanishing on every point in $X \setminus U''$. Consider $g_i := G_i + I(X)$. Then $P, Q \in X_{g_1 g_2} \subset U''$ and we are done. □

Remark 5.2.57. By induction we find that there is an open affine neighborhood containing k distinct points.

Lemma 5.2.58. *Let X be a variety and P, Q be two distinct points on X. There is an $f \in K(X)$ defined at both P and Q satisfying $f(P) = 0$ and $f(Q) \neq 0$. It thus follows that $\mathcal{O}_P(X)$ and $\mathcal{O}_Q(X)$ are distinct subrings of $K(X)$.*

Proof. There is some open affine variety U in X that is a neighborhood of P and Q. Then $U \simeq V$ for some affine variety $V \subset \mathbb{A}^n$. By Corollary 3.1.43 there is a polynomial $g \in K[x_1, \ldots, x_n]$ such that $g(\varphi(P)) = 0$ and $g(\varphi(Q)) = 1$. Since $\widetilde{\varphi} : \Gamma(V) \to \Gamma(U)$ is an isomorphism, there is an $f \in \Gamma(U)$ such that $f = \widetilde{\varphi}(g)$, hence $f(P) = \widetilde{\varphi}(g)(P) = 0$ and $f(Q) = \widetilde{\varphi}(g)(Q) = 1$. Then $f \in \mathfrak{m}_P(X)$ and hence $\frac{1}{f} \notin \mathcal{O}_P(X)$, while $f \notin \mathfrak{m}_Q(X)$, meaning $\frac{1}{f} \in \mathcal{O}_Q(X)$. □

Lemma 5.2.59. *Suppose $\varphi : X \to Y$ is a surjective morphism. Then $\widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$ is injective.*

Proof. Suppose $f \in \ker \widetilde{\varphi}$ and let $P \in X$. Note that $P = \varphi(Q)$ for some $Q \in Y$. Then $f(P) = f(\varphi(Q)) = (\widetilde{\varphi}(f))(Q) = 0$, hence $f = 0$. □

Lemma 5.2.60. *Consider the following commutative diagram of sets*

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\psi} & \\ Y & \xrightarrow{\phi} & Z \end{array}$$

1. *If X, Y, Z are topological space, π, ψ are continuous and π is open, then ϕ is continuous.*

2. *If, in addition, X, Y, Z are varieties, π, ψ are morphisms and π is surjective, then ϕ is a morphism.*

Proof. 1. Let $U \subset Z$ be open. Then $\pi^{-1}(\phi^{-1})(U)) = \psi^{-1}(U)$ is open, hence using that π is open, $\phi^{-1}(U) = \pi(\pi^{-1}(\phi^{-1}(U)))$ is open.
2. By 1. ϕ is continuous. Let $U \subset Z$ be open and let

$$\alpha : \operatorname{im} \widetilde{\pi} \subset \Gamma\left(\psi^{-1}(U)\right) = \Gamma\left(\pi^{-1}\left(\phi^{-1}(U)\right)\right) \to \Gamma\left(\phi^{-1}(U)\right)$$

be the inverse of $\widetilde{\pi}$ as a function onto its image, which exists by the assumption that π is surjective due to the prior lemma. Let $f \in \Gamma(U)$. Then

$$(\widetilde{\phi}(f))(P) = (\alpha\widetilde{\pi}\widetilde{\phi}(f))(P) = (\alpha\widetilde{\psi}(f))(P),$$

and since $\alpha\widetilde{\psi}(f) \in \Gamma\left(\phi^{-1}(U)\right)$, the result follows. □

Lemma 5.2.61. *The map $\varphi^{(n)} := \varphi : H_\infty \subset \mathbb{P}^n \to \mathbb{P}^{n-1}, [v_1, \ldots, v_n, 0] \mapsto [v_1, \ldots, v_n]$ is an isomorphism. If V is a variety in $\mathbb{P}^n$ contained in H_∞, then $\varphi|_V$ is an isomorphism to it image, which is a variety in $\mathbb{P}^{n-1}$. As a consequence, any projective variety is isomorphic to some closed subvariety $V \subset \mathbb{P}^n$ not contained in a hyperplane.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{n-1} \setminus 0 \times \{0\} & \xrightarrow{\phi} & \mathbb{A}^{n-1} \setminus 0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ H_\infty & \xrightarrow{\varphi} & \mathbb{P}^{n-1} \end{array}$$

where ϕ is the isomorphism $(v, 0) \mapsto v$, and π, τ are the appropriate restrictions of the quotient maps, which we note are surjective, open morphisms. Checking that φ is a bijection is easy. The prior lemma shows that φ and φ^{-1} are morphisms. We know that isomorphisms map varieties to varieties, hence the second statement follows. The third statement follows from any projective variety being isomorphic to some projective variety $V \subset \mathbb{P}^n$ for some n. V can be contained in at most n hyperplanes (cf. Lemma 4.1.54) call these $H_1, \ldots, H_m$. Alternately applying an appropriate change of coordinates and an appropriate choice of $\varphi^{(i)}$, we get that the image of V under this composition is a closed subvariety of $\mathbb{P}^{n-m}$ isomorphic to V. □

5.3 Developing Theory

5.3.1 Products & Graphs

Proposition 5.3.1. *Let $V \subset A$ and $W \subset B$ be closed subvarieties of some multispaces A and B. Then $V \times W \subset A \times B$ is a closed subvariety.*

Proof. That $V \times W$ is a closed set in $A \times B$ is obvious since it is just given by the vanishing set of the polynomials defining V and W considered as elements of a larger polynomial ring. Suppose $V \times W = X_1 \cup X_2$ with $X_i \subset A \times B$ closed. Set

$$U_i := \{w \in W : V \times \{w\} \not\subset X_i\}.$$

Note that in general if a closed set Z is contained in the union of two closed sets Z_1 and Z_2 but not in either of the two sets by themselves, then $Z = (Z_1 \cap Z) \cup (Z_2 \cap Z)$ is a composition of Z into the union of two non-trivial closed sets, meaning Z is reducible. Since $V \times \{w\}$ is a closed variety for every $w \in W$, it follows that $U_1 \cap U_2 = \emptyset$. Since W is a variety and no points in a variety can be separated by open sets, it follows that if U_1 and U_2 are open, then U_1 or U_2 is empty, hence $V \times W = X_1$ or $V \times W = X_2$. To prove that U_i is open, note that for $X_i = V(F_1, \ldots, F_k)$, there is for each $w \in U_i$ a j and a $v \in V$ such that $F_j(v, w) \neq 0$. Set $G_j := F_j(v, y)$. Then $w \in \{u \in W : G_j(u) \neq 0\} \subset U_i$, which is an open neighborhood of w in U_i, hence U_i is open. □

Remark 5.3.2. The product of open subvarieties $X \times Y \subset V \times W$ is an open subvariety in $A \times B$ since

$$A \times B = A \times W \cap V \times B$$

and $V \times W \setminus V \times B = V \times W \setminus B$ which is closed. A similar argument shows that $A \times W$ is closed.

Proposition 5.3.3. *1. If $\varphi : Z \to X$ and $\phi : Z \to Y$ is a morphism, then $(\varphi, \phi) : Z \to X \times Y, z \mapsto (\varphi(z), \phi(z))$ is a morphism.*

2. If $\psi : X' \to X$ and $\xi : Y' \to Y$ is a morphism, then $\psi \times \xi : X' \times Y' \to X \times Y, (x, y) \mapsto (\psi(x), \xi(y))$ is a morphism.

3. The diagonal:

$$\Delta_X := \{(x, y) \in X \times X : x = y\},$$

is a closed subvariety of $X \times X$. The map

$$\delta_X : X \to \Delta_X$$
$$x \mapsto (x, x)$$

is an isomorphism.

Proof. 1. It is sufficient to prove the statement in the case $X := \mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $Y = \mathbb{A}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$. Covering X by $U_{n,i_1,\ldots,i_k} := \mathbb{A}^n \times U_{1i_1} \times \cdots \times U_{mi_m}$ and Y by $U_{m,j_1,\ldots,j_l} := \mathbb{A}^m \times U_{1j_1} \times \cdots \times U_{lj_l}$ we get an open covering $X \times Y$ by $\{U_{n,i_1,\ldots,i_k} \times U_{m,j_1,\ldots,j_l}\}$ and hence an open covering of Z by $\{\varphi^{-1}(U_{n,i_1,\ldots,i_k}) \cup \phi^{-1}(U_{m,j_1,\ldots,j_l})\}$ satisfying the condition. So we may assume $X = \mathbb{A}^n$ and $Y = \mathbb{A}^m$ (cf. Lemma 5.2.36). We can cover Z by open affine varieties $\{U_\alpha\}$. Setting $V_\alpha := X \times Y$, again applying Lemma 5.2.36, we may assume that $Z \subset \mathbb{A}^r$ is affine. Then $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\phi = (\phi_1, \ldots, \phi_m)$ are

polynomial maps, hence $(\varphi, \phi) = (\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_m)$ is a polynomial map.
2. Consider the morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Then $(\psi\pi_1, \xi\pi_2)(v, w) = (\psi\pi_1(v, w), \xi\pi_2(v, w)) = (\psi(v), \xi(w)) = \psi \times \xi(v, w)$, hence $\psi \times \xi$ is a morphism by 1.
3. In the case $X = \mathbb{P}^n$, $\Delta_X = V(x_i y_j - x_j y_i : 1 \leq i, j \leq n+1)$. In the general case we may assume that $X \subset V \subset \mathbb{P}^n$ for some n and some closed subvariety V, hence $X = V(x_i y_j - x_j y_i : 1 \leq i, j \leq n+1) \cap V$ is closed. $\delta_X = (\mathrm{id}_X, \mathrm{id}_X)$ is a morphism whose inverse is π_1. It follows that Δ_X is irreducible. □

Corollary 5.3.4. *If $\varphi, \phi : X \to Y$ are morphisms, then $\{x \in X : \varphi(x) = \phi(x)\}$ is closed in X. If φ and ϕ agree on a dense set, then $\varphi = \phi$.*

Proof. The set in question is equal to $(\varphi, \phi)^{-1}(\Delta_Y)$. Let A be a dense set on which φ and ϕ agree. Then $A \subset \{x \in X : \varphi(x) = \phi(x)\}$, hence $X = \mathrm{cl}(A) = \mathrm{cl}(\{x \in X : \varphi(x) = \phi(x)\}) = \{x \in X : \varphi(x) = \phi(x)\}$. □

Definition 5.3.5. Let $\varphi : X \to Y$ be a morphism. *The graph of* φ is the set

$$G_\varphi := \{(x, f(x)) \in X \times Y : x \in X\}.$$

Proposition 5.3.6. *Let $\varphi : X \to Y$ be a morphism. $G_\varphi \subset X \times Y$ is a closed subvariety and $\pi|_{G_\varphi} : G_\varphi \to X$ is an isomorphism.*

Proof. One sees that $G_\varphi = (\varphi \times \mathrm{id}_Y)^{-1}(\Delta_Y)$. The inverse to $\pi|_{G_\varphi} : G_\varphi \to X$ is (id_X, φ). □

Lemma 5.3.7. *Let $\varphi : X \to Y$ be a morphism of varieties.*

1. *Suppose $\varphi(X)$ is dense in Y. Then $\widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$ is injective*
2. *Suppose Y is affine. Then $\varphi(X)$ is dense in Y if and only if φ is injective*

Proof. 1. Suppose $f \in \Gamma(Y)$ is given such that $0 = \widetilde{\varphi}(f) = f \circ \varphi$. Then 0 and f agree on $\varphi(X)$, hence by Corollary 5.3.4 they agree on all of Y, i.e. $f = 0$.
2. " $\Longrightarrow$ ": follows from 1.
" $\Longleftarrow$ ": Assume WLOG $Y \subset \mathbb{A}^m$ is a closed subvariety and there is $P \in Y \setminus \mathrm{cl}_Y(\varphi(X))$. We may then pick $f \in \Gamma(Y)$ vanishing on every point in $\mathrm{cl}_Y(\varphi(X))$ and $f(P) = 1$. Then $0 \neq f \in \ker \widetilde{\varphi}$. □

Remark 5.3.8. When Y is not affine then the "if"-part of the 2. is not true. Indeed, consider $\varphi : \mathbb{P}^n \to \mathbb{P}^{n+1}, [v] \mapsto [v, 0]$ whose image is the algebraic set $H_\infty \subsetneq \mathbb{P}^{n+1}$, but $\widetilde{\varphi} : \Gamma(\mathbb{P}^{n+1}) = K \to K = \Gamma(\mathbb{P}^n), a \mapsto a$ is injective.

Proposition 5.3.9. *Let U, V be open subvarieties of a variety X.*

1. *Then $U \cap V \simeq (U \times V) \cap \Delta_X$.*

2. *Suppose U and V are affine. Then $U \cap V$ is affine.*

Proof. 1. Indeed, define $\iota_U : U \cap V \to U, u \mapsto u$ and $\iota_V : U \cap V \to V, v \mapsto v$. Note that $\iota_U(x) = \iota_V(x)$ for every $x \in U \cap V$, hence $(\iota_U, \iota_V)(x) \in \Delta_X$ and clearly $(\iota_U, \iota_V)(x) \in U \times V$. Define $\varphi := \pi_1|_{U \times V \cap \Delta_X}$. This is easily seen to be the inverse of (ι_U, ι_V).
2. Pick $V', W' \subset \mathbb{A}^n$ affine varieties such that $V \overset{\varphi}{\simeq} V'$ and $W \overset{\phi}{\simeq} W'$. The inverse of $\varphi \times \phi|_{U \times V} \cap \Delta_X$ is $\varphi^{-1} \times \phi^{-1}|_{U' \times V' \cap \Delta_{\mathbb{A}^{2n}}}$ By 1. $U \cap V \simeq U \times V \cap \Delta_X \simeq V' \times W' \cap \Delta_{\mathbb{A}^{2n}}$. □

Proposition 5.3.10. *Let $d \geq 1$ and set $N := \frac{(d+1)(d+2)}{2}$ and let $M_1, \ldots, M_d$ be the monomials generating $V(d, 3)$. Consider $V := V\left(\sum_1^N M_i t_i\right) \subset \mathbb{P}^2 \times \mathbb{P}^{N-1}$. Let π denote the morphism given by restriction to V of the projection onto the first coordinate. For each $[v] \in \mathbb{P}^{N-1}$, let $C_{[v]} := V\left(\sum_1^N v_i M_i\right) \subset \mathbb{P}^{N-1}$ denote the associated curve. Then V is an irreducible algebraic set in $\mathbb{P}^2 \times \mathbb{P}^{N-1}$ and $\pi^{-1}([v]) = C_{[v]} \times \{[v]\}$, hence every curve can be identified with some fiber under π.*

Proof. V being irreducible follows from the polynomial in question being linear in $K[x, y, z][t_1, \ldots, t_N]$. Suppose $(P, [v]) \in \pi^{-1}([v]) \subset V$. Then $P \in V\left(\sum_1^N v_i M_i\right)$. Conversely if $(P, [v]) \in C_{[v]} \times \{[v]\}$, then $P \in V\left(\sum_1^N v_i M_i\right)$, hence $(P, [v]) \in V$, hence $(P, [v]) \in \pi^{-1}([v])$. □

The following lemma is useful

Lemma 5.3.11. *(Main Theorem of Elimination Theory) Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed. Then $\pi_1(Z)$ is closed. It follows that for any variety $X \subset \mathbb{P}^n \times \mathbb{P}^m$, if $\pi_1(X)$ is a variety, then closed subvarieties of X are mapped to a closed set by π_1.*

Proof. Since we can cover $\mathbb{P}^n \times \mathbb{P}^m$ by sets $U_i \times \mathbb{P}^m \simeq \mathbb{A}^n \times \mathbb{P}^m$ it is sufficient to prove that closed sets in $\mathbb{A}^n \times \mathbb{P}^m$ are projected to closed sets. Let $V = V(f_1, \ldots, f_k) \subset \mathbb{A}^n \times \mathbb{P}^m$ be closed where f_i is homogeneous in $K[\mathbf{x}][\mathbf{y}]$. We prove that the complement of $\pi_1(V)$ is open. Note that $v \notin \pi_1(V)$ if and only if for every $[w] \in \mathbb{P}^m$ there is an $i_{[w]}$ such that $[w]$ is not a zero of $f_{i_{[w]}}$, or equivalently $V(f_1(v, \mathbf{y}), \ldots, f_k(v, \mathbf{y})) = \emptyset$. By the projective Nullstellensatz we then get that $v \notin \pi_1(V)$ if and only if there is a $d \geq 0$ such that $\langle y_1, \ldots, y_{m+1} \rangle^d \subset \langle f_1(v, \mathbf{y}), \ldots, f_k(v, \mathbf{y}) \rangle$, hence

$$\mathbb{A}^n \setminus \pi_1(V) = \bigcup_{d \geq 0} A_d$$

where $A_d := \left\{v \in \mathbb{A}^n : \langle y_1, \ldots, y_{m+1}\rangle^d \subset \langle f_1(v, \mathbf{y}), \ldots, f_k(v, \mathbf{y})\rangle\right\}$. It is therefor sufficient to prove that A_d is open for each $d \geq 0$. For $l < 0$, $q \geq 1$, and a commutative ring R, set $V_R(l, q) := 0$. Fix $d \geq 0$ and set $d_i := \deg_{\mathbf{y}} f_i$ for each i. Define

$$T^{(d)} : \bigoplus_1^k V_{K[\mathbf{x}]}(d - d_i, m) \to V_d$$

$$(g_1, \ldots, g_k) \mapsto \sum_1^k f_i g_i$$

which is a $K[\mathbf{x}]$-linear map, and hence induced by some matrix $(T_{ij}^{(d)}) \in M_{n_d \times m_d}(K[\mathbf{x}])$. For each $v \in \mathbb{A}^n$, $v \in A_d$ if and only if $T^{(d)}(v) = (T_{ij}^{(d)}(v))$ is surjective, which is equivalent to the existence of m_d linearly independent rows of $(T_{ij}^{(d)}(v))$, i.e. the existence of a non-zero minor of $(T_{ij}^{(d)}(v))$ whose determinant is non-zero. We thus get that

$$A_d = \bigcup_{M \text{ minor of } (T_{ij}^d)} \mathbb{A}^n \setminus V(\det\ M),$$

which shows that A_d is open. □

Lemma 5.3.12. *The image of a morphism $\kappa : X \subset \mathbb{P}^n \to Y \subset \mathbb{P}^m$ is closed*

Proof. By Lemma 5.3.6 the graph of κ, G_κ, is a closed subvariety of $X \times Y$, by the prior lemma it follows that im $\kappa = \pi_2(G_\kappa)$ is closed. □

5.3.2 A Necessary and Sufficient Condition for the Existence Final Syzygies over $\mathbb{C}$

In Theorem 3.2.40 we saw a Gröbner basis method for checking whether a final syzygy exists for some finite set of polynomials. One may also find topological conditions that are necessary and sufficient for the existence of a final syzygy. For polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ we for purposes of this discussion define the polynomial map

$$\varphi := \varphi_{f_1,\ldots,f_n} : \mathbb{A}^n \to \mathbb{A}^m$$
$$v \mapsto (f_1(v), \ldots, f_m(v))$$

We can state the weak Nullstellensatz in the following way

Theorem 5.3.13. *Consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. These polynomials admit a final polynomial if and only if $0 \notin$ im φ.*

We can deform the setup this theorem to one about final syzygies:

Lemma 5.3.14. *Let K be an infinite field and consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. These polynomials admit a final syzygy if and only if $0 \notin$ cl(im φ)*

Proof. " $\implies$ ": Suppose $0 \in \text{cl}(\text{im } \varphi)$. Let $p \in K[y_1, \ldots, y_m]$ be given such that $p(f_1, \ldots, f_m) = 0$. Then p vanishes on im φ, hence im $\varphi \subset V(p)$. It follows that

$$\text{cl}(\text{im } \varphi) \subset V(p),$$

hence $p(0) = 0$, meaning p is not a final syzygy. It follows that there are no polynomial in $K[\mathbf{y}]$ satisfying both conditions for being a syzygy.
" $\impliedby$ ": Suppose $0 \notin \text{cl}(\text{im } \varphi)$. Then for some $0 \in U = \mathbb{A}^m \setminus V(p_1, \ldots, p_k) \subset \mathbb{A}^m$ open, $U \cap \text{im}\varphi = \emptyset$, hence im $\varphi \subset V(p_1, \ldots, p_k)$, hence for each i, $(f_1(v), \ldots, f_m(v)) \in$ im φ

$$p_i(f_1(v), \ldots, f_m(v)) = 0,$$

implying that $p_i(f_1, \ldots, f_m) = 0$ since K is infinite. Note that for some j, $p_j(0) \neq 0$ since $0 \notin \mathbb{A}^m \setminus U = V(p_1, \ldots, p_m)$, which means $p := p_j$ is a final syzygy for $f_1, \ldots, f_m$. $\square$

In the special case $K = \mathbb{C}$, one can reformulate the above condition to one about the euclidean topology on $\mathbb{A}^m$. To do this need to consider how the projective Zariski topology on $\mathbb{P}^m$, the affine Zariski topology on $\mathbb{A}^m$, the euclidean quotient topology on $\mathbb{P}^m$ and the euclidean topology on $\mathbb{A}^m$ relate to each other. In the canonical way we may consider φ as a map $U_{n+1} \to U_{m+1}$. Moreover, we can extend φ to a map

$$\begin{aligned} \phi : \mathbb{P}^n = U_{n+1} \sqcup H_\infty &\to \mathbb{P}^m \\ [v] &\mapsto [v_{n+1} f_1^*(v), \ldots, v_{n+1} f_m^*(v), 1] \end{aligned}$$

Indeed, for $P = [v_1, \ldots, v_n, 1] \in U_{n+1}$, $\phi(P) = [f_1^*(v), \ldots, f_m^*(v), 1] = [f_1(v), \ldots, f_m(v), 1]$. Note for $P \in H_\infty$, $\phi(P) = [0, \ldots, 0, 1]$. This is a morphism since it fits in the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{n+1} \setminus 0 & \xrightarrow{\psi} & \mathbb{A}^m \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \varphi_{m+1}} \\ \mathbb{P}^n & \xrightarrow{\phi} & \mathbb{P}^m \end{array}$$

where $\psi : v \mapsto (v_{n+1} f_1^*(v), \ldots, v_{n+1} f_m^*(v))$, due to Lemma 5.2.60. For the next result we will need some notation. Let the Zariski closure in $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{A}^m(\mathbb{C})$ be denoted by $\text{cl}_\mathcal{Z}$ and the closure with respect to respectively the quotient topology on $\mathbb{P}^n(\mathbb{C})$ induced by the Euclidean topology on $\mathbb{A}^{m+1}(\mathbb{C}) \setminus 0$ and the Euclidean topology on $\mathbb{A}^m(\mathbb{C})$ be denoted by $\text{cl}_\mathcal{E}$. We write $\text{cl}_{\mathcal{Z},A}$, $\text{cl}_{\mathcal{E},A}$ for the closure in some affine or projective subset A with respect to respectively the Zariski and Euclidean topologies.

Lemma 5.3.15. *Set $K = \mathbb{C}$ and let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then* $\text{cl}_\mathcal{Z}(\text{im } \varphi) = \text{cl}_\mathcal{E}(\text{im } \varphi)$.

Proof. Note that im ϕ is Zariski closed by Lemma 5.3.1 and therefor also closed with respect to the Euclidean quotient topology. We then get that

$$\mathrm{cl}_{\mathcal{Z}}(\phi(\mathrm{cl}_{\mathcal{Z}}(U_{n+1}))) = \mathrm{cl}_{\mathcal{Z}}(\phi(\mathbb{P}^n)) = \mathrm{cl}_{\mathcal{E}}(\phi(\mathbb{P}^n)) = \mathrm{cl}_{\mathcal{E}}(\phi(\mathrm{cl}_{\mathcal{E}}(U_{n+1}))),$$

by continuity of ϕ in both topologies, we have that $\phi(\mathrm{cl}(A)) \subset \mathrm{cl}(\phi(A)) \subset \mathrm{cl}(\phi(\mathrm{cl}(A)))$, hence $\mathrm{cl}(\phi(A)) = \mathrm{cl}(\phi(\mathrm{cl}(A)))$, hence

$$\mathrm{cl}_{\mathcal{Z}}(\phi(U_{n+1})) = \mathrm{cl}_{\mathcal{E}}(\phi(U_{n+1})).$$

We then get that

$$\mathrm{cl}_{\mathcal{Z}}(\text{im } \varphi) = \varphi_{m+1}(\mathrm{cl}_{\mathcal{Z}}(\phi(U_{n+1})) \cap U_{m+1}) = \varphi_{m+1}(\mathrm{cl}_{\mathcal{E}}(\phi(U_{n+1})) \cap U_{m+1}) = \mathrm{cl}_{\mathcal{E}}(\text{im } \varphi),$$

where we use the fact that φ_{m+1} is a homeomorphism of $\mathbb{A}^m$ with U_{n+1} in both topologies. □

Theorem 5.3.16. *Set $K = \mathbb{C}$ and consider $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$. Then $f_1, \ldots, f_m$ admit a final syzygy if and only if $0 \notin \mathrm{cl}_{\mathcal{E}}(\text{im } \varphi)$*

Proof. This follows from Lemma 5.3.14 and the prior lemma □

Definition 5.3.17. A system of polynomial equations over $\mathbb{C}$, $V(f_1, \ldots, f_m) \subset \mathbb{A}^n(\mathbb{C})$ is called *stably inconsistent* if there is a $\epsilon > 0$ such that for every $\delta_1, \ldots, \delta_m \in (-\epsilon, \epsilon)$, $V(f_1 + \delta_1, \ldots, f_m + \delta_m)$

Remark 5.3.18. One thus find that $f_1, \ldots, f_m$ have a final syzygy if and only if $V(f_1, \ldots, f_m)$ is stably inconsistent.

Example 5.3.19. Some examples

5.3.3 A Little Something about Algebraic Groups

Definition 5.3.20. A variety V is called an *algebraic group* if $(V, +)$ is a group such that $V \times V \to V, (v, w) \mapsto vw$ and $V \to V, v \mapsto v^{-1}$ are morphisms.

Example 5.3.21.

1. $\mathbb{A}^1 = K$ with addition is an algebraic group. Indeed, $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (a, b) \mapsto a + b$ is a polynomial map and so is $\mathbb{A}^1 \to \mathbb{A}^1, a \mapsto -a$. This group is also denoted $\mathbb{G}_a$.

2. $\mathbb{A}^1 \setminus 0$ with multiplication is an algebraic group. Multiplication is just a restriction of the polynomial map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (a, b) \mapsto ab$. Let $V := V(f_1, \ldots, f_m) \setminus 0 \subset$

$\mathbb{A}^1 \setminus 0$. For each i pick $n_i \geq 0$ such that $x^{n_i} f(1/x) \in K[x]$. Then if a is in the preimage of V under $\bullet^{-1}$, a^{-1} is a zero of each f_i, hence a is a zero of $f(1/x)$ and therefor also of $x^{n_i} f(1/x)$. Let $a \in V(x^{n_i} f_i(1/x) : 1 \leq i \leq m) \setminus 0$. Then $a^{n_i} f_i(a^{-1}) = 0$ for each i, hence $f_i(a^{-1}) = 0$, hence a is in the preimage of V under $\bullet^{-1}$. Let $U \subset \mathbb{A}^1 \setminus 0$ be open. Define

$$\alpha : \Gamma(U) \to \Gamma\left(\text{im}^{-1} \bullet^{-1}(U)\right)$$
$$f \mapsto f(1/x)$$

is a well-defined K-algebra map and it is readily verified to agree with $\widetilde{\bullet^{-1}}$. We also denote this group $\mathbb{G}_m$.

3. $\mathbb{A}^n$ is an algebraic group with addition. Indeed, $(v, w) \mapsto v + w = (v_1 + w_1, \ldots, v_n + w_n)$ is a polynomial map and so is $v \mapsto -v = (-v_1, \ldots, -v_n)$.

4. The group $\text{GL}_n(K)$ is an algebraic group. Indeed, $\text{GL}_n(K) = M_n(K) \setminus V(\det(x_{ij})) \subset M_n(K) = \mathbb{A}^{2n}$. One sees that $((a_{ij}), (b_{ij})) \mapsto (\sum_k a_{ik} b_{kj})$ is the restriction of a polynomial map where the i, j'th coordinate function is defined by the polynomial $\sum_k x_{ik} y_{kj}$. Note that $A \mapsto \text{adj}(A)$ is a polynomial map hence this restricts to a morphism on $\text{GL}_n(K)$. To prove that $A \mapsto A^{-1}$ defines a morphism, it, due to Cramer's rule, suffices to prove that $A \mapsto 1/\det(A)$ is a morphism, for then the operation of taking inverse is given by the composition,

$$\mu \circ (1/\det, \text{adj}),$$

where $\mu : (a, v) \mapsto av$ is scalar multiplication which is clearly the restriction of a polynomial map. Note that det is a polynomial map and that $1/\det = \nu \circ \det|_{\text{GL}_n(K)}$, where $\nu : \mathbb{A}^1 \setminus 0 \to \mathbb{A}^1 \setminus 0, a \mapsto a^{-1}$ which we know to be a morphism from 2. It follows that $1/\det$ is a morphism.

5.3.4 Dimension of Varieties

In this section we develop the notion of dimension to our new general notion of variety. The definition is obvious.

Definition 5.3.22. Let X be a variety. Then *the dimension of* X, denoted dim X, is defined to be $\text{trdeg}_K\ K(X)$.

Remark 5.3.23. A one-dimensional variety is called a curve and a two-dimensional variety is called a surface. This will make sense in a second.

Lemma 5.3.24. *1. Let $U \subset X$ be an open subvariety of a variety X. Then* dim U = dim X.

2. *Let V be an affine variety. Then* dim V = dim V^*.

3. *The only zero-dimensional varieties are points.*

4. *The only proper closed sub varieties of curves are points.*

5. *The only one-dimensional varieties in $\mathbb{A}^2$ (resp. $\mathbb{P}^2$) are affine (resp. projective) curves.*

Proof. 1. follows from the fact $K(U) = K(X)$ by definition.
2. follows from the fact that $K(V) \simeq K(V^*)$.
3. Any variety is isomorphic to some open projective subvariety of projective variety V and the dimension these varieties coincide. Moreover, V is isomorphic to some W^* for some affine variety W whose dimension coincide with that of W. So all in all we lose no generality by assuming that our given variety, X say is affine.
If dim $X = 0$, then $K(X) \supset K$ is algebraic, so by Lemma 1.9.21, $K(X) = K$, hence $\Gamma(X) = K$. By a corollary of the Nullstellensatz we already know that X is a point if and only if $K = \Gamma(X)$. This fact also proves the converse implication.
4. Again we may assume a curve V is affine. Let $W \subset V$ be a proper, closed subvariety. Then $\Gamma(W) \simeq \Gamma(V)/I_V(W)$. Then we are in a situation where we can apply 3. of the prior lemma: Let $R = \Gamma(V)$, $L = K(V)$. Since $\emptyset \neq W \subsetneq V$, $0 \subsetneq I_V(W) \subsetneq \Gamma(V)$. Then $\Gamma(V)/I_V(W) \simeq K$. So $\dim(K(W)) = 0$ and W is a point.
5. We can always reduce to the case $V \subset \mathbb{A}^2$ or $V \subset \mathbb{P}^2$. In the case $V \subset \mathbb{A}^2$, V is a point, a plane curve $V = V(f)$, or $V = \mathbb{A}^2$. If dim $V = 1$, then V is not a point, since this would be 0 dimensional and it is not the entire plane since this is a surface. Suppose $V = V(f)$ is a plane curve. Note that $K(V) = K(x + I(V), y + I(V))$. Note that $f(x + I(V), y + I(V)) = 0$, hence $K(x + I(V), y + I(V)) \supset K(x + I(V))$ is algebraic, meaning dim $V \in \{0, 1\}$ and since V is not a point, dim $V = 1$. In the case $V \subset \mathbb{P}^2$, we apply the affine case to V_*. □

Proposition 5.3.25. *Let Y be a closed subvariety of a variety X.* dim $Y \leq$ dim X *and* dim X = dim $Y \iff X = Y$.

Proof. We can reduce the problem to $Y \subset X \subset \mathbb{P}^n$, and then again to $Y \subset X \subset \mathbb{A}^n$ where X is also an affine variety by the prior lemma. We have already proven this case in Lemma 3.4.71. □

Proposition 5.3.26. *Let $M \supset L$ be a module-finite field extension over a characteristic 0 field L. Let $V \subsetneq \mathbb{A}^n(K)$ (K is still algebraically closed) be an algebraic set. Then $M = L\left(\sum_1^n \lambda_i a_i\right)$ for some $(\lambda_1, \ldots, \lambda_n) \in \mathbb{A}^n \setminus V$.*

Proof. The case $n = 1$ is trivial. Assume the statement true for some $n \geq 1$. Consider $M = L(a_1, \ldots, a_{n+1})$. Write $V = V(f_1, \ldots, f_m)$. Since V is proper we may WLOG assume that there is a $\mu_{n+1} \neq 0$ and an i such that $f_i(x_1, \ldots, x_n, \mu_{n+1}) \neq 0$, i.e. setting $g_j := f_j(x_1, \ldots, x_n, \mu_{n+1})$, $W := V(g_1, \ldots, g_m)$ is a proper algebraic set in $\mathbb{A}^n$. In the proof of the prime element theorem we saw that we can choose $(\lambda_1, \ldots, \lambda_n) \in \mathbb{A}^n$ such that $c := \sum_1^n \lambda_i a_i$ is such that $L(a_1, \ldots, a_n) = L(c)$ and $M = L(\nu c + \mu_{n+1} a_{n+1})$ for some $\nu \neq 0$. By induction there is also $(\mu_1, \ldots, \mu_n) \in \mathbb{A}^n \setminus W$ where $c' := \sum_1^n \mu_i a_i$ is such that $L(a_1, \ldots, a_n) = L(c') = L(\nu c)$. Then $(\mu_1, \ldots, \mu_n, \mu_{n+1}) \in \mathbb{A}^{n+1} \setminus V$ and

$$M = L(c', a_{n+1}) = L(\nu c + \mu_{n+1} a_{n+1}) = L(c' + \mu_{n+1} a_{n+1}) = L\left(\sum_1^{n+1} \mu_i a_i\right).$$

□

Proposition 5.3.27. *Consider a function field $K(\alpha_1, \ldots, \alpha_n) \supset K$ in r variables. There is an affine variety $V \subset \mathbb{A}^n$ such that we may identify $K(\alpha_1, \ldots, \alpha_n)$ with $K(V)$.*

Proof. $K[\alpha_1, \ldots, \alpha_n] \simeq K[x_1, \ldots, x_n]/\ker \mathrm{ev}_{\alpha_1, \ldots, \alpha_n}$, and since $K[\alpha_1, \ldots, \alpha_n]$ is an integral domain, $\ker \mathrm{ev}_{\alpha_1, \ldots, \alpha_n}$ is prime, so upon setting $V := V(\ker \mathrm{ev}_{\alpha_1, \ldots, \alpha_n})$, we are done. □

Proposition 5.3.28. *(*char $K = 0$*). In the same setup as above, we may find an affine variety $V \subset \mathbb{A}^{r+1}$ such that $K(\alpha_1, \ldots, \alpha_n)$ may be identified with $K(V)$.*

Proof. We may find $y_1, \ldots, y_r \in K(\alpha_1, \ldots, \alpha_n)$ that are algebraically independent over K. Then $K(\alpha_1, \ldots, \alpha_n) = K(y_1, \ldots, y_r, \alpha)$ for some $\alpha \in K(\alpha_1, \ldots, \alpha_n)$. Then

$$K[x_1, \ldots, x_{r+1}]/\ker \mathrm{ev}_\alpha \simeq K[y_1, \ldots, y_r, \alpha].$$

Picking $V := V(\ker \mathrm{ev}_\alpha)$, we are done. □

5.3.5 Rational Maps & Birational Equivalence

Definition 5.3.29. Let X, Y be varieties and $U_1, U_2 \subset X$ open subvarieties. Two morphisms $\varphi_i : U_i \to Y$ are equivalent if $\varphi_1(v) = \varphi_2(v)$ for every $v \in U_1 \cap U_2$.
An equivalence class of such morphisms is called *a rational map from X to Y*.
We define the *domain* of a rational map Φ to be the union over the domain of all representatives of Φ. We denote this by dom Φ

Example 5.3.30. A regular function is an example of a rational function.

Remark 5.3.31. Given a rational map $\Phi = [\varphi_\alpha]$ with domain U pick for each $P \in U$, an α_P such that $P \in U_{\alpha_P}$. We define

$$\varphi : U \to A \times U \to Y$$
$$P \mapsto (\alpha_P, P) \to \varphi_\alpha(P)$$

If $(\alpha, P) = (\beta, Q)$, then $\varphi_\alpha(P) = \varphi_\beta(Q)$, hence the above is well-defined. It is a morphism since a $\varphi|_\alpha = \varphi_\alpha$ is a morphism for each α. Note that $\varphi \in \Phi$. Thus a rational map can equivalently be considered as a morphism $\varphi : U \to Y$ for some open subvariety in X that cannot be extended to a morphism from any larger open subset of X to Y.

Definition 5.3.32. A rational map $\Phi = [\varphi] : X \to Y$ is said to be *dominant* if $\varphi(U) \subset Y$ is dense.

Remark 5.3.33. Note that this is independent of the choice of representative of Φ. Indeed if (φ_1, U_1) and (φ_2, U_2) are two representations of Φ, then using the continuity of φ_i a couple of times (Note in particular that we use the fact that $U_1 \cap U_2 \neq \emptyset$)

$$\begin{aligned}\mathrm{cl}_Y(\varphi_1(U_1)) = \mathrm{cl}_Y(\varphi_1(\mathrm{cl}_{U_1}(U_1 \cap U_2))) = \mathrm{cl}_Y(\varphi_1(U_1 \cap U_2)) = \mathrm{cl}_Y(\varphi_2(U_1 \cap U_2)) \\ = \mathrm{cl}_Y(\varphi_2(\mathrm{cl}_{U_2}(U_1 \cap U_2))) = \mathrm{cl}_Y(\varphi_2(U_2)).\end{aligned}$$

hence if the image of one representative is dense in Y, then this is also the case for the image of any other representative.

Definition 5.3.34. Let A, B be a pair of local rings with $B \supset A$. We say that B *dominates* A if the maximal ideal of B contains the maximal ideal of A.

Proposition 5.3.35. *1. Let $\Phi : X \to Y$ be a dominant rational map. Let $U \subset X$, $V \subset Y$ be open affine varieties, $\varphi : U \to V$ a representative of Φ. Then $\widetilde{\varphi} : \Gamma(V) \to \Gamma(U)$ is injective and it extends to an injective K-algebra homomorphism, $\widetilde{\Phi} : K(Y) = K(V) \to K(X) = K(U)$. This homomorphism is independent of the choice of such a (φ, U, V).*

2. *Consider a point $P \in \mathrm{dom}\,\Phi$ and set $Q := \Phi(P)$. Then $\mathcal{O}_P(X)$ dominates $\widetilde{\Phi}(\mathcal{O}_Q(Y))$. Conversely, if $P \in X$, $Q \in Y$ and $\mathcal{O}_P(X)$ dominates $\widetilde{\Phi}(\mathcal{O}_Q(Y))$, then $P \in \mathrm{dom}\,\Phi$ and $\Phi(P) = Q$.*

3. *For every injective $\sigma \in \mathrm{Hom}^{\mathrm{K-Alg}}(K(Y), K(X))$, there is a unique dominant rational map $\Phi : X \to Y$ such that $\widetilde{\Phi} = \sigma$.*

Proof. 1. Lemma 5.3.7 shows that $\widetilde{\varphi} : \Gamma(V) \to \Gamma(U)$. Since $\widetilde{\varphi}(f) = 0 \iff f = 0$ it follows that $\widetilde{\Phi} : K(V) \to K(U), f/g \mapsto \widetilde{\varphi}(f)/\widetilde{\varphi}(g)$. Consider another representative $\varphi' : U' \to V'$ of Φ. Note that $Q(\Gamma(V \cap V')) = K(Y)$. So we may write $f \in K(Y)$ as a ratio a/b, where $a, b \in \Gamma(V \cap V')$, $b \neq 0$. Note that the induced maps of φ and φ' agree on $\Gamma(V \cap V')$, so $\frac{\widetilde{\varphi}(a)}{\widetilde{\varphi}(b)} = \frac{\widetilde{\varphi'}(a)}{\widetilde{\varphi'}(b)}$.

2. The maximal ideal in $\widetilde{\Phi}(\mathcal{O}_Q(Y))$ is equal to $\{\widetilde{\Phi}(f) \in \widetilde{\Phi}(\mathcal{O}_Q(Y)) : \widetilde{\Phi}(f)(P) = 0\}$ which is clearly a subset of the maximal ideal of $\mathcal{O}_P(X)$. Pick affine neighborhoods $V \ni P$ and $W \ni Q$. Then $\Gamma(W)$ is a ring-finite extension of K, with generators $z_1, \ldots, z_n$. Since $\widetilde{\Phi}\Big|_{\Gamma(W)} : \Gamma(W) \to K(V) = Q(\Gamma(V))$ is well-defined, we see that upon writing $\widetilde{\Phi}(z_i) = \frac{a_i}{b_i}$ for suitable $a_i, b_i \in \Gamma(V)$, $b_i(P) \neq 0$. Set $b := \prod b_i$. Then $\widetilde{\Phi}(f) = \frac{g}{b^m}$ for some $m \geq 0$, $g \in \Gamma(V)$ for every $f \in \Gamma(W)$. Then $\widetilde{\Phi}(\Gamma(W)) \subset \Gamma(V_b)$, hence $\sigma := \widetilde{\Phi}\Big|_{\Gamma(W)} : \Gamma(W) \to \Gamma(V_b)$ is well-defined. Since W and V_b are affine, there is a unique morphism $\varphi : V_b \to W$ inducing σ. Then $P \in V_b \subset \operatorname{dom} \Phi$. Let $f \in \mathfrak{m}_Q(Y) \cap \Gamma(W) = I_W(\{Q\})$. Then $\widetilde{\Phi}(f) \in \mathfrak{m}_P(X)$, since $\mathcal{O}_P(X)$ dominates $\widetilde{\Phi}(\mathcal{O}_Q(Y))$, which means $f \in \mathfrak{m}_{\Phi(P)}(X) \cap \Gamma(W) = I_W(\{\Phi(P)\})$. Then by maximality $I_W(\{Q\}) = I_W(\{\Phi(P)\})$, hence $Q = \Phi(P)$.

3. We may assume that X, Y are affine, since X and Y in any case contains open affine sets and $K(\bullet)$ maps an open subset to the same field. Let $\sigma : K(Y) \to K(X)$ be an injective K-algebra map. By restricting to $\Gamma(Y)$, as in 2. we get an injective K-algebra map $\tau = \sigma|_{\Gamma(Y)} : \Gamma(Y) \to \Gamma(X_b)$ for a suitable $b \in \Gamma(X)$. Then τ is induced by a morphism $\varphi : X_b \to Y$, where $\operatorname{cl}(\varphi(X_b)) = Y$ (cf. Lemma 5.3.7). Then setting $\Phi := [\varphi] : X \dashrightarrow Y$, we are done. □

Definition 5.3.36. A rational map $\Phi : X \to Y$ is called *birational* if there are open subvarieties $U \subset X$, $V \subset Y$ and an isomorphism $\varphi : U \to V$ that is a representative of Φ.

We say that two varieties X, Y are *birationally equivalent* if there exists a birational map $\Phi : X \to Y$. If this is the case we write $X \sim_{\text{bir}} Y$.

The following result is given to book keep the functoriality of the K-algebra maps induced by rational maps

Theorem 5.3.37. *Varieties with dominant rational maps is a category. The assignment* $(X, \Phi : Y \to Z) \mapsto (K(X), \widetilde{\Phi} : K(Z) \to K(Y))$ *to the category of algebraic function fields*$/K$ *with injective* K*-algebra maps is a fully faithful functor. Hence* $K(X) \simeq K(Y) \iff X \sim_{\text{bir}} Y$.

Proof. We define composition of two rational maps $\Phi : X \dashrightarrow Y$ and $\Psi : Y \dashrightarrow Z$ to be

$$\Psi \circ \Phi = \left[\Psi \circ \Phi|_{\Phi^{-1}(\text{dom } \Psi)}\right] : X \dashrightarrow Z,$$

where $\Phi^{-1}(\text{dom } \Psi)$ is to be interpreted to be the preimage of dom Y under the map $\Phi : \text{dom } \Phi \to Y$. One readily verifies that this operation is associative and that $[\text{id}_A : A \to A]$ is the identity with respect to this composition. Note that since we consider dominant rational maps, $\text{dom } \Psi \cap \Phi(\text{dom } \Phi) \neq \emptyset$, hence composition is never the empty map, which is not always the case when we consider varieties with rational maps. The functor is contravariant. Indeed, consider $\Phi : X \dashrightarrow Y$, $\Psi : Y \dashrightarrow Z$. Then

$$\widetilde{\Psi \circ \Phi} = \Psi \circ \widetilde{\Phi|_{\Phi^{-1}(\text{dom } \Psi)}} = \widetilde{\Phi|_{\Phi^{-1}(\text{dom } \Psi)}} \circ \widetilde{\Psi} = \widetilde{\Phi} \circ \widetilde{\Psi}.$$

In the second to last equality, we use faithfulness of the functor $(R, \sigma : S \to T) \mapsto (Q(R), \sigma : Q(S) \to Q(T))$. It is obvious that $\widetilde{\text{id}_A} = \text{id}_{K(A)}$. It follows from Proposition 5.3.35 3. that this functor is fully faithful. □

Corollary 5.3.38. *Every curve V is birationally equivalent to a plane curve V'.*

Proof. By Lemma 1.9.53 1. $K(V) = K(a, b)$. Set $I := \ker \text{ev}_{a,b} \subset K[x, y]$, which is a prime ideal. Then $V' := V(I) \subset \mathbb{A}^2$ is a variety and $K(V') = Q(\Gamma(V')) = Q(K[x, y]/I) \simeq K(a, b) = K(V)$, hence $V \sim_{\text{bir}} V'$. Then $\dim V' = 1$, hence V' is a plane curve by Lemma 5.3.24. □

Definition 5.3.39. A variety is said to be *rational* if it is birationally equivalent to some affine or projective space.

Example 5.3.40. $\mathbb{A}^n \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is isomorphic to $\mathbb{A}^n \times \mathbb{P}^{(n_1+1)\cdots(n_m+1)-1}$. In particular $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is rational.

Example 5.3.41. Let L, L' be lines in $\mathbb{P}^1$. Consider points $P_L, P_{L'}$ which L resp. L' do not pass through. Then $L \times \{P_{L'}\} \cap \{P_L\} \times L' = \emptyset$ and $L \times \{P_{L'}\} \simeq L$, $\{P_L\} \times L'$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$ since for a general isomorphism $\varphi : V \to \mathbb{P}^2$, if $\Lambda, \Lambda \subset V$ are dimension 1 closed subvarieties, then so are $\varphi(\Lambda), \varphi(\Lambda')$ in $\mathbb{P}^2$, hence $\varphi(\Lambda) \cap \varphi(\Lambda') = P$ for some $P \in \mathbb{P}^2$.

Example 5.3.42. The following is a continuation of Example 3.3.21.Consider $f = x^2 - y^3$ and $g = y^2 - z^3$ and set $V := V(f, g) \subset \mathbb{A}^3$. Consider also the bijective polynomial map $\varphi : \mathbb{A}^1 \to V, t \mapsto (t^9, t^6, t^4)$. $\Gamma(V) \to K[T]$ is injective. Note that $t = \frac{t^9}{t^8} = \widetilde{\varphi}(\frac{\overline{x}}{\overline{z}^2})$, so $K(V) \simeq K(t)$, hence φ is a birational map, so V is rational. V does not, however, have an open subvariety about $P := (0, 0, 0)$ isomorphic to an open subvariety of a plane curve:

In general, if two open subvarieties U, U' of a pair of varieties W, W' are isomorphic, then $\widetilde{\psi} : \Gamma(U) \simeq \Gamma(U')$ which restricts to an isomorphism $\mathcal{O}_P(U) = \mathcal{O}_Q(W) \simeq \mathcal{O}_{\psi(Q)}(W')$, meaning $\dim_K \mathfrak{m}_P(W)/\mathfrak{m}_P(W)^2 = \dim_K \mathfrak{m}_P(W')/\mathfrak{m}_P(W')^2$. In Example 3.4.44 we saw that $\dim_K \mathfrak{m}_P(V)/\mathfrak{m}_P(V)^2 = 3$, while for a plane curve C and a point R on C, $\dim_K \mathfrak{m}_R(C)/\mathfrak{m}_R(C)^2 \in \{1, 2\}$ (cf. Proposition 3.4.43).

Proposition 5.3.43. *Suppose there is a dominant rational map, $\Phi : X \dashrightarrow Y$. Then* $\dim Y \leq \dim X$.

Proof. Φ induces an injective K-algebra map $\widetilde{\Phi} : K(Y) \hookrightarrow K(X)$. Then by Lemma 1.9.50

$$\dim Y = \text{trdeg } K(Y) \leq \text{trdeg } K(X) = \dim X.$$

□

Proposition 5.3.44. *Every r-dimensional variety X over a characteristic 0 field* ***How to generalize to positive characteristic? Every function field being seperable would suffice*** *is birationally equivalent to a hypersurface in $\mathbb{A}^{n+1}$ or $\mathbb{P}^{n+1}$.*

Proof. By definition $K(X)$ is a function field in r variables. We proceed as in the proof of Proposition 5.3.28. It follows that we just need to argue that $\ker \text{ev}_\alpha$ is principal. Indeed let f be the minimal irreducible monic polynomial in $K(y_1, \ldots, y_r)[x]$ vanishing on α. We may assume that the for each non-zero coefficient of f the numerator and denominator are co-prime. We may find a $g = cf \in K[y_1, \ldots, y_r, x] \setminus 0$ where c is the least common multiple of the denominators and this is primitive in $K[y_1, \ldots, y_r][x]$ (cf. Lemma 1.8.109) and vanishes on α, so $g \in \ker \text{ev}_\alpha$. Let $h \in \ker \text{ev}_\alpha$. Then h is in the kernel of the map $K(y_1, \ldots, y_r)[x] \to K(X), a \mapsto a(\alpha)$ which is generated by f, hence $f \mid h$, which implies that $g = cf \mid h$ in $K(y_1, \ldots, y_n)[x]$ and so by Gauss' lemma (cf. Lemma 1.8.110) $g \mid h$ in $K[y_1, \ldots, y_n, x]$. We thus get that $\ker \text{ev}_\alpha = \langle g \rangle$. So $K(X) \simeq K(V(g))$, hence $X \simeq V(g)$, since $(Y, \Phi : Z \dashrightarrow W) \mapsto (K(Y), \widetilde{\Phi} : K(W) \hookrightarrow K(Z))$ is fully faithful. The projective version is accomplished by taking projective closure. □

Proposition 5.3.45. *Suppose X, Y are varieties, $P \in X, Q \in Y$ with $\mathcal{O}_P(X)$ is K-algebra isomorphic to $\mathcal{O}_Q(Y)$. Then there are open neighborhoods $U \ni P$ and $V \ni Q$ such that $U \simeq V$.*

Proof. WLOG X, Y are affine. First note that $\sigma : \mathcal{O}_Q(Y) \simeq \mathcal{O}_P(X)$ implies $\sigma : K(Y) \simeq K(X)$. Then there is a unique birational $\Phi : X \sim_{\text{bir}} Y$ such that $\widetilde{\Phi} = \sigma$. Since $\mathcal{O}_P(X)$ dominates $\sigma(\mathcal{O}_Q(Y))$ it follows that $Q = \Phi(P)$ (cf. Proposition 5.3.35 2.). As in the proof

of Proposition 5.3.35 We may find $b \in \Gamma(X)$ and a $y \in \Gamma(Y)$ such that $\sigma(\Gamma(Y)) \subset \Gamma(X_b)$ and $\sigma^{-1}(\Gamma(X)) \subset \Gamma(Y_d)$. Then we also have that $\sigma(\Gamma(Y)) \subset \Gamma((X_b)_{\sigma(d)})$ and that $\sigma^{-1}(\Gamma(X)) \subset \Gamma((Y_d)_{\sigma^{-1}(b)})$. In general recall that $\Gamma(Z_\mu) = \Gamma(Z)[1/\mu]$ for Z affine (cf. Proposition 5.2.51). Hence if $f = \frac{a}{d^n\sigma^{-1}(b)^m} \in \Gamma((Y_d)_{\sigma^{-1}(b)})$, then $\sigma(f) = \frac{\sigma(a)}{\sigma(d)^n b^m} \in \Gamma((X_b)_{\sigma(d)})$. Conversely, if $g = \frac{c}{\sigma(d)^n b^m} \in \Gamma((X_b)_{\sigma(d)})$, we get that $g = \sigma\left(\frac{\sigma^{-1}(c)}{d^n\sigma^{-1}(b)^m}\right)$. This means $\sigma : \Gamma((Y_d)_{\sigma^{-1}(b)}) \to \Gamma((X_b)_{\sigma(d)})$ is a K-algebra isomorphism, which is induced by a representative $\varphi : U := (X_b)_{\sigma(d)} \xrightarrow{\sim} V := (Y_d)_{\sigma^{-1}(b)}$ of Φ. Note that by the construction of b, $b(P) \neq 0$ and $\sigma(d)(P) = \widetilde{\Phi}(d)(P) = d(\Phi(P)) = d(Q) \neq 0$ by the construction of d, so $P \in U$. The fact that $Q = \Phi(P) = \varphi(Q)$ shows that $Q \in V$. □

Proposition 5.3.46. *Let C be a projective curve, $P \in C$. There is a birational map $\Phi : C \dashrightarrow C'$ where C' is projective plane curves such that $\Phi^{-1}(\Phi(P)) = P$.*

Proof. We may assume $C \subset \mathbb{P}^{n+1}$ (in the case $n = 0$, $\mathbb{P}^1 \simeq L_\infty, [a, b] \mapsto [a, b, 0]$). Denote the $n+2$ coordinate hyperplanes by $y, x_1, \ldots, x_n, z$. After a change of coordinates we may choose that $C \not\subset y$, meaning $C \cap y$ is finite since it is closed in C and every component is a point (cf. Lemma 5.3.24 4.); we may choose that the C do not intersect $y \cap z$ (again we can arrange that $C \not\subset y \cap z$ and then move potential points of intersection); we may choose that $P = [0, \ldots, 0, 1]$. As a consequence $K(C) \supset K(\overline{y}/\overline{z})$ (by our choices z does not vanish on C and hence $\overline{z} \neq 0$) is algebraic.
For $v = (v_1, \ldots, v_n) \in \mathbb{A}^n$ define

$$\varphi_v : C \to \mathbb{P}^2$$
$$[\mu, w_1, \ldots, w_n, \nu] \mapsto [\mu, \sum_1^n v_i w_i, \nu]$$

This is well-defined since for every $[\mu, w_1, \ldots, w_n, \nu] \in C$, $\mu \neq 0$ or $\nu \neq 0$, since $C \cap y \cap z$ is empty. It is a morphism since it is covered by $U_1 \cap C, U_{n+2} \cap C$, on which it restricts to a polynomial map to the first resp. third copy of the affine plane in $\mathbb{P}^2$. Observe that $\varphi_v(P) = [0, 0, 1]$. Set $C_v := \mathrm{cl}(\varphi_v(C))$. We now just need to argue that we can choose v such that φ_v becomes a birational map onto C'_v. First of all, C'_v is a closed subvariety of $\mathbb{P}^2$, since if $C'_v = A \cup B$ a decomposition into a union of closed sets, then we assume $C = \varphi_v^{-1}(A)$. Then $\varphi(C) = \varphi_v(\varphi_v^{-1}(A))$, hence $C' = \mathrm{cl}(\varphi_v(\varphi_v^{-1}(A))) \subset A$, implying $C' = A$. Note that φ_v is a dominant rational map and therefor induces a well-defined (injective) field extension map over K, $\widetilde{\varphi_v} : K(C'_v) \to K(C), \frac{\overline{f}}{\overline{g}} \mapsto \frac{\overline{f(y, \sum_1^n v_i x_i)}}{\overline{g(y, \sum_1^n v_i x_i)}}$. We now choose $v \in \mathbb{A}^{n+1}$ such that $K(C) = K(u, \sum_1^n v_i \frac{\overline{x_i}}{\overline{z}})$. Then $K(C'_v) \simeq K(C)$, so φ_v. Write $C \cap y = \{Q_1, \ldots, Q_m, P\}$. Set $h := \sum_1^n y_i x_i \in K[\mathbf{x}, \mathbf{y}]$, $h_i := h(\mathbf{y}, Q_i)$ and $V := \bigcup V(h_1, \ldots, h_m)$. We may pick $v \in \mathbb{A}^n \setminus V$ so that φ_v is birational. This

means that if $Q = [t, w, s] \in C \cap y$, then $\sum_1^n v_i w_i \neq 0$ or $w = 0$. In particular, if $\varphi_v([t, w_1, \ldots, w_n, s]) = [0, 0, 1]$, then $[t, w, s] \in C \cap y$ and $\sum_1^n v_i w_i = 0$, hence $w = 0$, and $[t, w, s] = P$. □

Definition 5.3.47. We say that a rational map is *finite* if the induced K-algebra homomorphims between the algebraic function fields is finite.

Lemma 5.3.48. *Let C and C' be curves, $\Phi : C' \dashrightarrow C$ some rational map.*

1. *Φ is dominating or constant.*
2. *If Φ is dominating, then $\widetilde{\Phi}$ is finite.*

Proof. 1. Note that $\mathrm{cl}(\Phi(\mathrm{dom}\ \Phi))$ is a closed subvariety of C and thus is either equal to C or, due to Lemma 5.3.24, a point.
2. Since $K(C') \supset K$ is field finite, then so is $K(C') \supset \widetilde{\Phi}(K(C))$. Let $f \in K(C) \setminus K$, then $\widetilde{\Phi}(f) \in K(C) \setminus K$, hence $K(C') \supset K(\widetilde{\Phi}(f))$ is algebraic by Lemma 1.9.53 1. hence $K(C') \supset \widetilde{\Phi}(K(C)) = \widetilde{\Phi}(K(C))(\widetilde{\Phi}(f))$ is algebraic. It follows that $K(C') \supset \widetilde{\Phi}(K(C))$ is module-finite, hence Φ is finite. □

5.3.6 Dimension of Topological Spaces

Definition 5.3.49. Let X be a topological space. We define for $n \in \mathbb{N}$,

$$\mathrm{IrrChain}_n(X)$$

to be the set of strictly increasing chains $\{Z_i\}_{i=0}^n$ of irreducible closed sets in X. Then the *dimension of X* is

$$\dim\ X := \sup(\{n \in \mathbb{N} \cup \{\infty\} : \mathrm{IrrChain}_n(X) \neq \emptyset\}).$$

Definition 5.3.50. A topological space X is called *Noetherian* if every descending chain of closed sets in X stabilizes.

Lemma 5.3.51. *Let X be topological space covered by a family of subsets $\{Y_i\}_1^n$ such that each Y_i with the trace topology with respect to X is Noetherian. Then X is Noetherian.*

Proof. Consider a descending chain $Z_1 \supset Z_2 \supset \ldots$ of closed sets in X. Then for each i, $Y_i \cap Z_1 \supset Y_i \cap Z_2 \supset \ldots$ is a descending chain of closed sets Y_i, hence it stabilizes. Then since $Z_j = \bigcup Y_i \cap Z_j$, it follows that $Z_1 \supset Z_2 \supset \ldots$ stabilizes. □

Lemma 5.3.52. *Let X be a Noetherian topological space, then any subspace $Y \subset X$, where Y is closed in X is Noetherian.*

Proof. Let $Z_1 \supset Z_2 \supset \dots$ be a descending chain of closed sets in Y. This is also a descending chain of closed sets in X, hence it stabilizes. □

Lemma 5.3.53. *A variety is Noetherian.*

Proof. We first prove that $\mathbb{A}^n$ is Noetherian for each n. Consider a descending chain of closed sets in $\mathbb{A}^n$, $Z_1 \supset Z_2 \supset \dots$. Then

$$I(Z_1) \subset I(Z_2) \subset \dots$$

is an ascending chain of ideals in $K[x_1, \dots, x_n]$ which is a Noetherian ring, hence it stabilizes. Since $Z_i = V(I(Z_i))$, it follows that $Z_1 \supset Z_2 \supset \dots$ stabilizes.
Any algebraic set is Noetherian by the prior lemma. In particular, an affine variety is thus Noetherian.
Consider an arbitrary variety X. This is covered by open affine varieties $\{U_i\}1^n$. It thus follows from Lemma 5.3.51 that X is Noetherian. □

Proposition 5.3.54. *A noetherian topological space can be written uniquely as a finite union of irreducible subsets*

$$\bigcup_1^n Y_i,$$

with $Y_i \not\supset Y_j$

Proof. Not really necessary but can be used to prove the decomposition of algebraic sets into irreducibles □

Theorem 5.3.55. *Let V be a variety. Then the dimension of V is equal to the Krull dimension of $\Gamma(V)$.*

Proof. □

Corollary 5.3.56. *The topological dimension of varieties described here corresponds to the algebro geometric described earlier.*

5.4 The Study of Curves & Resolution of Singularities

5.4.1 Rational Maps of Curves

Definition 5.4.1. Let P be a point on a curve C. P is called *simple* if $\mathcal{O}_P(C)$ is a DVR.

Remark 5.4.2. This agrees with the definition on plane curves by Theorem 3.4.36. We denote the order function on $K(C)$ defined by $\mathcal{O}_P(C)$ by $\text{ord}_P := \text{ord}_P^C$.

Definition 5.4.3. A curve is called *non-singular* if every point on it is simple.

Definition 5.4.4. Let $L \supset K$ be a field extension, and A a local ring. We say that A is *a local ring of* L if it is contained in L, $Q(A) = K$ and $A \supset K$. Similarly, *a DVR of* L is a DVR that is a local ring of L.

Theorem 5.4.5. *Let C be a projective curve, $L := K(C)$. Suppose $M \supset L$ is some field and R is a DVR of M. Suppose $R \not\supset K$. Then there is a unique $P \in C$ such that R dominates $\mathcal{O}_P(C)$.*

Proof. Denote the maximal ideal of R by $\mathfrak{m}$. **Uniqueness:** Suppose for a contradiction that there are two distinct points P, Q on C such that R dominates $\mathcal{O}_P(C)$ and $\mathcal{O}_Q(C)$. We may construct a rational function $f \in K(C)$ such that $f \in \mathfrak{m}_P(V) \subset \mathfrak{m}$ and $\frac{1}{f} \in \mathcal{O}_Q(C)$ (cf. Lemma 5.2.58). Then $\text{ord}(f) > 0$, and since f is a unit in the local ring of C at Q so therefor also a unit in R, meaning $\text{ord}(f) \geq 0$; this is a contradiction.
Existence: WLOG $C \subset \mathbb{P}^n$ for some $n \geq 1$. We may also assume that C is not contained in any coordinate hyperplane, hence that $C \cap U_i \neq \emptyset$ for every i (cf. Lemma 5.2.61). We have that $\Gamma^h(C) = K[x_1, \ldots, x_{n+1}]/I^{\mathbb{P}}(C) = K[\overline{x_1}, \ldots, \overline{x_{n+1}}]$. Set $N = \max_{i,j} \text{ord}(\overline{x_i}/\overline{x_j})$. WLOG $\text{ord}(\overline{x_j}/\overline{x_{n+1}}) = N$ for some j. Note that for each i

$$\text{ord}(\overline{x_i}/\overline{x_{n+1}}) = \text{ord}((\overline{x_j}/\overline{x_{n+1}})(\overline{x_i}/\overline{x_j})) = N - \text{ord}(\overline{x_j}/\overline{x_i}) \geq 0.$$

It then follows that $R \supset K\left[\frac{\overline{x_1}}{\overline{x_{n+1}}}, \ldots, \frac{\overline{x_n}}{\overline{x_{n+1}}}\right] \simeq \Gamma(C_*)$, where C_* is the affine curve corresponding to $C \cap U_{n+1}$. Set $J := \mathfrak{m} \cap \Gamma(C_*)$. Then J is a prime ideal and hence corresponds to some subvariety of C_*, W say. If $W = C_*$, then $J = 0$, hence $\Gamma(C_*) \setminus 0$ are all units in R, which would imply that $R \supset L$. So W is a proper subvariety of C_* and thus is a point, $\{P\}$ say, by Lemma 5.3.24 4. We may write elements of $\mathcal{O}_P(C)$ as a fraction $\frac{\alpha}{\beta} \in K(C)$, where $\alpha, \beta \in \Gamma(C_*)$, where $\beta(P) \neq 0$. Then $\beta \notin J$, hence $\beta \notin \mathfrak{m}$. It thus follows that β is a unit in R, hence $\frac{\alpha}{\beta} \in R$. Moreover if $\frac{\alpha}{\beta} \in \mathfrak{m}_P(V)$, then $\alpha \in J$, hence $\alpha \in \mathfrak{m}$, so $\frac{\alpha}{\beta} \in \mathfrak{m}$. In conclusion, P is a point in C such that R dominates $\mathcal{O}_P(C)$. □

Corollary 5.4.6. *Let C, C' be curves, C projective, $\Phi : C' \dashrightarrow C$ some rational map. Then the set of simple points in C' is contained in* dom Φ. *Then immediately it follows that if C' is nonsingular then Φ is a morphism, i.e* dom $\Phi = C'$.

Proof. In the case of Φ not being dominant, it is constant (cf. Lemma 5.3.48 1.), hence dom $\Phi = C'$. Suppose Φ is dominant. Then $\widetilde{\Phi}$ is injective, so we can identify $K(C)$ with a subfield of $K(C')$. Let P be a simple point of C' and set $R := \mathcal{O}_P(C')$, which is then a DVR of $K(C')$. If $R \not\supset K(C)$, then by the prior theorem there is a $Q \in C$ such

that $\mathcal{O}_Q(C)$ (which we have identified with $\widetilde{\Phi}(\mathcal{O}_Q(C))$) is dominated by R, which means $P \in \text{dom } \Phi$ by Lemma 5.3.35 2. Suppose for a contradition that $K(C) \subset R \supset K(C')$. Note that $K(C') \supset K(C)$ is module-finite by Lemma5.3.48 2. Then R is a field by Proposition 1.9.24, leading to a contradiction with the fact that R is a DVR and therefor a non-field. □

Corollary 5.4.7. *Let $\mathcal{C}$ be the category nonsingular curves with dominant morphisms. Then the restriction $\mathcal{C}$ of the $\widetilde{\bullet}$-functor to the category of 1 variable algebraic function fields over K with K-algebra homomorphisms is a fully faithful functor.*

Definition 5.4.8. Let C be a nonsingular curve. Define

$$\text{DVR}(C) := \{R \subset K(C) : R \text{ is a DVR of } K(C)\}.$$

Corollary 5.4.9. *Let C be a nonsingular curve. The map*

$$\mathcal{O}_\bullet(C) : C \to \text{DVR}(C)$$
$$P \mapsto \mathcal{O}_P(C)$$

is a bijection.

Proof. It is obviously well-defined. Let $R \in \text{DVR}(C)$. By the prior theorem there is a unique $P_R \in C$ such that R dominates $\mathcal{O}_{P_R}(C)$. Then $R = \mathcal{O}_{P_R}(C)$ by Proposition 1.6.125. So the inverse of $\mathcal{O}_\bullet(C)$ is $R \mapsto P_R$. □

Proposition 5.4.10. *Let C be a nonsingular curve. The family of subsets of* $\text{DVR}(C)$*,*

$$\{\emptyset\} \cup \{U \subset \text{DVR}(C) : \text{DVR}(C) \setminus U \textit{ is finite } \}$$

defines a topology on $\text{DVR}(C)$ *which turns $\mathcal{O}_\bullet(C)$ into a homeomorphism.*

Proof. Let $\{U_\alpha\}$ be a family non-empty subsets of $\text{DVR}(C)$ such that the complement of U_α is finite. The Demorgan's law shows that the complement of $\bigcup U_\alpha$ is also finite. Let $U_1, U_2 \subset \text{DVR}(C)$ be non-empty subsets. Again by Demorgan's law the complement of $U_1 \cap U_2$ is finite. So the family of subset is indeed a topology. Note that if S is a closed subset of $\text{DVR}(C)$ it is either all of $\text{DVR}(C)$ or a finite set. In the first case $\mathcal{O}_\bullet^{-1}(S) = C$ and in the second $\mathcal{O}_\bullet^{-1}(S)$ is a finite union of points. In either case it is a closed subset of C. If $V \subset C$ is closed, it is either equal to C or a finite subset. In either case $\mathcal{O}_\bullet$ maps it to a closed subset of $\text{DVR}(C)$. We then conclude that $\mathcal{O}_\bullet$ is a homeomorphism. □

Lemma 5.4.11. *Let $C \subset V$ be a curve and Z a non-empty proper closed subset of C. Then Z is a finite collection of points in C.*

Proof. Note that $Z = Y \cap C$ where $Y \subsetneq V$ is closed. Then $Y = Y_1 \cup \ldots \cup Y_m$ for irreducible closed sets $Y_i \subsetneq V$. Since dim V = dim $C = 1$, it follows that $Y_i = \{P_i\}$ for some $P_i \in V$ for each i. Then Z is a collection of points. □

Proposition 5.4.12. *A curve C has only finitely many multiple points.*

Proof. C is birationally equivalent to a plane curve which may be identified with an open subset of a projective plane curve C'. So there is a birational equivalence $\Phi : C \to C'$. All simple points are contained in $U :=$ dom Φ by Corollary 5.4.6. The complement of U in C is a proper closed subset of C, which by the prior lemma is finite. We have an isomorphism $\Phi : U \to U'$ where U' is open in C'. This isomorphism establishes a bijection of multiple points of C' in U' (of which there are finitely many) and multiple points of C in U. It follows that C has only finitely many multiple points. □

Definition 5.4.13. Let C be a projective curve. *A resolution of the singularities of C* is a non-singular projective curve X with a birational map $\Phi : X \dashrightarrow C$.

In the next subsection we are going to work on how to "blow up" multiple points on a curve. The easy case is affine plane curves.

5.4.2 Blowing up Points in $\mathbb{A}^2$

We start by working through an example to sketch the idea.

Example 5.4.14. Consider the curve $C := V(y^2 - x^2(x+1))$, $P := (0,0) \in \mathbb{A}^2$, $U := \{(a,b) \in \mathbb{A}^2 : a \neq 0\} = \mathbb{A}^2 \setminus V(x)$. Consider a morphism

$$\varphi : U \to \mathbb{A}^1$$
$$(a,b) \mapsto \frac{b}{a}$$

In the section introducing algebraic groups we saw that multiplication and taking multiplicative inverese are morphisms and φ is just the composition, $(a,b) \mapsto (a^{-1},b) \mapsto a^{-1}b$. Set

$$G := G_\varphi \subset U \times \mathbb{A}^1 \subset \mathbb{A}^3$$

(the graph of φ). Then

$$G = \left\{(a,b,c) \in \mathbb{A}^3 : a \neq 0, b = ac\right\}.$$

Set

$$B := \left\{(a,b,c) \in \mathbb{A}^3 : b = ac\right\}.$$

Then $G = B \setminus V(x)$. Define

$$\pi : B \to \mathbb{A}^2$$
$$(a, b, c) \mapsto (a, b)$$

We see that for $(a, c) \in \mathbb{A}^2$ that $(a, ac) \in U$ when $a \neq 0$ and $(a, ac) = P$ when $a = 0$, hence

$$\pi(B) = U \cup \{P\}.$$

Set $L := \pi^{-1}(P) = \{(0, 0, c) \in \mathbb{A}^2 : c \in K\}$ and note that $\pi^{-1}(U) = G$. Then

$$\pi|_{\pi^{-1}(U)} : G \to U$$

defines an isomorphism with inverse $(a, b) \mapsto (a, b, b/a)$. Futhermore, $G \subset \mathbb{A}^3$ is an open subvariety of B which is an affine variety. Then

$$\text{cl}(G) = B.$$

We also see that L is a closed subvariety of B. Note that away from P, points in C are in U, hence $\pi^{-1}(C \setminus \{P\}) = \{(a, b, c) \in \mathbb{A}^3 : b^2 = a^2(a + 1), b = ac\} \setminus L$. Upon setting $C' = \text{cl}_B(\pi^{-1}(C \setminus \{P\}))$ the hope is that C' is a nonsingular curve with two points lying over P. Indeed, we can check that

$$C' = \text{cl}_B\left(V(y^2 - x^2(x + 1)) \setminus L \cap B\right) = V(z^2 - (x+1)) \cap B = V(z^2 - (x+1)) \cap V(y - xz)$$

Call the right-hand side V.

$$\phi : \mathbb{A}^2 \to B$$
$$(a, c) \mapsto (a, ac, c)$$

this is an isomorphism with inverse $(a, b, c) \mapsto (a, c)$. Restricting ϕ to $V(z^2 - (x+1)) \subset \mathbb{A}^2$ we get that $V(z^2 - (x+1)) \cap B$ is a proper closed subvariety of B of dimension 1. Then given W, an arbitrary closed subvariety of B contained in V, contaning $V(y^2 - x^2(x+1)) \setminus L \cap B$, we get dim $= W = 1 =$ dim V, since $V(y^2 - x^2(x+1)) \setminus L \cap B$ is an infinite set. Since the closed subvarities of B form a closed basis for the Zariski topology on B, it follows that V is the smallest closed set containing $V(y^2 - x^2(x + 1)) \setminus L \cap B$, which means $C' = V$. We see that C' intersects L in $(0, 0, \pm 1)$. The main take away from this example is that we have taken the double point P and essentially "split" it into two points which we moved along L, taking C with us in the process, in a birational way such that the resulting curve is algebraic. Moreover, C' is isomorphic to the non-singular curve $V(z^2 - (x + 1))$, so C is birationally equivalent to this curve since $\pi : \pi^{-1}(C \setminus \{P\}) \to C \setminus \{P\}$ defines an isomorphims and hence a birational map $C' \dashrightarrow C$. Moreover, $\phi : V(z^2 - (x + 1)) \to C'$ defines an isomorphism. Then $\pi\phi : V(z^2 - (x + 1)) \dashrightarrow C$ is a birational equivalence.

To generalize the approach in the example, Let $C = V(f) \subset \mathbb{A}^2$ be some affine plane curve, keep B, P,π, U, L and ϕ as already defined. We set $C_0 := C \cap U$. We define a morphism

$$\psi := \pi\phi : \mathbb{A}^2 \to \mathbb{A}^2$$
$$(a, c) \mapsto (a, ac, c) \mapsto (a, ac)$$

Set $E := \psi^{-1}(P) = \phi^{-1}(L) = V(x)$. Then the restriction ψ to $\mathbb{A}^2 \setminus E = U$ is an ismorphism of U with U. Then ψ defines a birational morphism of $\mathbb{A}^2$ with itself. Set $C_0' := \psi^{-1}(C_0)$ and $C' := \mathrm{cl}_{\mathbb{A}^2}(C_0')$. Define

$$\theta := \psi|_{C'} : C' \to C$$

This is well-defined by continuity of ψ. Then θ is a birational equivalence of C' with C by restricting it to $C' \setminus E$. Note θ induces an automorphism $K(x, y) = K(C) \simeq K(C') = K(x, z)$, where y is identified with xz, i.e $z := y/x \in K(C)$. We need to prove two things: Firstly, that C' is actually an affine plane curve. Secondly: that C' has nicer multiple points. The phrasing "nicer" is of course informal. One could hope for C' to be non-singular. This however is not the case, but each point in preimage of P under θ will have only one tangent counted with multiplicity.

Lemma 5.4.15. *Write $C = V(f)$,*

$$f = \sum_r^n f_i$$

where $f_i \in K[x, y]$ is a form of degree i, $r = m_P(C)$ and $n = \deg\ C$. Then $C' = V(f')$ is an affine plane curve, where

$$f' = \sum_r^n x^{i-r} f_i(1, z) \in K[z][x].$$

Proof. First note that

$$f(x, xz) = \sum_r^n x^i f_r(1, z) = x^r \sum_r^n x^{i-r} f_r(1, z) = x^r f'.$$

Suppose for a contradiction that $f' = gh$ for non-units $g_1, g_2 \in K[z][x]$ with $d_i := \deg_z\ g_i$. Note that in general for $\gamma \in R[t]$ that

$$x_1^{\deg\ \gamma} \gamma\left(x_1, \frac{x_2}{x_1}\right) \in K[x_1, x_2] \subset K(x_1, x_2)$$

It thus follows that $x^{d_i} g_i\left(\frac{y}{x}\right) \in K[x, y]$ are non units with

$$f = f\left(x, x\frac{y}{x}\right) = x^r f'\left(x, \frac{y}{x}\right) = \left(x^{d_1} g_1\left(x, \frac{y}{x}\right)\right)\left(x^{d_2} g_2\left(x, \frac{y}{x}\right)\right),$$

which implies f is reducible leading to a contradiction. We thus conclude that f' is irreducible. Note that if $(a,c) \in C_0'$, then $(a,ac) = \varphi(a,c) \in C$, hence $a \neq 0$ and

$$a^r f'(a,c) = f(a,c) = 0.$$

Therefore, $C_0' \subset V(f')$, hence $C' = V(f')$ by a dimension argument. □

We have thus produced an affine curve C' birationally isomorphic to C. It remains for us to check that the points of $\theta^{-1}(P)$ are all nice points. To be precise, that the multiple point P is blown up into $m_P(C)$ points with one tangent each possibly counted with multiplicity.
WLOG x is not tangent to f at P and

$$f_r = \prod_1^s (y - \alpha_i x)^{r_i}.$$

With the above assumption we have that following lemma

Lemma 5.4.16. *$\theta^{-1}(P) = \{P_1, \ldots, P_s\}$ where $P_i = (0, \alpha_i)$ and*

$$m_{P_i}(C') = I(P_i, E \cap C') = r_i.$$

Moreover, if P is an ordinary multiple point, then each P_i is a simple point of C' and $\text{ord}_{P_i}^{C'}(x) = 1$ *(cf. Proposition 3.4.54)*

Proof. Indeed,

$$\begin{aligned}\theta^{-1}(P) &= \{(a,b) \in C' : a = 0\} = \{(a,b) \in \mathbb{A}^2 : f'(0,b) = 0\} \\ &= \{(a,b) \in \mathbb{A}^2 : f_r(1,b) = 0\} = E \cap C' = \{P_1, \ldots, P_s\}.\end{aligned}$$

Moreover,

$$\begin{aligned}m_{P_i}(C') \leq I\left(P_i, C' \cap E\right) &= I\left(P_i, \left(\prod_1^s (z - \alpha_j)^{r_j}\right) \cap x\right) \\ &= \sum_1^s r_j I(P_i, (z - \alpha_j) \cap x) = r_i.\end{aligned}$$

If P is an ordinary multiple point, then $r_i = 1$ for all i, hence $m_{P_i}(C') = 1$, hence P_i is simple. □

Lemma 5.4.17. *There is an open affine neighborhood W of P in C such that $W' := \theta^{-1}(W)$ is an affine open subvariety in C', $\theta(W') = W$, $\Gamma(W')$ is module finite over $\Gamma(W)$ and $x^{r-1}\Gamma(W') \subset \Gamma(W)$. Here we have implicitly made the identification of $\Gamma(W)$ as a subring of $K(C)$.*

Proof. Write

$$f = \sum_{(i,j):i+j\geq r} a_{ij}x^i y^j.$$

Set

$$h := \sum_{j\geq r} a_{0j}y^{j-r}$$

and denote the image of h in $\Gamma(C)$ by $\overline{h}$. Note that

$$h(P) = h(0,0) = a_{0r} = 1 \neq 0,$$

hence $P \in C_{\overline{h}}$. Set $W := C_{\overline{h}}$. We claim that $W' = C'_{\overline{h}}$. Indeed, if $Q = (v_1, v_2) \in \theta^{-1}(W)$. $h(v_1, v_1v_2) \neq 0$. Upon thinking of $\overline{h}$ as an element of $K(C')$, it follows that $Q \in C'_{\overline{h}}$. If $Q = (v_1, v_2) \in C'_{\overline{h}}$, then $\overline{h}(v_1, v_2) = h(v_1, v_1v_2) \neq 0$, note then $\widetilde{\theta}(\overline{h}) \in K(C)$ does not vanish on Q, hence $Q \in \theta^{-1}(W)$.
Note that from the K-algebra map

$$\widetilde{\theta} : \Gamma(W') \to \Gamma(W)$$

takes a rational function $f \in \Gamma(W')$ to the rational function $f(x, xz) \in \Gamma(W)[z]$. Conversely, for $f = \sum_0^d f_i(x,y)z^i \in \Gamma(W)[z]$ consider $g = \sum_0^d f_i(x,z)\left(\frac{z}{x}\right)^i$. Then

$$\widetilde{\theta}(g) = g(x, xz) = \sum_0^d f_i(x, xz)\frac{x^i z^i}{x^i} = \sum_0^d f_i(x,y)z^i = f$$

It is then sufficient to prove that z is integral over $\Gamma(W)$. Note that

$$f'(x,z) = \frac{f(x,xz)}{x^r} = \sum_{(i,j):i+j\geq r} a_{ij}x^{i+j-r}z^j = \sum_{(i,j):i+j=r} a_{ij}y^{i+j-r}z^{r-i}.$$

For $0 < i < r$ set $b_i := \left(\sum_{j\geq r} a_{ij}y^{i+j-r}\right)/\overline{h}$ and $b_r = \left(\sum_{i,j\geq r} a_{ij}x^{i-r}y^j\right)/\overline{h}$. Note that for each $0 < i < r$,

$$z^{r-i}b_i = \left(\sum_{j\geq r} a_{ij}x^{i+j-r}z^j\right)/\overline{h}$$

and that

$$b_r = \sum_{i,j\geq r} a_{ij}x^{i+j-r}z^j$$

and lastly that

$$z^r\overline{h} = z^r\sum_{j\geq 0} a_{0j}y^{j-r} = z^r\sum_{j\geq 0} x^{0+j-r}z^j.$$

It follows that

$$z^r + \sum_1^r b_i z^{r-i} = \frac{f'(x,z)}{\overline{h}} = 0.$$

Note in particular that this also shows that $\Gamma(W')$ is a finitely generated $\Gamma(W)$ module with generators $1, z, \ldots, z^{r-1}$. Since for $i \leq r-1$,

$$x^{r-1}z^i = x^{r-1}\frac{y^i}{x^i} = x^{r-1-i}y^i \in \Gamma(W),$$

it follows that $x^{r-1}\Gamma(W') \subset \Gamma(W)$. □

Remark 5.4.18. In general, if U is an open neighborhood and U' is an open set in C' containing $\{P_1, \ldots, P_s\}$. Construct W as in the above proof and take $g \in \Gamma(W)$ such that $g(P) \neq 0$ and $g(Q) = 0$ whenever $Q \in W \setminus U \cup \theta(W' \setminus U')$. Then W_g is an affine open neighborhood of P contained in U and $W'_g = \theta^{-1}(W_g)$ is an affine open set contained in U'. We can thus choose W and W' in the above lemma arbitrarily small.

Remark 5.4.19. We can make it so that W includes any finite set of points on C. First note that if $Q \in C_0 \setminus v(h)$, this is already achieved in the lemma. When this is not the case, the idea is to apply a change of coordinates T that fixes P, but moves Q to $T(Q)$ in such a way that we can apply the lemma to f^T, we to an open affine variety $W^\#$ that contains both P and $T(Q)$, pulling back via T then gives us the desired affine open neighborhood containing both P and Q.
Consider first the case, where $Q \in C \setminus C_0$. I.e. $Q \in V(x)$. Consider $T_\alpha : (v_1, v_2) \mapsto (v_1 + \alpha v_2, v_2)$ for $\alpha \in K \setminus 0$. This fixes P and $T_\alpha(Q) \notin V(x)$.
Consider then the case where $Q \in C_0$ but where the h obtained in the proof of the lemma vanishes on Q. Consider a coordinate change $T_\beta : (v_1, v_2) \mapsto (v_1, v_2 + \beta v_1)$ such that $h(T_\beta(Q)) \neq 0$ ($V(h) \cap C_0$ is finite so we are spoiled for choice). By construction $T_\beta(Q) \notin V(x)$ and $T_\beta(Q) \in f^{T_\beta}$, we are thus in a position to apply the lemma to obtain an affine open set $V(f^{T_\beta})_h$ that will result in the desired affine open set when pulling back via T_β.
Given $Q_1, \ldots, Q_{k+1} \in C$, we may by induction assume that $Q_1, \ldots, Q_k \in C_0 \setminus V(h)$ via a series of coordinate changes. If $Q_{k+1} \in C_0$ we are done. If $Q_{k+1} \in V(x)$, we first apply a change of coordinates of type T_α to Q_{k+1} taking care to pick α such that $T_\alpha(Q_i) \in V(x) \cap V(h)$ for $i \leq k$. We can clearly do this since there are only finitely many choices of α where this fails. We can thus reduce to the case $Q_{k+1} \in V(h)$. Here we apply a change of coordinates of T_β, taking care to pick β such that $T_\beta(Q_i) \notin V(x) \cap h$. We construct an affine open set and pullback via the composition of all the change of coordinates made to obtain the desired W.

Proposition 5.4.20. *Suppose P is an ordinary multiple point on C and that $\theta^{-1}(P) = \{P_1, \ldots, P_r\}$, with $P_i = (0, \alpha_i)$.*

1. $f_y = \sum_i \prod_{j \neq i}(y - \alpha_i x) + \sum_{r+1}^{n}(f_k)_y$.
2. *For $\overline{x}, \overline{y} \in K(C)$,*

$$f_y(\overline{x}, \overline{y}) = \overline{x}^{r-1}\left(\sum_i \prod_{j \neq i}(z - \alpha_j) + a_{r1}\overline{x} + \ldots\right).$$

3. $\mathrm{ord}_{P_i}^{C'}(f_y(x, y)) = r - 1$.

Proof. 1. This just follows by induction and Leibniz' rule applied to f_r.
2. Indeed,

$$\begin{aligned}(f_r)_y(\overline{x}, \overline{y}) &= \sum_i \prod_{j \neq i}(\overline{y} - \alpha_j \overline{x}) = \sum_i \prod_{j \neq i}(\overline{x}z - \alpha_j \overline{x}) = \sum_i \prod_{j \neq i} \overline{x}(z - \alpha_j) \\ &= \sum_i \overline{x}^{r-1} \prod_{j \neq i}(z - \alpha_j) = \overline{x}^{r-1} \sum_i \prod_{j \neq i}(z - \alpha_j)\end{aligned}$$

a term in f_{r+k} for $k \geq 1$ is of the form $a_{i,j}x^i y^j$ where $i + j = r + k$. In case $i = r + k$, $D_y(a_{i,j}x^i y^j) = 0$. Otherwise,

$$\overline{D_y(a_{ij}x^i y^j)} = j a_{ij} \overline{x}^{i+j-1} z^{j-1} = \overline{r-1} j a_{ij} \overline{x}^{i+j-r} z^{j-1},$$

so

$$f_y(\overline{x}, \overline{y}) = \overline{x}^{r-1}\left(\sum_i \prod_{j \neq i}(z - \alpha_j) + a_{r1}\overline{x} + \text{ some other terms}\right).$$

3. Since x is not tangent to C' at P_i and since each P_i is simple, $\overline{x}$ is the uniformizing parameter for the DVR $\mathcal{O}_{P_i}(C')$ (cf. Theorem 3.4.36). In 2. we found that

$$f_y(\overline{x}, \overline{y}) = \overline{x}^{r-1} u$$

where u is a unit in $\mathcal{O}_{P_i}(C')$, hence $\mathrm{ord}(f_y(\overline{x}, \overline{y})) = r - 1$. □

Proposition 5.4.21. *Let P be an ordinary multiple point on C, $\theta^{-1}(P) = \{P_1, \ldots, P_r\}$ with $P_i = (0, \alpha_i)$ and $f_r = \prod_1^r l_i$ with $l_i = y - \alpha_i x$. Let g be a plane curve and denote the image of g in $\Gamma(C') \subset \Gamma(C)$ by $\overline{g}$.*

1. $\mathrm{ord}_{P_i}(\overline{g}) \geq m_P(g)$ *with equality if L_i is not tangent to g at P.*
2. *If $s \leq r$, and $\mathrm{ord}_{P_i}^{C'}(\overline{g}) \geq s$ for each $i = 1, \ldots, s$, then $m_P(g) \geq s$.*

Proof. 1. In case $m_P(g) = 0$ we are done. Write

$$g = \sum_{m_P(g)}^{d} g_i.$$

Then

$$\overline{g} = g(\overline{x}, \overline{x}z) = x^{m_P(g)} \underbrace{\left(\sum_{m_p(g)}^{r} \overline{x}^i g_i(1, z) \right)}_{\theta}.$$

Then

$$\text{ord}_{P_i}(\overline{g}) = \text{ord}_{P_i}^{C'}(\overline{x}^{m_P(g)}) + \text{ord}_{P_i}^{C'}(\theta) \geq m_P(g).$$

Suppose $L_i = y - \alpha_i x$ is not tangent to P. Write $g_{m_P(g)} = \prod_1^{m_P(g)} \Lambda_j := \gamma_j y - \beta_j x$. Assume first $\gamma_j \neq 0$ for each j. Then WLOG $\Lambda_j = y - \beta_j x$. Then $\beta_j \neq \alpha_i$. When $\beta_j \neq 0$,

$$\Lambda_j(z, 1)(P_i) = \alpha_j - \beta_j \neq 0$$

and when $\beta_j = 0$, this evaluates to $\alpha_i \neq 0$. When $\gamma_j = 0$, we have that $\beta_j \neq 0$, hence

$$\Lambda_j(z, 1)(P_i) = \beta_j \neq 0.$$

It follows that

$$\theta(P_i) = g_{m_P(g)}(1, z)(P_i) \neq 0 \implies \theta \notin \mathfrak{m}_{P_i}(C')$$

which means θ is a unit in $\mathcal{O}_{P_i}(C')$, so $\text{ord}_{P_i}^{C'}(\overline{g}) = m_P(g)$.
2. Suppose for a contradiction that $m_P(g) = k$ for some $k < s$. Then for each $i = 1, \ldots, s$ we have $\text{ord}_{P_i}^{C}(\overline{g}) > k$, hence L_i is a tangent of g at P. But then $m_P(g) \geq r$, leading to a contradiction. □

5.4.3 Blowing up Points in $\mathbb{P}^2$

We proceed to describe the process of blowing up any finite sequence of points on a projective plane curve. Fix points $P_1, \ldots, P_t \in \mathbb{P}^2$. Assume that $P_i = [a_{i1}, a_{i2}, 1] \in U_3$ for each i. We do not lose any generality with this assumption since given any t distinct points there is a change of coordinates moving these points to t distinct points in U_3. We set

$$U := \mathbb{P}^2 \setminus \{P_1, \ldots, P_t\}$$

We define a morphism for each i,

$$f_i : U \to \mathbb{P}^1$$
$$[v_1, v_2, v_3] \mapsto [v_1 - a_{i1}v_3, v_2 - a_{i2}v_3]$$

and set

$$f := (f_1, \ldots, f_t) : U \to \prod_1^t \mathbb{P}^1, u \mapsto (f_1(u), \ldots, f_t(u))$$

Let $G \subset U \times \prod_1^t \mathbb{P}^1$ denote the graph of this morphism. We set

$$B := V\left(\{\, y_{i1}(x_2 - a_{i2}x_3) - y_{i2}(x_1 - a_{i1}x_3) |\, i = 1, \ldots, t\}\right),$$

where x_1, x_2, x_3 are the homogeneous coordinates for $\mathbb{P}^2$ and y_{i1}, y_{i2} are homogeneous coordinates for the i'th copy of $\mathbb{P}^1$. Note that each polynomials that define B are t+1-forms of t + 1-degree $(1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in the i'th position in $K[\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_t]$. For now, we assume that $B = \mathrm{cl}(G)$ (we later see that this true) so that B is a closed subvariety of $\mathbb{P}^2 \times (\mathbb{P}^1)^t$ (cf. Proposition 5.3.6 and Lemma 5.2.7 1.). Let $\pi : B \to \mathbb{P}^2$, be defined by

$$\pi(R, Q_1, \ldots, Q_t) = R \quad \left((R, Q_1, \ldots, Q_t) \in \mathbb{P}^2 \times (\mathbb{P}^1)^t\right)$$

So π is the restriction to B of the projection of $\mathbb{P}^2 \times (\mathbb{P}^1)^t$ onto $\mathbb{P}^2$. Set $E_i := \pi^{-1}(P_i)$. Note that

$$E_i = \{P_i\} \times \{f_1(P_i)\} \times \cdots \times \{f_{i-1}(P_i)\} \times \mathbb{P}^1 \times \{f_{i+1}(P_i)\} \times \cdots \times \{f_t(P_i)\}.$$

Indeed, $R = (P_i, f_1(P_i), \ldots, Q, \ldots, f_t(P_i)) \in B$, since for $j \neq i$

$$\mathrm{ev}_R(y_{1j}(x_2 - a_{j2}x_3) - y_{2j}(x_1 - a_{j1}x_3)) = (a_{i1} - a_{j1})(a_{i2} - a_{j2}) - (a_{i2} - a_{j2})(a_{1i} - a_{j1}) = 0,$$

when $j = i$

$$\mathrm{ev}_R(y_{1j}(x_2 - a_{j2}x_3) - y_{2j}(x_1 - a_{j1}x_3)) = Q_1(a_{i2} - a_{i2}) - Q_2(a_{i1} - a_{i1}) = 0.$$

Conversely, if $(P_1, [b_{11}, b_{21}], \ldots, [b_{t1}, b_{t2}]) \in \mathbb{P}^2 \times (\mathbb{P}^1)^t$ is given with

$$b_{1j}(a_{i2} - a_{j2}) = b_{2j}(a_{1j} - a_{i1})$$

for each $j = 1, \ldots, t$. For each $i \neq j$, since either $b_{kj} \neq 0$ for some $k = 1, 2$ we may assume

$$a_{1j} - a_{i1} = \frac{b_{1j}}{b_{2j}}(a_{i2} - a_{j2})$$

Then

$$f_i(P_j) = [a_{j1} - a_{i1}, a_{j2} - a_{i2}] = \left[\frac{b_{1j}}{b_{2j}}(a_{i2} - a_{j2}), a_{j2} - a_{i2}\right] = [b_{1j}, b_{2j}].$$

Lemma 5.4.22.

$$B \setminus \bigcup_1^t E_i = B \cap \left(U \times \left(\mathbb{P}^1\right)^t\right) = G$$

So $\pi|_{B \setminus \bigcup_1^t E_i} : B \setminus \bigcup_1^t E_i \to U$ is an isomorphism.

Proof. "$\subset$" is obvious for the first inclusion. If $P \in B$ is not in $U \times (\mathbb{P}^1)^t$, then $P = \{P_i\} \times Q$ for some i and some $Q \in (\mathbb{P}^1)^t$, hence $P \in E_i$. Clearly each point in G vanish on

$$y_{1j}(x_2 - a_{j2}x_3) - y_{2j}(x_1 - a_{j1}x_3)$$

Note additionally that P_i is not in U, and is therefore not a point on the graph of f, hence any point in G must be on the form $(Q, f_1(Q), \ldots, f_t(Q)) \in U \times (\mathbb{P}^1)^t$. If $(Q, Q_1, \ldots, Q_t) \in B \cap \left(U \times (\mathbb{P}^1)^t\right)$, then

$$Q_{i1}(Q_2 - a_{i2}Q_3) = Q_{i2}(Q_1 - a_{i1}Q_3)$$

hence

$$Q_1 - a_{i1}Q_3 = \frac{Q_2 - a_{i2}Q_3}{Q_{i2}}Q_{i1}$$

so

$$f_i(Q) = [Q_1 - a_i 1 Q_3, Q_2 - a_{12}Q_3] = \left[\frac{Q_2 - a_{i2}Q_3}{Q_{i2}}Q_{i1}, Q_2 - a_{12}Q_3\right] = [Q_{i1}, Q_{i2}],$$

hence $(Q, Q_1, \ldots, Q_t) \in G$. □

Lemma 5.4.23. *Consider a projective change of coordinates on $\mathbb{P}^2$. Set $P_i' = T(P_i)$, $U = \mathbb{P}^2 \setminus \{P_1', \ldots, P_t'\}$. Define*

$$f_i' : U' \to \mathbb{P}^1$$
$$[v_1, v_2, v_3] \mapsto [v_1 - a_{1i}'v_3, v_2 - a_{i2}'v_3]$$

with $P_i' = [a_{i1}', a_{i2}', a_{i3}']$. Set $f = (f_1', \ldots, f_t')$. For each i there is a unique projective change of coordinates on $\mathbb{P}^1$, T_i say, such that

$$\begin{array}{ccc} U & \xrightarrow{f_i} & \mathbb{P}^1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T_i} \\ U' & \xrightarrow[f_i']{} & \mathbb{P}^1 \end{array}$$

commutes. In other words

$$\begin{array}{ccc} U & \xrightarrow{f} & (\mathbb{P}^1)^t \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T_1 \times \cdots \times T_t} \\ U' & \xrightarrow[f']{} & (\mathbb{P}^1)^t \end{array}$$

commmutes. Setting $G' := G_{f'}$, $B' := \mathrm{cl}(G')$ and $E_i' := pi^{-1}(P_i')$, we see that $T \times T_1 \times \cdots \times T_t$ define isomorphism of these varieties with respectively G, B and E.

Proof. Consider first $Q = [0, 0, 1]$ and $Q' = SQ$ for some change of coordinates S on $\mathbb{P}^2$ and write

$$S = (ax + by + b_{11}z, cx + dy + b_{12}z, ex + fy + z)$$

We first prove the following claim:

$$S_i := ((a - b_{11}e)x + (b - b_{11}f)y, (c - b_{12}e)x + (d - b_{12}f)y)$$

satisfies $S_1 f_Q = f_{Q'} S$. We see this from direct computation:

$$\begin{aligned} S_1 f_Q([v_1, v_2, v_3]) &= S_1([v_1, v_2]) \\ &= [(a - b_{11}e)v_1 + (b - b_{11}f)v_2, (c - b_{12}e)v_1 + (d - b_{12}f)v_2] \\ &= [av_1 + bv_2 + b_1 1 v_3 - b_{11}(ev_1 + fv_2 + v_3), \\ &\quad cv_1 + dv_2 + b_{12}v_3 - b_{12}(ev_1 + fv_2 + v_3)] \\ &= f_{Q'}([av_1 + bv_2 + b_{11}v_3, cv_1 + dv_2 + b_{12}v_3, ev_1 + fv_2 + v_3]) \\ &= f_{Q'} T([v_1, v_2, v_3]). \end{aligned}$$

Consider the 3×3 matrix

$$A := \begin{pmatrix} 1 & 0 & -a_{i1} \\ 0 & 1 & -a_{i2} \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $AP_i = Q$ and $f_i = f_Q A$. Setting $S = TA^{-1}$ we thus get a change of coordinates satisfying,

$$T_i f_i = T_i f_Q A = f_i' S A = f_i' T.$$

For uniqueness, note that $T_i^{(1)} f_i = f_i' T = T_i^{(2)} f_i$, if and only if

$$T_i^{(1)} f_Q = T_i^{(2)} f_Q$$

which is equivalent

$$T_i^{(1)} v = T_i^{(2)} v$$

for each $v \in \mathbb{A}^2$, hence $T_i^{(1)} = T_i^{(2)}$. This shows that $T \times T_1 \times \cdots \times T_t$ maps G, B and E_i onto G', B' and E_i'. □

Lemma 5.4.24. *For an i if T_i is a projective change of coordinates on $\mathbb{P}^1$ then there is a projective change of coordinates T on $\mathbb{P}^2$ such that $T(P_i) = P_i$ and*

$$\begin{array}{ccc} U & \xrightarrow{f_i} & \mathbb{P}^1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T_i} \\ U & \xrightarrow[f_i]{} & \mathbb{P}^1 \end{array}$$

commutes.

Proof. Consider first $Q = [0, 0, 1]$ and a projective change of coordinates A on $\mathbb{P}^1$. Write

$$A = (ax + by, cx + dy).$$

Then

$$\Lambda = (ax + by, cx + dy, z)$$

satisfies

$$\Lambda(Q) = Q$$

and

$$\begin{aligned} Af_Q([v_1, v_2, v_3]) = A([v_1, v_2]) &= [av_1 + bv_2, cv_1 + dv_2] \\ &= f_Q([av_1 + bv_2, cv_1 + dv_2, v_3]) = f_Q\Lambda([v_1, v_2, v_3]) \end{aligned}$$

hence

$$T_i f_Q = f_Q T.$$

Then (see proof of the prior lemma for definition of A)

$$T_i f_i = T_i f_Q A = f_Q T A = f_Q A(A^{-1}TA) = f_i(A^{-1}TA).$$

Note that $A^{-1}TA(P_i) = A^{-1}T(Q) = A^{-1}(Q) = P_i$. □

Apply a change of coordinates so that $P_1 = [0, 0, 1]$ and the remaining P_i's remain unchanged. This is possible by the two lemmas above. Fix $Q \in E_1$. Assume $Q = [\lambda, 1]$ for $\lambda \in K$. We lose no generality with this assumption, since we can always apply a suitable change of coordinates to generalize to this case.
What we aim to do now is construct a specific neighborhood of a point in an E_1. Let φ_3 denote the identification of $\mathbb{A}^2$ with $U_3 \subset \mathbb{P}^2$. Set $V = U_3 \setminus \{P_2, \ldots, P_t\}$, $W := \varphi_3^{-1}(V)$ and $W' := \psi^{-1}(W) = (\psi \circ \varphi_3)^{-1}(V)$. Consider also the morphism

$$\begin{gathered} \varphi : W' \to \mathbb{P}^2 \times \left(\mathbb{P}^1\right)^t \\ (x, z) \mapsto ([x, xz, 1], [z, 1], f_2([x, xz, 1]), \ldots, f_t([x, xz, 1])) \end{gathered}$$

Note that $\psi \circ \varphi_3 = \pi \circ \varphi$. Recall that ψ is the map $(a, c) \mapsto (a, ac)$ Set $V' := \varphi(W')$. We immediately see that

$$V' = B \setminus \left(\bigcup_{i \geq 2} E_i \cup V(x_3) \cup V(y_{12}) \right).$$

We thus conclude that V' is an open neighborhood of Q in B since $\varphi(0, \lambda) = Q$. This shows the case $i = 1$. To produce open neighborhoods for points in the other E_i, we first

move P_1 back to its original position via a change of coordinates, and follow the same constuction for $i = 2$, $i = 3$ and so on. We move on to prove the claim that B is the closure of G.

Lemma 5.4.25.

$$B = \text{cl}(G),$$

hence B is a variety.

Proof. Let $S \supset G$ be a closed set in $\mathbb{P}^2 \times (\mathbb{P}^1)^t$. Then $\varphi^{-1}(S)$ is closed in W', containing $\varphi^{-1}(G)$, which is easily checked to be equal to $W' \setminus V(x)$, hence $\varphi^{-1}(G)$ is open in W' and thus dense in W'. We then find that $\varphi^{-1}(S) = W'$, implying $S = V'$. □

Lemma 5.4.26. *Consider the morphism*

$$\phi : \mathbb{P}^2 \times \left(\mathbb{P}^1\right)^t \setminus V(x_3 y_{12}) \to \mathbb{A}^2$$
$$([x_1, x_2, x_3], [y_{11}, y_{12}], \ldots, [y_{t1}, y_{t2}]) \mapsto \left(\frac{x_1}{x_3}, \frac{y_{12}}{y_{11}}\right)$$

This morphism restricted to V' is the inverse of φ. We thus have the following diagram

$$\begin{array}{ccccc} A^2 \supset & W' & \xrightarrow{\varphi} & V' & \subset B \\ {\scriptstyle\psi}\downarrow & \downarrow & & \downarrow & \downarrow{\scriptstyle\pi} \\ A^2 \supset & W & \xrightarrow{\varphi_3} & V & \subset \mathbb{P}^2 \end{array}$$

π thus locally looks like ψ

Proof. Indeed, for $P = ([x, xz, 1], [1, z], f_2([x, xz, 1]), \ldots, f_t([x, xz, 1])) \in V'$,

$$\varphi\phi(P) = \varphi(\frac{x}{1}, \frac{z}{1}) = P$$

and for $(x, z) \in \mathbb{A}^2$,

$$\phi\varphi(x, z) = \phi([x, xz, 1], [1, z], f_2([x, xz, 1]), \ldots, f_t([x, xz, 1])) = (x, z).$$

Fix a point $Q \in B$. When Q sits in some E_i the commutativity of the diagram shows that in a suitable neighborhood around Q, π corresponds to ψ. When $Q \in G \cap \pi^{-1}(U_3)$, set $W := \varphi_3^{-1}(U \cap U_3)$ and $W' = \psi^{-1}(W)$ define

$$\varphi : W' \to B$$
$$(x, z) \mapsto ([x, xz, 1], (f_1, \ldots, f_t)(x, z))$$

Setting $V' := \varphi(W')$ we obtain local description of π at Q as something that looks like ψ in an affine chart. □

Lemma 5.4.27. *Let C be an irreducible projective curve in $\mathbb{P}^2$. Set $C_0 := C \cap U$, $C_0' := \pi^{-1}(C_0)$ and $C' := \mathrm{cl}_B(C_0')$. π restricts to a birational morphism $\Psi : C' \to C$, given by the restriction of π to C', birationality given via the isomorphism from C_0' to C_0. Locally this looks like the map θ from the affine case.*

Proof. Indeed, U and $\pi^{-1}(U)$ are isomorphic via π, so this is also the case for C_0 and C_0'. Let $Q \in C_0$ in some neighborhood $V' \cap C_0$ around Q, Ψ looks like f by the prior lemma. □

Theorem 5.4.28. *Let C be an irreducible projective plane curve with all multiple points ordinary. There is a non-singular projective curve C' birationally equivalent to C with the underlying morphism being onto.*

Proof. Pick $P_1, \ldots, P_t$ to be the multiple points of C and construct C_0, C_0' and C' as in the above lemma. We want to investigate the behavior of $P \in \pi^{-1}(P_i) \cap C' = E_i \cap C'$. Note that $\varphi_3^{-1}(P_i)$ is a multiple point of C_*. Thus $\theta^{-1}\varphi_3^{-1}(P_i)$ contain only simple points by Lemma 5.4.16 on a suitable curve C_*'. On some suitable open neighborhood V' of P in C', $\varphi_3\theta = \pi\varphi$, where $\varphi : W' \to V'$ is an isomorphism for some open set $W' \subset C_*'$. Thus $\varphi^{-1}(P)$ is a simple point in $\theta^{-1}\varphi_3^{-1}(P_i)$ contained in W'. Now note $\mathcal{O}_{\varphi^{-1}(P)}(C_*') = \mathcal{O}_{\varphi^{-1}(P)}(W') \simeq \mathcal{O}_P(V') = \mathcal{O}_P(C')$, which means P is simple. That it is onto is simple: C_0 and C_0' are isomorphic, while $C \cap E_i$ maps to P_i. □

Remark 5.4.29. While this is indeed an answer to the original goal of this section, note that C' is not a plane curve. This means we cannot apply theorems about plane curves to argue about C' and transfer these findings to results about C, which is the reason to resolve singularities in the first place.

5.4.4 Quadratic Transformations

Definition 5.4.30. The *fundamental points in* $\mathbb{P}^2$ are

$$P := [0,0,1], \quad P' := [0,1,0], \quad P'' := [1,0,0].$$

The *exceptional lines in* $\mathbb{P}^2$ are

$$l := V(z), \quad l' := V(y), \quad l'' := V(x)$$

Remark 5.4.31. l' and l'' intersect in P, and l passes through P' and P''.

For this section, we set

$$U := \mathbb{P}^2 \setminus (l \cup l' \cup l'') = \mathbb{P}^2 \setminus V(xyz)$$

We define a morphism

$$Q : \mathbb{P}^2 \setminus \{P, P', P''\} \to \mathbb{P}^2$$
$$[v_1, v_2, v_3] \mapsto [v_2v_3, v_1v_3, v_1v_2]$$

The image of Q is $U \cup \{P, P', P''\}$. Indeed, when $[v] \neq P, P', P''$, then $v_i, v_j \neq 0$ for $i \neq j$. If $v_k \neq 0$ for each k, then $[v]$ does not vanish on xyz. If $v_k = 0$ for some k, then $v_kv_i = 0$ and $v_kv_j = 0$, hence $Q([v]) \in \{P, P', P''\}$. Conversely, if $[v] = [v_1, v_2, v_3] \in U$, then $v_i \neq 0$ for each $i = 1, 2, 3$. Then

$$[v] = Q(Q([v])).$$

Note that $P = Q([1, 1, 0])$, $P' = Q([1, 0, 1])$ and $P'' = Q([0, 1, 1])$. Moreover,

$$Q^{-1}(P) = l \setminus \{P', P''\}, Q^{-1}(P') = l' \setminus \{P, P''\}, Q^{-1}(P'') = l'' \setminus \{P, P'\}.$$

Note that the following lemma is true

Lemma 5.4.32. *Q is an automorphism of U. It induces a birational map $Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.*

Remark 5.4.33. Q is called the *standard quadratic transformation* or the *standard Cremona transformation*.

We fix a projective plane curve C that is not an exceptional line. $C \cap U$ is an open set in C and a closed in U. Note that $Q(C \cap U)$ is then a closed curve in U. Define

$$C' := \text{cl}\left(Q^{-1}(U \cap C)\right) = \text{cl}\left(Q(U \cap C)\right).$$

With this definition Q restricts to a birational morphism $C' \setminus \{P, P', P''\} \dashrightarrow C$.

Lemma 5.4.34.

$$C'' = C$$

Proof. First note, that $C \cap U \subset \text{cl}(Q(\text{cl}(Q(C \cap U)))) = C'' \implies C = \text{cl}(C \cap U) \subset C''$. Conversely, by continuity,

$$C'' \subset \text{cl}(QQ(C \cap U)) = \text{cl}(C \cap U) = C.$$

□

Fix an irreducible, homogeneous $f \in K[x, y, z]$ with $C = V(f)$ and set $n := \deg f$. We define the *algebraic transform of f* to be

$$f^Q = f(yz, xz, xy),$$

which is homogeneous of degree $2n$. Set $r := m_P(f)$.

Lemma 5.4.35. *r is the largest natural number such that $z^r \mid f^Q$.*

Proof. For suitable forms $f_r, \ldots, f_n \in K[x,y]$ with $f_r \neq 0$, we may write

$$f = \sum_{i=r}^{n} f_i z^{n-i},$$

hence

$$f^Q = \sum_{i=r}^{n} f_i(yz, xz)(xy)^{n-i} = z^r \sum_{i=r}^{n} z^{r-i} f_i(y,x)(xy)^{n-i},$$

and since $z \nmid f_r(y,x)$ and $z \mid f^{r-i} f_i(y,x)$ for every other i the result follows. □

Setting $r' := m_{P'}(f)$ and $r'' := m_{P''}(f)$, we get analogous results for these quantities. I.e., r' is the largest number such that $y^{r'} \mid f^Q$ and r'' the largest numbers such that $x^{r''} \mid f^Q$. We can thus write

$$f^Q = x^{r''} y^{r'} z^r f'$$

where $f' \in K[x,y,z]$ is form not divisible by x, y or z. This polynomial is called the *proper transform of f*.

Lemma 5.4.36. *The proper transform of f has degree $2n - (r + r' + r'')$, $(f')' = f$ and f' is an irreducible polynomial such that $C' = V(f')$.*

Proof. That deg $f' = 2n - (r + r' + r'')$ is apparent from the definition of f'. Note that

$$(f^Q)^Q = (f(yz, xz, xy))^Q = f(xyzx, xyzx, xyzx) = (xyz)^n f,$$

hence $(f')^Q = x^{n-(r+r')} y^{n-(r+r'')} z^{n-(r'+r'')} f$, implying $(f')' = f^Q$. Consider an irreducible factor p of f'. Then $f \mid p^Q$, hence $x^{r''} y^{r'} z^r f' \mid (xyz)^{\deg\, p} p$. It follows that $f' \mid p$, hence $f' = p$. Let $v = [v_1, v_2, v_3] \in Q^{-1}(C \cap U)$. Then $f^Q(v) = 0$, and since x, y and z do not vanish at v, we get $f'(v) = 0$, hence $V(f') \supset Q^{-1}(C \cap U) = Q(C \cap U)$, so by the typical dimension argument, $V(f') = \mathrm{cl}(Q(C \cap U)) = C'$ □

Lemma 5.4.37.

$$m_P(f') = n - (r' + r''), \quad m_{P'}(f') = n - (r + r''), \quad m_{P''}(f') = n - (r + r').$$

Proof. Note that

$$f' = \sum_{i=0}^{n-r} f_{r+i}(y,x) x^{n-(r+r'+i)} y^{n-(r+r''+i)} z^i$$

The lowest degree term in the homogeneous decomposition of f'_* is thus $f_n(y,x) x^{-r'} y^{-r''}$, which is of degree $n - (r' + r'')$, hence $m_P(f') = n - (r' + r'')$. The other cases follow completely analogously. □

Definition 5.4.38. A projective plane curve C is at a *good position* if no excepetional line is tangent to C at an exceptional point.

Lemma 5.4.39. *If C is in good position, so is C'*

Proof. We prove only that L is not tangent to C' at P'. The proof for L through P'' follows by symmetry. Proofs for L' with P and P'', and L'' with P and P' can also be found by symmetry.
L is tangent at P' if and only if $I(P', f' \cap z) > m_{P'}(f') = n - (r + r'')$ or equivalently

$$I\left(P', f_r(y,x)x^{n-(r+r')}y^{n-(r+r'')} \cap z\right) > n - (r + r'')$$

or equivalently (since $I(P', y^{n-(r+r'')} \cap z) = 0$ and $I(P', x^{n-(r+r')} \cap z) = n - (r + r')$)

$$I\left(P', f_r(y,x) \cap z\right) > 0,$$

and this only happens when $P \in f_r(y,x) \cap z$, or when $f_r(1,0) = 0$. Since C is in good position, y is not tangent to f at P, hence by a similar string of arguments to the above, $f_r(1,0) \neq 0$, so we conclude that z is not a tangent of C' at P'. □

We now assume that C is in good position and that $P \in C$. Set

$$C_0 := (C \cap U) \cup \{P\}.$$

Define

$$f'_* := f(x, xz, 1)x^{-r}, \quad C'_* = V(f'_*).$$

Consider the morphism

$$\begin{aligned} \psi_* : \mathbb{A}^2 \supset C'_* &\to C_* \subset \mathbb{A}^2 \\ (x,z) &\mapsto f_*(x, xz) \end{aligned}$$

We further define $\eta := Q|_{C'_0} : C'_0 \to C_0$ where $C'_0 = C' \setminus V(xy)$.

Lemma 5.4.40. *There is a neighborhood $W \subset C^*$ of $(0,0)$ and isomorphisms $\varphi : W \to C_0$ and $\varphi' : W' := \eta^{-1}(W) \to C'_0$ such that the diagram*

$$\begin{array}{ccc} C'_* \supset W' & \xrightarrow{\varphi'} & C'_0 \subset C' \\ {\scriptstyle\psi_*}\downarrow \quad \downarrow & & \downarrow{\scriptstyle\eta} \\ C_* \supset W & \xrightarrow{\varphi} & C_0 \subset C \end{array}$$

Proof. Take $W := (C_* \setminus V(xy)) \cup \{(0,0)\}$,

$$\varphi : W \to C_0$$
$$(v_1, v_2) \mapsto [v_1, v_2, 1]$$

whose inverse is $[v_1, v_2, v_3] \mapsto (v_1, v_2)$. Indeed, a point in C_0 is $[v_1, v_2, v_3]$ where all $v_i \neq 0$ or $v_1 = v_2 = 0$ and $v_3 = 1$. Thus, $C_* \setminus V(xy)$ is in bijection with $C \setminus V(xyz)$ and $(0,0)$ is mapped to P. Furthermore, we define

$$\varphi' : W' \to C_0'$$
$$(v_1, v_3) \mapsto [v_3, 1, v_1 v_3]$$

whose inverse is $[v_1, v_2, v_3] \mapsto (v_3/v_1, v_1/v_2)$. Note that

$$\eta\varphi'(a,c) = \eta([c,1,ac]) = [ac, ac^2, c] = [a, ac, 1] = \varphi(a, ac) = \varphi\psi_*(a,c).$$

□

Lemma 5.4.41. *Suppose $P_1, \ldots, P_s \in C' \cap L$ are not fundamental points. Then*

$$m_{P_i}(C') \leq I(P_i, C' \cap L) \text{ and } \sum_i I(P_i, f' \cap z) = r.$$

Proof. The first statement is a simple fact about intersection numbers. For the second,

$$\sum_i I(P_i, f' \cap z) = \sum_i I(P_i, f_r(y,x) \cap z) = r.$$

□

Remark 5.4.42. In the case $P \notin C$, note that $m_P(C) = 0$, hence $\sum_v I(v, f' \cap z) = 0$, hence in this case there are no non-fundamental points in $C' \cap L$.

Definition 5.4.43. A curve C is in *excellent position*, if it is in good position, L intersects C in n distinct points all different from the fundamental points, and L' and L'' intersect C in $n - r$ distinct points all not fundamental.

We proceed with a classification of the multiple points of C' when C is in excellent position.

Lemma 5.4.44. *Suppose C is in excellent position. C' has the following multiple points.*

1. *Those in $C' \cap U$ are in one-to-one correspondence with those in $C \cap U$ and this correspondence preserves multiplicity and ordinary multiple points.*

2. *P, P', P'' are multiple points of C' with $m_P(C') = n$, $m_{P'}(C') = m_{P''}(C') = n - r$.*

3. *$C' \cap L'$ and $C' \cap L''$ contain only non-fundamental points. Let $P_1, \ldots, P_s$ be the non-fundamental points on $C' \cap L$. Then $m_P(C') \leq I(P_i, C' \cap L)$ and $\sum_1^s I(P_i, C' \cap L) = r$.*

Proof. 1. $C \cap U \simeq C' \cap U$ via η. The local rings of these varieties thus agree at each point, so their multiplicities agree. Furthermore, being an ordinary multiple point is also completely determined by properties of the associated local ring Haven't done this exercise yet! Do at some point, hence this is also preserved by the isomorphism.

2. A prior lemma shows that since C is in good position so is C' and that $(C')' = C$. The assumption of C being in excellent position implies

$$m_P(C') = \sum_{v \neq P, P', P''} I(v, (C')' \cap L) = \sum_{v \neq P, P', P''} I(v, C \cap L) = n$$

a similar computation shows that $m_{P'}(C') = m_{P''}(C') = n - r$. Why are they ordinary

3. Note that $r \leq I(P, C \cap L')$ and that $\sum_v I(v, C \cap L') = n$. By the assumption that C is excellent position, we must have $I(P, C \cap L') \leq r$, hence $I(P, C \cap L') = r$. It thus follows that $I(P'', C \cap L') = 0$ and hence $P'' \notin C$, which implies $C' \cap L'$ contains no non-fundamental points. A symmetric argument shows that $C' \cap L''$ contains no non-fundamental points. $m_P(C') \leq I(P_i, C' \cap L)$ and $\sum_1^s I(P_i, C' \cap L) = r$ follows from Lemma 5.4.41 □

Definition 5.4.45. For an irreducible projective curve C of degree n, we define

$$g^*(C) := \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C) - 1)}{2}$$

Lemma 5.4.46. *If C is excellent position, then*

$$g^*(C') = g^*(C) - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C') - 1)}{2},$$

where $P_1, \ldots, P_s$ are the non-fundamental points in $C' \cap L$

Proof. Indeed,

$$\begin{aligned}
g^*(C') &= \frac{(2n-r-1)(2n-r-2)}{2} - \sum_{v\in C'} \frac{m_v(C')(m_v(C')-1)}{2} \\
&= \frac{(n-1)(n-2)}{2} + \frac{3n^2-3n-4nr+3r+r^2}{2} - \sum_{v\in C'\cap U} \frac{m_v(C)(m_v(C)-1)}{2} \\
&- \frac{n(n-1)}{2} - \frac{2(n-r)(n-r-1)}{2} - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C')-1)}{2} \\
&= \frac{(n-1)(n-2)}{2} - \sum_{v\in C\cap U} \frac{m_v(C)(m_v(C)-1)}{2} - \frac{r(r-1)}{2} - \\
&\sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C')-1)}{2} \\
&= \frac{(n-1)(n-2)}{2} - \sum_{v\in C\cap U} \frac{m_v(C)(m_v(C)-1)}{2} \\
&- \sum_{v\in C\cap V(xyz)} \frac{m_v(C)(m_v(C)-1)}{2} - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C')-1)}{2} \\
&= \frac{(n-1)(n-2)}{2} - \sum_{v\in C} \frac{m_v(C)(m_v(C)-1)}{2} - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C')-1)}{2} \\
&= g^*(C) - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C')-1)}{2}
\end{aligned}$$

□

We now abandon P, P', P'' and L, L', L'' being notation for special objects. The whole point of considering a curve in excellent position is that this can easily be transformed into a curve with only ordinary multiple points. The upshot is that any irreducible projective plane curve is can be transformed into one in excellent position. This puts us in a position where we can resolve the singularities of any projective plane curve.

Lemma 5.4.47. *Suppose* char $K = 0$. *Let f be any irreducible projective plane curve and $P \in f$ any point. There is a change of coordinates T = such that f^T is in excellent position and $T([0,0,1]) = P$.*

Proof. If we can find lines l, l', l'' such that $L' \cdot C = rP + \sum_1^{n-r} P'_i$ and $L'' \cdot C = rP + \sum_1^{n-r} P''_i$, $L \cdot C = \sum_1^n P_i$ with $P, P_1, \ldots, P_n, P'_1, \ldots, P'_{n-r}, P''_1, \ldots, P''_{n-r}$ are all distinct, then by Lemma 4.1.55 we can find a change of coordinates T such that $L^T = z$, $(L')^T = y$ and $(L'')^T = x$. Then since $T(P) \in x \cap y$, hence $T(P) = [0,0,1]$. By Proposition 4.3.50 (this is where the assumption of char $K = 0$ is used) we can find L' and L''. For L just take any simple point on C different from P and any P'_i and P''_i and apply the same result. □

Definition 5.4.48. For a projective change of coordinates T on $\mathbb{P}^2$ as in the above lemma, the *quadratic transform* of T is QT. Let f be any irreducible projective plane curve and $P \in f$ any point, the irreducible projective plane curve

$$(f^T)'$$

is the *quadratic transformation of* f. If f^T is in excellent position and $T([0,0,1]) = P$, then the quadratic transformation of f is said to be *centered at* P.
Given a sequence of irreducible projective plane curves, $f = f_1, \ldots, f_m = g$ such that f_i is a quadratic transformation of f_{i-1}, then we say that f is *transformed* g via finite sequence of quadratic transformations.

Theorem 5.4.49. *Suppose* char $K = 0$. *Any irreducible projective plane curve C may be transformed into an irreducible projective plane curve with only ordinary multiple points via a finite sequence of quadratic transformations.*

Proof. We describe an algorithm for producing such a sequence of quadratic transformations.

1. If C has only ordinary multiple points, terminate, otherwise continue to 2.
2. Pick a non-ordinary multiple point P on C and change the value of C to be a quadratic transformation centered at P (one such exists by the prior lemma), go to 1.

Each step maps P to the ordinary multiple point $[0,0,1]$. If C' is the quadratic transformation of C centered at P, then if C' has the same number of non-ordinary multiple points as C by Lemma 5.4.44 one of these must be among the non-fundamental points $P_1, \ldots, P_s$ of C', hence

$$g^*(C') = g^*(C) - \sum_1^s \frac{m_{P_i}(C')(m_{P_i}(C)-1)}{2} < g^*(C).$$

It thus follows that this algorithm must terminate in most $N + g^*(C)$ steps. □

5.4.5 Non-singular Models of Curves

Lemma 5.4.50. *Suppose* char $K = 0$. *Let f be any irreducible projective plane curve and $P \in f$ any point.*

1. *Let $S \subset f$ be any finite set of points not containing P. There is a quadratic transform T of f centered at P with $T^{-1}(S) \cap V(xyz) = \emptyset$.*

2. *When T is given as in 1. there is a neighborhood of S on f isomorphic to an open set in $(f^T)'$.*

3. *If S contains only simple points of f, then there is a curve f' that only has ordinary multiple points and neighborhood U of S on f isomorphic to an open set U' on f' containing only simple points on f'.*

Proof. 1. In the proof of Lemma 5.4.47 we may also pick L, L', L'' such that non of these pass through any point in S, since are also infinitely many such lines.
2. $f \setminus (L \cup L' \cup L'')$ is an open neighborhood of S, and

$$T^{-1}(f \setminus (L \cup L' \cup L'')) = f^T \setminus V(xyz) = f^T \cap (\mathbb{P}^2 \setminus V(xyz)) \\ \simeq (f^T)' \cap (\mathbb{P}^2 \setminus V(xyz))$$

where the last isomorphism is via Q. Check
3. In the algorithm described in Theorem 5.4.49 only pick quadratic transformations of the form described in 1. i.e. at each step apply 1. on f_{i-1} and S_{i-1} with $S_1 = S$ and $S_i = T^{-1}(S_{i-1})$ when $i > 1$, and $f_1 = f$ and $f_i = (f_{i-1}^T)'$. □

Example 5.4.51. The goal of this example, for a given quadratic transformation T centered at a point $P \in C$, is to compute deg $(C^T)'$ and $g^*((C^T)')$ when

1. P is an ordinary multiple point on C.

2. P is simple on C

3. P is not in C.

1. In this case deg $(C^T)' = 2n - m_P(C)$, since $[1,0,0], [0,1,0] \notin C^T$. Moreover, locally around P, Q looks like θ, and $((C^T)')_* \simeq (C_*^T)'$, hence by the affine case $\theta^{-1}((0,0))$ consist of only simple points. These correspond to the non-fundamental points of $(C^T)' \cap z$, hence

$$g^*((C^T)') = g^*(C)$$

2. Here deg $(C^T)' = 2n - 1$. We compute,

$$g^*((C^T)') = g^*(C^T) - \sum_1^s \frac{m_{P_i}((C^T)')(m_{P_i}((C^T)') - 1)}{2} = g^*(C)$$

Note for the above that $[1,0,0]$ and $[0,1,0]$ are ordinary multiple points of multiplicity $n-1$ summing to $2n-2$. There is thus at most one non-fundamental point in $(C^T)' \cap z$

and if it exists it must be simple.
3. When $P \notin C$, we get deg $(C^T)'$ = $2n$. Moreover,

$$g^*((C^T)') = g^*(C^T) - \sum_1^s \frac{m_{P_i}((C^T)')(m_{P_i}((C^T)') - 1)}{2} = g^*(C^T) = g^*(C)$$

The last equality follows from the fact that when $[0, 0, 1] \notin C^T$, there are no non-fundamental points in $(C^T)' \cap z$

The upshot of having considered this example is the following

Lemma 5.4.52. *We may pick f' in Lemma 5.4.50 3. of arbitrarily large degree. I.e. given any $N \geq 0$ there is an f' satisfying the conditions of Lemma 5.4.50 with* deg $f' \geq N$.

Proof. Given the f' obtained in 3. apply a series of suitable quadratic transformation centered at any point (at this point there will be no non-ordinary multiple points) until the degree is larger than N. □

Definition 5.4.53. *A nonsingular model* of a projective curve C is a nonsingular, projective curve X_C with a birational map $\Phi : X_C \dashrightarrow C$.

Theorem 5.4.54. *Let C be any projective curve. There is a nonsingular model of C, $\Phi : X_C \dashrightarrow C$. The underlying morphism of any such nonsingular model is onto. If $\Phi' : Y_C \dashrightarrow C$ is another such pair, then the following diagram commutes*

$$\begin{array}{ccc} X_C & & \\ {\scriptstyle \exists! \Psi}\,\vdots\downarrow & \searrow^{\Phi} & \\ Y_C & \xrightarrow[\Phi']{} & C \end{array}$$

X_C is thus the intial object in the category $C/\text{NonSingCur}_K^{\text{dom}}$ where $\text{NonSingCur}_K^{\text{bir}}$ are nonsingular curves over K with dominant rational maps as morphisms.

Proof. **Uniqueness (up to unique isomorphism):** Two such birational morphisms Φ and Φ', induce isomorphisms $\widetilde{\Phi} : K(C) \to K(X_C)$, $\widetilde{\Phi'} : K(C) \to K(Y_C)$, hence $\widetilde{\Psi} := \widetilde{\Phi}\widetilde{(\Phi')}^{-1}$. This is the unique map of fields such that

$$\begin{array}{ccc} K(X_C) & & \\ \uparrow & \nwarrow^{\widetilde{\Phi}} & \\ K(Y_C) & \xrightarrow[\widetilde{\Phi'}^{-1}]{} & K(C) \end{array}$$

commutes. There is a unique birational map Ψ inducing $\widetilde{\Psi}$. Ψ^{-1} is constructed using the same argument with $\widetilde{\Phi}^{-1}$ and $\widetilde{\Phi'}$ and noting that all simple points of X_C and Y_C are all

contained in the domain of Ψ resp. Ψ^{-1}.

Existence: By Corollary 5.3.38, C is birationally equivalent to a plane curve. We can pick this with only ordinary multiple points by Theorem 5.4.49, hence by Theorem 5.4.28, C is birationally equivalent to a nonsingular curve.

Φ **is onto:** Consider first the case $C \subset \mathbb{P}^2$. Let $P \in C$ by 2. of the prior lemma, we get a quadratic transformation, with $(C^T)' \dashrightarrow C^T$ having $T^{-1}(P)$ in its domain. The repeated application of this fact as in 3. of the prior lemma shows that there is C' with only ordinary multiple points birationally equivalent to C with a point P' in the domain of this birational map such that this is mapped to P. Constructing X_C from C' by Theorem 5.4.28 results in a birational map $X_C \dashrightarrow C'$ which is onto, hence $\Phi : X_C \to C$ maps some P' to P. The universal property of nonsingular models shows that for any other nonsingular model $\Phi' : Y_C \dashrightarrow C$, there is a point P'', with $\Phi'(P'') = P$. Any nonsingular model must then have an underlying map that is onto.

We extend this to the general case, $C \subset \mathbb{P}^n$. Let $P \in C$. We may choose a birational map $\Theta : C \to C_{\text{Plane}}$ with C_{Plane} being a plane curve such that $\Theta^{-1}(\Theta(P)) = \{P\}$ (cf. Proposition 5.3.45). Let $\Phi : X \to C_{\text{Plane}}$ be a nonsingular model for C_{Plane}. For some P', $\Phi(P') = \Theta(P)$, hence $\Theta^{-1}\Phi(P') = P$. □

Corollary 5.4.55. $K(_) : \text{NonSingCur}_K \to \text{AlgFuncField}^1_K$ *is fully faithul, here the codomain is Algebraic function fields over* K *in one variable.*

Remark 5.4.56. The points of a nonsingular model $\Phi : X \dashrightarrow C$ of a curve C are in one-to-one correspondence with discrete valuation rings of $K(C)$ (here we canonically identify $K(C)$ with $K(X)$). Note then that $\Phi(Q) = P \in C$ exactly when $\mathcal{O}_Q(X)$ dominates $\mathcal{O}_P(C)$.

Definition 5.4.57. Let (X, Φ) be a nonsingular model of curve C. Then a point on X is called a *place* on C or on $K(C)$. A place $Q \in X$ is *centered* at $P \in C$ if $\Phi(Q) = P$.

Lemma 5.4.58. *Let* C *be a projective plane curve,* (X, Φ) *a nonsingular model of* C *and* $P \in C$*. There is an affine neighborhood open* U *of* P *such that*

1. $U' := \Phi^{-1}(U)$ *is an affine open subvariety of* X*.*
2. $\Gamma(U')$ *is finite as a* $\Gamma(U)$*-module.*
3. *There is a non-zero* $t \in \Gamma(U)$ *such that* $t\Gamma(U) \subset \Gamma(U')$*.*
4. $\dim_K \Gamma(U')/\Gamma(U) < \infty$*.*

U *may be chosen such that it excludes an arbitrary set points* $S \subset C$ *with* $P \notin S$*.*

Proof. We first argue that we can find a series of quadratic transformations $C_1 = C, \ldots, C_N$ such that there are affine open sets $W_1 \subset C_1, \ldots, W_N \subset C_N$ such that

a) $P \in W_1$ and $S \cap W_1 = \emptyset$.

b) The underlying morphism of $f_i : C_{i+1} \dashrightarrow C_i$ are $f_i : W_{i+1} \to W_i$.

c) $W_{i+1} = f^{-1}(W_i)$ are affine and $\Gamma(W_{i+1})$ is module finite over $\Gamma(W_i)$ and $t\Gamma(W_i) \subset \Gamma(W_{i+1})$ for some $t \neq 0$.

Consider first the case where $N = 2$. First pick an open neighborhood V of P in C and a quadratic transformation (C_2, T) as in Lemma 5.4.50 2. Note that when restricted to $(C_2)_* = (C^T)'_*$, $Q : (C_2)_* \to C^T_*$ looks like $\theta : (a, c) \mapsto (a, ac)$. By Lemma 5.4.17 there is an affine, open neighborhood V_0 of $[0, 0, 1]$ in C^T_* such that $W_2 := \theta^{-1}(V_0)$ an affine neighborhood with $\Gamma(W_2)$ module finite over $\Gamma(V_0)$ and $t\Gamma(V_0) \subset \Gamma(W_2)$ for some $t \neq 0$. In the remark following this lemma, we know that can pick $V_0 \subset T^{-1}(V)$ and $W_2 \subset Q|^{-1}_{C_2} T^{-1}(V)$, hence picking $W_1 := T(V_0)$, we get the desired C_1, C_2, W_1, W_2 and $f_1 : C_1 \dashrightarrow C_2$ when taking into account that the Zariski topology is Frechét, which means we can pick this neighborhood away from S. Now suppose a), b) and c) for some sequence of quadratic transformations are satisfied, using the same argument as before we find a quadratic transformation of C_N centered at $f_{N-1} \cdots f_1(P)$ with an open affine neighborhood $W' \subset W_N$ and an affine open neighborhood W_{N+1} satisfying b) and c). Replace W_{N-1} with $f_i^{-1}(W')$, W_{N-2} with $f_{i-1}^{-1}(W_{N-1})$ and so on, and we get the desired sequence of quadratic transformations.

We pick such a sequence $C_1, \ldots, C_N$ with C_N having only ordinary multiple points. Take $f_N : X \dashrightarrow C_N$ to be the nonsingular model of C_N. Let $W_{N+1} := f_N^{-1}(W_N)$, we shrink the preceding neighborhoods as before so that W_{N+1} has the desired compatibility with the preceding W_i. f_N looks like $(a, c) \mapsto (a, ac)$ locally, so we get the module finiteness condition for these open affine sets as well. Set $U' := W_{N+1}$ and $U = W_1$. Being module finite is a transitive property, so $\Gamma(U')$ is finite as a $\Gamma(U)$-module.

write

$$\Gamma(U') := \sum_1^m \Gamma(U) z_i \quad (z_i \in \Gamma(U')).$$

Since $Q(\Gamma(U)) = Q(\Gamma(U'))$, we may find a non-zero $t \in \Gamma(U)$ such that $tz_i \in \Gamma(U)$. It thus follows that $t\Gamma(U) \subset \Gamma(U')$.

We define

$$\rho : \Gamma(U')/\Gamma(U) \to \Gamma(U)/\langle t \rangle$$
$$z + \Gamma(U) \mapsto tz + \langle t \rangle$$

which is an injective K-linear map. It thus suffices to prove that $\Gamma(U)/\langle t\rangle$ is finite-dimensional by rank-nullity. Since t has only finitely many zeroes (its pole set is an algebraic subset of U, hence it is finite) it follows from the Nullstellensatz (more specifically Corollary 3.2.26) that dim $\Gamma(U)/\langle t\rangle$. □

Proposition 5.4.59. *Let $C = V(f)$ be an irreducible projective plane curve, $P \in C$, $\Phi : X \dashrightarrow C$ a nonsingular model of C. Let g be a plane curve. Then*

$$I(P, g \cap C) = \sum_{v \in \Phi^{-1}(P)} \text{ord}_v(g)$$

Proof. Let $\overline{g}$ denote the image of $g_* \in K(\mathbb{P}^2)$ in $K(C)$. Pick U and U' as in Lemma 5.4.58 with $P \in U$ and $U \setminus \{P\} \subset \{P' : \overline{g}(P') \neq 0\}$. We may do this since there are only finitely many points on which $\overline{g}$ vanish, hence for an initial choice of U, replace it with something contained in U excluding these points. We have thus chosen U such that $\overline{g} \in \mathcal{O}_{P'}(C)$ is a unit for each $P' \in U$ with $P' \neq P$. Then

$$I(P, C \cap g) = \text{dim } \mathcal{O}_P(\mathbb{P}^2)/\langle f_*, g_*\rangle = \text{dim } \mathcal{O}_P(C)/\langle \overline{g}\rangle = \text{dim } \Gamma(U)/\langle \overline{g}\rangle,$$

where the second equality follows from Proposition 3.3.69. For the last equality note that

$$\Gamma(U) \twoheadrightarrow \mathcal{O}_P(C)/\mathcal{O}_P(C)\overline{g}$$

has kernel $\Gamma(U)\overline{g}$. Indeed, if $z \in \Gamma(U)$ is in $\mathcal{O}_P(C)\overline{g}$, then $z = h\overline{g}$ for some $h \in \mathcal{O}_P(C)$. Note that $z \in \mathcal{O}_{P'}(C)$ for $P' \in U \setminus \{P\}$, hence $h = z\overline{g}^{-1} \in \mathcal{O}_{P'}$, meaning $z \in \Gamma(U)\overline{g}$. Fulton claims that Corollary 3.3.77 applies here. I sort of does, I guess, but really I think we need something like Corollary 3.3.79, which is the not mandatory problem 2.47. This approach is less simple than what I did above. Set $V = \Gamma(U)$ and $V' = \Gamma(U')$. Consider $\rho : V \to V', z \mapsto \overline{g}z$. Since V/V' is finite dimensional, we get

$$\text{dim } V/\langle \overline{g}\rangle = \text{dim } V'/\langle \overline{g}\rangle$$

By (an appropriate version of) Corollary 3.3.77 and Lemma 1.6.135

$$\text{dim } V'/\langle \overline{g}\rangle = \sum_{v \in \Phi^{-1}(P)} \text{dim } \mathcal{O}_v/\langle \overline{g}\rangle = \sum_{v \in \Phi^{-1}(P)} \text{ord}_v(\overline{g}).$$

□

Lemma 5.4.60. *Suppose P is an ordinary multiple point on an irreducible projective plane curve C of multiplicity r with nonsingular model $\Phi : X \dashrightarrow C$. Write $\Phi^{-1}(P) = \{P_1, \ldots, P_r\}$. If $z \in K(C)$ with* $\text{ord}_{P_i}(z) \geq r - 1$, *then $z \in \mathcal{O}_P(C)$.*

Proof. Pick $U \ni P$ such that $z \in \mathcal{O}_{P'}(C)$ for each $P' \in U \setminus \{P\}$ and such that $\Phi : \Phi^{-1}(U) \to \Phi$ looks like θ in Lemma 5.4.17. By Lemma 5.4.16, $\text{ord}_{P_i}(x) = 1$, hence $zx^{1-r} \in \Gamma(U')$. Since $x^{r-1}\Gamma(U') \subset \Gamma(U)$, we get $z = x^{r-1}(zx^{1-r}) \in \Gamma(U) \subset \mathcal{O}_P(C)$. □

Proposition 5.4.61. *Let f be an irreducible projective plane curve, P an ordinary point of multiplicity r on f. Let $P_1, \ldots, P_r$ be the places centered at P. For plane curves g, h, Noether's conditions are satisfied at P with respect to f, g, h, if* $\text{ord}_{P_i}(h) \geq \text{ord}_{P_i}(g) + r - 1$ *for each i.*

Proof. Note that $h_* \in \langle f_*, g_* \rangle \subset \mathcal{O}_P(\mathbb{P}^2)$ if and only if $z := \frac{\overline{h_*}}{\overline{g_*}} \in \mathcal{O}_P(f)$. Since $\text{ord}_{P_i}(z) = r - 1$ for each i, it follows that $z \in \mathcal{O}_P(f)$ by the above lemma. □

Lemma 5.4.62. *Let $\Phi : X \to C$ be the nonsingular model of an irreducible projective plane curve. Let $P \in C$ be an ordinary multiple point with multiplicity r and places $Q_1, \ldots, Q_r$ on X centered at P. Let G be a projective plane curve and $s \leq r$. $m_P(g) \geq s$ if and only if* $\text{ord}_{Q_i}(\overline{g}) \geq s$ *for $i = 1, \ldots, r$.*

Proof. We pick $U \ni P$ in the usual way such that Φ locally looks like θ (from the affine case). By Lemma 5.4.21 2. $\text{ord}_{Q_i}(\overline{g}) \geq s$ for each i implies $m_P(g) \geq s$. Conversely, for any i, if $m_P(g) \geq s$ we get that $\text{ord}_{Q_i}(g) \geq s$, since $\text{ord}_{P_i}(g) \geq m_P(g)$ by 1. of Lemma 5.4.21. □

Lemma 5.4.63. *Let X be a nonsingular projective curve, $P_1, \ldots, P_s \in X$.*

1. *There is a projective plane curve with only ordinary multiple points whose nonsingular model is $\Phi : X \to C$ with $\Phi(P_i)$ simple in C for each i.*
2. *For any finite sequence $m_1, \ldots, m_s \in \mathbb{Z}$ there is a $z \in K(X)$ such that* $\text{ord}_{P_i}(z) = m_i$.
3. *C of part 1. may be found of arbitrarily large degree.*

Proof. 1. There is a birational map $\Psi : X \dashrightarrow Y$ to some irreducible projective plane curve Y. Suppose we can construct a curve C and a birational map $\phi : C \dashrightarrow Y$ such that $\phi^{-1}(\Psi(P_i))$ contain only simple points for each i. Let $\Phi : X' \dashrightarrow Y$ be a non-singular model of Y that factors through C via some $\Phi' : X' \dashrightarrow C$ and ϕ. Then we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Phi'} & C \\ {\scriptstyle \exists! \varphi}\uparrow & \nearrow_{\eta} & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{\Psi} & Y \end{array}$$

where $\eta := \Phi\varphi$. Then

$$\eta(P) = \Phi(\varphi(P_i)) \in \phi^{-1}(\Psi(P_i)),$$

since

$$\phi\eta(P_i) = \Psi(P_i).$$

It thus follows that $\eta : X \dashrightarrow C$ is a birational map to projective plane curve with only ordinary multiple points such that $\eta(P_i)$ is simple for each i.

Consider first the case $n = 1$ and set $P := P_1$. When $\Psi(P)$ is simple, we can apply Lemma 5.4.50 with respect to $S = \{\Psi(P)\}$ to obtain the desired curve C such that $\phi^{-1}(\Psi(P))$ is a simple point. Suppose $\Psi(P)$ is an ordinary multiple point. Take a quadratic transformation centered at $\Psi(P)$ and denote the resulting quadratic transformation of Y by C_1 and the birational map $C_1 \dashrightarrow C$ by ϕ. Note that $\phi^{-1}(\Psi(P)) = C_1 \cap z$ contain only simple points. This is seen from Lemma 5.4.40 and Lemma 5.4.16. More precisely, locally the standard quadratic transformation looks like $(a, c) \mapsto (a, ac)$ and the fiber of an ordinary multiple point with respect to this map consists exclusively of simple points. We obtain the desired curve C by applying Lemma 5.4.50 to C_1 with respect to $S = \phi^{-1}(\Psi(P))$. Suppose $\Psi(P)$ is a non-ordinary multiple point. Apply Lemma 5.4.50 3. to obtain a plane curve C_1 with only ordinary multiple points and a birational map $\phi_1 : C_1 \dashrightarrow C$. Let $\{R_1, \ldots, R_t\}$ be the subset of ordinary multiple points in $\phi_1^{-1}(\Psi(P))$. Take a quadratic transformation centered at R_t, resulting in a birational map $\phi_2 : C_2 \dashrightarrow C_1$ with $\phi_2^{-1}(R_t)$ all simple. Note that $\phi_2^{-1}(\{R_1, \ldots, R_t\})$ is a set containing at most $t - 1$ points, by induction there is a plane curve C with only ordinary multiple points and a birational map $\phi' : C \to C_1$ such that $(\phi')^{-1}(\{R_1, \ldots, R_t\})$ contain only simple points. We then set $\phi := \phi'\phi_1$.

2. From 1. there is an irreducible projective plane curve C and a birational map $\Phi : X \to C$ such that $\Phi(P_1), \ldots, \Phi(s)$ are simple in C. It Follows from Lemma 4.3.27 that there is a $z \in K(C) \simeq K(X)$ such that

$$\mathrm{ord}_{P_i}(z) = \mathrm{ord}_{\Phi(P_i)}(z) = m_i$$

for each i, where we on the left-hand side mean the order of the element in $K(X)$ corresponding to z.

3. Each time we apply Lemma 4.3.27, apply Lemma 5.4.52 instead. □

5.5 Riemann-Roch

5.5.1 Divisors

Definition 5.5.1. Let $X \to C$ be a non-singular model of an irreducible projective curve C. A *divisor on* X is an element in $\mathbb{Z}[X]$.
The degree of a divisor $d = \sum_{P \in X} n_P P \in \mathbb{Z}[X]$, is the quantity

$$\deg\ d := \sum_P n_P$$

Remark 5.5.2. Note that $\deg\,(d + d') = \deg d + \deg d'$.

Definition 5.5.3. A divisor $d = \sum_{P \in X} n_P P$ is *effective* or *positive*, if $n_P \geq 0$ for each P.

Definition 5.5.4. For a pair of divisors $\sum_{P \in X} n_P P$ and $\sum_{P \in X} m_P P$ we write

$$\sum_{P \in X} n_P P \geq \sum_{P \in X} m_P P$$

if $n_P \geq m_P$ for each P.

Definition 5.5.5. Suppose C is a plane curve of degree n and g is a plane curve not containing C as a component of degree m. The *divisor of* G, is defined to be

$$\mathrm{div}(G) := \sum_{P \in X} \mathrm{ord}_P(g) P.$$

Remark 5.5.6. By Proposition 5.4.59 and Bezout,

$$\deg\ \mathrm{div}(g) = \sum_{P \in X} \mathrm{ord}_P(g) = \sum_{P' \in C} I(P', C \cap g) = nm.$$

Definition 5.5.7. Let $z \in K(X) \setminus 0$. *The divisor of* z is defined to be

$$\mathrm{div}(z) := \sum_{P \in X} \mathrm{ord}_P(z) P.$$

Remark 5.5.8. Note that $\{P : z(P) = 0\}$, corresponding to the points with positive order, and $\mathcal{P}(z)$, corresponding to points with negative order, are both finite sets, hence $\mathrm{div}(z) \in \mathbb{Z}[X]$. With this in mind

$$(z)_0 := \sum_{P \in X : \mathrm{ord}_P(z) > 0} \mathrm{ord}_P(z) P$$

is called the *divisors of zeros of* z, and

$$(z)_\infty := \sum_{P \in X : \mathrm{ord}_P(z) < 0} -\mathrm{ord}_P(z) P,$$

the *divisors of poles of* z. We thus have

$$\mathrm{div}(z) = (z)_0 - (z)_\infty.$$

Lemma 5.5.9. *For $z, z' \in K(X)$, we have* $\mathrm{div}(zz') = \mathrm{div}(z) + \mathrm{div}(z')$ *and* $\mathrm{div}(z^{-1}) = -\mathrm{div}(z)$

Proof. This follows from basic properties of the order function. □

Lemma 5.5.10. *For any $z \in K(X) \setminus 0$,*

$$\deg\ \mathrm{div}(z) = 0.$$

Proof. Write $z = \frac{f+I(C)}{g+I(C)} \in \Gamma^{\mathbb{P}}(C)$ for suitable homogeneous polynomials f, g of the same degree, m say. Say C has degree n. Then

$$\mathrm{div}(z) = \mathrm{div}(f) - \mathrm{div}(g) = mn - mn = 0.$$

□

Corollary 5.5.11. *For a $z \in K(X) \setminus 0$ the following are equivalent*

1. $\mathrm{div}(z) \geq 0$.
2. $z \in K$.
3. $\mathrm{div}(z) = 0$.

Proof. Suppose $\mathrm{div}(z) \geq 0$. Then $\mathrm{ord}_P(z) \geq 0$ for each P, hence $z \in \mathcal{O}_P(C)$ for each P. Fix $P_0 \in C$. Then $z - z(P_0) = 0$ or $\deg\ \mathrm{div}(z - z(P_0)) = 0$. Note that $z - z(P_0) \in \mathcal{O}_P(C)$ for each P, meaning $\mathrm{ord}_P(z - z(P_0)) \geq 0$ for each P, and that $\mathrm{ord}_{P_0} z - z(P_0) > 0$. Then $z - z(P_0) = 0$, hence $z \in K$. □

Corollary 5.5.12. *Consider $z, z' \in K(X) \setminus 0$. Then* $\mathrm{div}(z) = \mathrm{div}(z')$ *if and only if $z = \lambda z$ for some $\lambda \in K$.*

Proof. $\mathrm{div}(z) = \mathrm{div}(z')$ if and only if $\mathrm{div}(z(z')^{-1}) = 0$, which is equivalent to $z(z')^{-1} \in K$. This is equivalent to $z = \lambda z'$ for some $\lambda \in K$. □

Definition 5.5.13. Two divisors $d, d' \in \mathbb{Z}[X]$ are *linearly equivalent* if $d = d + \mathrm{div}(z)$ for some $z \in K(X)$. In this case we write $d \equiv d'$.

Remark 5.5.14. Being linearly equivalent is an equivalence relation. Indeed, $d = d + \mathrm{div}(1)$ for each $d \in \mathbb{Z}[X]$. Suppose $d = d' + \mathrm{div}(z)$ for some $z \in K(X) \setminus 0$. Then

$$d' = d - \mathrm{div}(z) = d + \mathrm{div}(z^{-1}).$$

Suppose $d, d', d'' \in \mathbb{Z}[X]$ are divisors with $d = d' + \mathrm{div}(z)$ and $d' = d'' + \mathrm{div}(z')$ for some $z, z' \in K(X) \setminus 0$. Then

$$d = d' + \mathrm{div}(z) = d'' + \mathrm{div}(z') + \mathrm{div}(z) = d'' + \mathrm{div}(zz').$$

Lemma 5.5.15. *The following statements hold for divisors*

1. *For each $d \in \mathbb{Z}[X]$, $d \equiv 0$ if and only if $d = \mathrm{div}(z)$ for some $z \in K(X) \setminus 0$.*
2. *For each $d, d' \in \mathbb{Z}[X]$, $d \equiv d'$ implies* deg d = deg d'.
3. *For each $d_1, d_1', d_2, d_2' \in \mathbb{Z}[X]$, if $d_1 \equiv d_1'$ and $d_2 \equiv d_2'$, then*

$$d_1 + d_2 \equiv d_1' + d_2'.$$

4. *For $d, d' \in \mathbb{Z}[X]$. $d \equiv d'$ if and only if there are two curves g, g' of the same degree such that $d + \mathrm{div}(g) = d' + \mathrm{div}(g')$.*

Proof. 1. This is trivial, since $d \equiv 0$ is equivalent to $d = 0 + \mathrm{div}(z) = \mathrm{div}(z)$ for some $z \in K(X)$ by definition.
2. If $d \equiv d'$, then for some $z \in K(X) \setminus 0$,

$$d = d' + \mathrm{div}(z)$$

hence

$$\text{deg } d = \text{deg } (d' + \mathrm{div}(z)) = \text{deg } d' + \text{deg } \mathrm{div}(z) = d'.$$

3. Indeed, if $d_1 \equiv d_1'$ and $d_2 \equiv d_2'$, then $d_1 = d_1' + \mathrm{div}(z_1)$ and $d_2 = d_2' + \mathrm{div}(z_2)$ for some $z_1, z_2 \in K(X) \setminus 0$. Then

$$d_1 + d_2 = d_1' + d_2' + \mathrm{div}(z_1 z_2),$$

hence $d_1 + d_2 \equiv d_1' + d_2'$.
4. If $d' = d + \mathrm{div}(z)$, then $z = \frac{\overline{g}}{\overline{g'}}$ where g, g' are projective plane curves of the same degree and $\overline{g}, \overline{g'}$ are images of g and g' in $\Gamma^h(C)$. Then $\mathrm{div}(z) = \mathrm{div}(g) - \mathrm{div}(g')$, hence $d + \mathrm{div}(g) = d' + \mathrm{div}(g')$.
Conversely, if $d + \mathrm{div}(g) = d' + \mathrm{div}(g')$ for same degree curves. Then choosing $z = \frac{g}{g'}$, we get $d = d' + \mathrm{div}(z)$. □

Definition 5.5.16. Suppose C has only ordinary multiple points. We define the *minimal adjoint of C* to be the divisor

$$E_C := \sum_{Q \in X} (m_{\Phi(Q)}(C) - 1)Q.$$

An *adjoint of C* is any projective plane curve g with

$$\mathrm{div}(g) \geq E_C$$

Remark 5.5.17. Note that

$$\begin{aligned}\deg E_C &= \sum_{Q\in X}(m_{\Phi(Q)}(C)-1) = \sum_{P\in C}\sum_{Q\in\Phi^{-1}(P)}(m_{\Phi(Q)}-1) = \sum_{P\in C}\sum_{Q\in\Phi^{-1}(P)}(m_P(C)-1)\\ &= \sum_{P\in C} m_P(C)(m_P(C)-1) \geq 0,\end{aligned}$$

hence E_C is effective, hence any adjoint of C is effective.

Lemma 5.5.18. *A projective plane curve G is adjoint to C if and only if*

$$m_P(g) \geq m_P(C) - 1$$

for every multiple point $P \in C$. In particular. if C is non-singular, then every curve is an adjoint to C.

Proof. For each multiple point P apply Lemma 5.4.62 with $s = m_P(C) - 1$. This says that $m_P(g) \geq s$ is equivalent to $\text{ord}_Q(g) \geq s$ for each place centered at P. For each multiple point $P \in C$, having $\text{ord}_Q(g) \geq m_P$ for each place centered at P is by definition equivalent to $g \geq E_C$. □

Theorem 5.5.19. *Consider linearly equivalent divisors, $d, d' \in \mathbb{Z}[X]$. Let g be an adjoint of C of degree m with*

$$\text{div}(g) = d + E_C + a$$

for some effective divisor $a \in \mathbb{Z}[X]$. Then there is an adjoint d' of C of degree m such that

$$\text{div}(g') = d' + E_c + a.$$

Proof. Since $d \equiv d'$, we may find curves h, h' such that $d + \text{div}(h) = d' + \text{div}(h')$. Note that

$$\text{div}(hg) = \text{div}(h) + d + E_c + a = \text{div}(h') + d' + E_c + a \geq \text{div}(h') + E_C.$$

This shows that $\text{ord}_Q(hg) \geq \text{ord}_Q(h') + m_P(C) - 1$ for each $Q \in \Phi^{-1}(P)$, hence Noether's condition is satisfied for each $P \in C$ by Proposition 5.4.61. Noether's theorem show that there are curves f', g' such that

$$gh = f'f + g'h'$$

with $\deg\ g' = m$. It follows that

$$\begin{aligned}\text{div}(g') &= \text{div}(gh) - \text{div}(ff') - \text{div}(h') = \text{div}(gh) - \text{div}(h') = \text{div}(h') + d' + E_C + a - \text{div}(h')\\ &= d' + E_C + a\end{aligned}$$

□

5.5.2 The Vector Spaces L_d

Definition 5.5.20. For a divisor $d = \sum_{P\in X} n_P P \in \mathbb{Z}[X]$, define

$$L_d := \{f \in K(X) : \text{ord}_P(f) \geq -n_P \text{ for every } P \in X\}$$

Lemma 5.5.21. *L_d is a vector space over K for each $d \in \mathbb{Z}[X]$.*

Proof. We prove that L_d is K subspace of $K(X)$. Write $d = \sum_{P\in X} n_P P$ and suppose $f, g \in L_d$ and $a, b \in K$. Then for each $P \in X$,

$$\text{ord}_P(af + bg) \geq \min(\text{ord}_P(af), \text{ord}_P(bg)) = \min(\text{ord}_P(f), \text{ord}_P(g)) \geq -n_P,$$

hence $af + bg \in L_d$. Trivially $\text{ord}_P(0) = \infty \geq -n_P$ for each $P \in X$, hence $0 \in L_d$. □

Remark 5.5.22. We denote the dimension of the vector space L_d by l_d.

Lemma 5.5.23. *1. For divisors d, d' on X, if $d \leq d'$, then $L_d \subset L_{d'}$ and*

$$\dim_K L_{d'}/L_d \leq \deg(d - d').$$

2. $L_0 = K$ and $L_d = 0$ for divisors d on X with $\deg d < 0$.

3. For a divisor d on X, if $\deg d \geq 0$, then $l_d \leq (\deg d) + 1$. It follows that for every divisor d on X, $\dim_K L_d < \infty$.

4. For a pair of divisors d, d' on X, if $d \equiv d'$, then $L_d = L_{d'}$.

Proof. For each instance of d write $d = \sum_P n_P P$ and each instance of d', $d' = \sum_P n'_P P$.
1. If $f \in L_d$, then $\text{ord}_P(f) \geq -n_P \geq -n'_P$ for each $P \in X$, hence $f \in L_{d'}$. Note there is a finite sequence of points $P_1, \ldots, P_s \in X$ such that

$$d' = d + \sum_1^s P_i,$$

where $s = \deg(d' - d)$. Set $V_0 := L_d$ and $V_i := L_{d+\sum_1^i P_i}$ for $1 \leq i \leq s$. Then

$$\dim L_{d'}/L_d = \sum_0^s \dim V_{i+1}/V_i.$$

To prove that $\dim L_{d'}/L_d \leq s$, it thus suffices to prove that $\dim L_{d+P}/L_d \leq 1$ for an arbitrary point $P \in X$. Let t denote the uniformizing parameter in $\mathcal{O}_P(X)$. We define a K-linear map

$$\lambda : L_{d+P} \to K$$
$$f \mapsto (t^{n_P+1} f)(P)$$

Since $\mathrm{ord}_P\left(t^{n_P+1}f\right) \geq n_P + 1 - n_P - 1 = 0$ for each $f \in L_{d+P}$, this is well-defined. If $f \in L_d$, then

$$\mathrm{ord}_P\left(t^{n_P+1}f\right) \geq n_P + 1 - n_P = 1,$$

it follows that $\lambda(f) = 0$. Conversely, if $f \in L_{d+P}$ is given with $(t^{n_P+1}f)(P) = 0$, then $1 \leq \mathrm{ord}_P(t^{n_P+1}f) = n_P + 1 + \mathrm{ord}_P(f)$, hence $\mathrm{ord}_P(f) \geq -n_P$. We thus conclude that $\ker\ \lambda = L_d$. It follows that

$$L_{d+P}/L_d \hookrightarrow K \implies \dim\ L_{d+P}/L_d = \dim\ \mathrm{im}\ \lambda \leq 1.$$

2. Corollary 5.5.11 shows that $\mathrm{div}(f) \geq 0$ if and only if $f \in K$, hence $L(0) = K$. Suppose $\deg\ d < 0$ and let $P \in X$. If $f \in L_d$, then $\mathrm{ord}_P(f) \geq -n_P > 0$, hence $f(P) = 0$, meaning $f = 0$.

3. Let $P \in X$ and set $d' := d - (\deg\ D + 1)P$. Then $\deg\ d' < 0$, hence $L_{d'} = 0$. Then

$$l_d = \dim\ L_d = \dim\ L_d/L_{d'} \leq \deg\ d + 1.$$

4. By assumption $d = d' + \mathrm{div}(g)$ for a curve g. Set

$$\lambda : L_d \to L_{d'}$$
$$f \mapsto fg$$

This is a K linear map with inverse $h \mapsto \frac{h}{g}$, hence $l_d = l_{d'}$ □

Definition 5.5.24. Let $S \subset X$ and $d = \sum_{P \in X} n_P P \in \mathbb{Z}[X]$. We define the *degree of* d *in* S to be

$$\deg^S\ d := \sum_{P \in S} n_P.$$

We also define

$$L_d^S := \{f \in K(X) : \mathrm{ord}_P(f) \geq -n_P \text{ for each } P \in S\}$$

Lemma 5.5.25. *Conider divisors on* X, d, d' *with* $d \leq d'$ *and a subset* $S \subset X$. *Then* $L_d^S \subset L_{d'}^S$. *If* S *is finite, then*

$$\dim\ L_{d'}^S/L_d^S = \left(\deg^S\ d - d'\right).$$

Proof. The first statement is proven similarly to the proof of

$$d \leq d' \implies L_d \subset L_{d'}$$

(cf. Lemma 5.5.23 1.). For the second statement it suffices to prove

$$\dim\ L_{d+P}^S/L_d^S = 1$$

for an arbitrary $P \in X$. We prove that

$$\lambda : L^S_{d+P} \to K$$

is surjective (again cf. the proof of Lemma 5.5.23). Write $d = \sum_Q n_Q Q$. It suffices to prove that λ is not the zero map. To do this it suffices to find a $z \in K(X)$ with $\text{ord}_P(z) = -n'_p P - 1$ and $\text{ord}_Q(z) \geq -n_Q$ for all $Q \in S$. Such a rational function exists by Lemma 5.4.63 2. □

Lemma 5.5.26. *Consider a divisor $d = \sum_{P \in X} n_P P$ on X. Let $z \in K(X)$ be non-constant. If $z \in L_d$ is, then $\text{ord}_P(z) < 0$ for some $P \in X$.*

Proof. Suppose $\text{ord}_P(z) \geq 0$ for each $P \in X$. Then in particular $\text{ord}_P(z) > 0$ for some P. Moreover, $\text{ord}_P(z^{n+1}) > \text{ord}_P(z^n)$ for each $n \geq 1$ and $\{z^n : n \geq 1\}$ is linearly independent over K. If $z \in L_d$, then $z^n \in L_d$ for each $n \geq 0$. Since L_d is finite dimensional it thus follows that $z \notin L_d$. □

Proposition 5.5.27. *Let $z \in K(X)$ that is non-constant. Set $n := [K(X) : K(z)]$. Then*

1. *$(z)_0 = \sum_{P \in X} n_P P$ is an effective divisor of degree $m \leq n$.*
2. *There is a constant integer τ such that*

$$l_{r(z)_0} \geq rn - \tau$$

   for all r.

3. *deg $(z)_0 = n$.*

Proof. 1. Set $S := \{P \in X : n_P > 0\}$. For $v \in L^S_0$ let $\overline{v}$ denote the image of v in $L^S(0)/L^S_{-(z)_0}$. Note that by Lemma 5.5.25

$$\text{dim } L^S_0/L^S_{-(z)_0} = m$$

Pick a basis $\{\overline{v_1}, \ldots, \overline{v_m}\}$ of $L^S(0)/L^S(-(z)_0)$ over $K(z)$. Suppose for a contradiction that

$$\sum_1^m \frac{a_i}{b_i} v_i = 0$$

for some $\frac{a_i}{b_i} \in K(z)$ not all 0. Multiplying the left-hand side by $\prod b_i$, we get

$$\sum_1^m g_i v_i = 0.$$

we may pick these g_i of the form

$$\lambda_i + zh_i$$

with $\lambda_i \in K$ and $h_i \in K[z]$ such that $\lambda_i \neq 0$ for some i or $g_i = 0$ for all i. It follows that

$$\sum_1^m \lambda_i v_i = -z \sum_1^m h_i v_i \in L^S_{-(z)_0}.$$

But then $\sum_1^m \lambda_i \overline{v_i} = 0$, leading to a contradiction. It thus follows that $m \leq n$.
2. Fix $r \geq 0$. Pick a basis $\{w_1, \ldots, w_n\}$ for $K(X)$ over K. By Lemma 1.9.53 1. each w_i is algebraic over $K(z)$. For a fixed i there is a polynomial $g_i = x^{n_i} + \sum_0^{n_i-1} a_{ij}x^j$ in $K(z)[x]$ vanishing on w_i. Multiplying by a sufficiently large power of z^{-1}, we may assume it in $K[z^{-1}][x]$. We then have $\text{ord}_P(w_i) \geq 0$ for $P \notin S$ by Lemma 1.6.131, since

$$w_i^{n_i} + \sum_0^{n_i-1} a_{ij} w_i^j = 0$$

and $\text{ord}_P(a_{ij}) \geq 0$, which implies

$$n_i \text{ord}_P(w_i) = \text{ord}_P(a_{ij}) + (n_i - 1)\text{ord}_P(w_i) \implies \text{ord}_P(w_i) = \text{ord}_P(a_{ij}) \geq 0.$$

Set

$$t := \max_{i, P \in S} \text{ord}_P(w_i).$$

Then for each i, $\div(w_i) \geq -t(z)_0$. It thus follows that for each $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, r\}$, that

$$\div(w_i z^{-j}) \geq -(t + r)(z)_0 \implies w_i z^{-j} \in L_{(t+r)(z)_0}.$$

Since $\{w_1, \ldots, w_n\}$ is linearly independent over $K(x)$ and $\{1, x^{-1}, \ldots, x^{-r}\}$ is linearly independent over K, it follows that

$$\left\{w_i x^{-j} : i \in \{1, \ldots, n\}, j \in \{0, \ldots, r\}\right\}$$

is linearly independent over K. Then

$$l_{(r+t)(z)_0} = \dim L_{(t+r)(z)_0} \geq n(r + 1).$$

Moreover,

$$l_{(r+t)(z)_0} = l_{r(z)_0} + \dim\ L_{(r+t)(z)_0}/L_{r(z)_0} \leq l(r(z)_0) + tm.$$

For the first equality we use rank-nullity and for the second we use Lemma 5.5.23 1. Setting $\tau := tm - n$ we are done, since then

$$l_{r(z)_0} \geq l_{(r+t)(z)_0} - tm \geq n(r + 1) - tm = nr - \tau.$$

3. By 3. of Lemma 5.5.23

$$rn - \tau \leq l_{r(z)_0} \leq rm + 1 \implies r(n-m) \leq \tau + 1.$$

Since this holds for any r, we must have $n = m$. □

Corollary 5.5.28. *The following are equivalent*

1. *C is rational.*
2. *X is isomorphic to $\mathbb{P}^1$.*
3. *There is a $z \in K(X)$ with* deg $(z)_0 = 1$.
4. *For some $P \in X$, $l_P > 1$.*

Proof. "1. $\iff$ 2.": C is rational if and only if it is birationally equivalent to $\mathbb{P}^1$ which is equivalent to X being birationally equivalent to $\mathbb{P}^1$. Two non-singular curves are birationally equivalent if and only if they are isomorphic.
"2. $\implies$ 3.": If X is birationally equivalent to $\mathbb{P}^1$, then $K(X) \simeq K(\frac{x}{y})$. In particular $[K(X) : K(\frac{x}{y})] = 1$, hence deg $(x)_0 = 1$.
"3. $\implies$ 2.": Let $z \in K(X)$ with deg $(z)_0 = 1$. Then $[K(X) : K(z)]$ = deg $(z)_0 = 1$, hence $K(X) = K(z) \simeq K(x) = Q(K[x])$, hence X is birationally equivalent to $\mathbb{P}^1$.
"4. $\implies$ 1.": Note that since $P \geq 0$, $l_P \leq 2$, hence $l_P = 2$. Then there is a non-constant $z \in L_P$. By Lemma 5.5.26, $(z)_\infty = P$. Then

$$\text{deg } (z)_0 = \text{deg } (z)_\infty = 1.$$

Then $[K(X) : K(z)] = 1$, hence C is rational.
"1. $\implies$ 4.": Suppose C is rational. Then $K(X) \simeq K(\mathbb{P}^1) = K(t)$ with $t = \frac{x}{y} \in Q(K[x, y])$. We have that div$(t) = [0, 1] - [1, 0]$, hence $l_{[1,0]} = 2$. □

Lemma 5.5.29. *For divisors $d, d' \in \mathbb{Z}[X]$, if $d \leq d'$, then $l_{d'} \leq l_d +$* deg $(d' - d)$

Proof. We apply rank-nullity to the surjective K-linear map

$$\pi : L_{d'} \twoheadrightarrow L_{d'}/L_d$$

and get

$$l_{d'} = \text{dim } (\text{ker } \pi) + \text{dim } (\text{im } \pi) = \text{dim } L_d + \text{dim } L_{d'}/L_d \leq l_d + \text{deg } (d' - d).$$

□

Lemma 5.5.30. *Consider a divisor $d \in \mathbb{Z}[X]$. $l_d > 0$ if and only if d is linearly equivalent to an effective divisor.*

Proof. If $l_d > 0$, then there is a non-zero $z \in L_d$. Then $\text{div}(z) \geq -d$, hence

$$d = (\text{div}(z) + d) + \text{div}(z^{-1}) \implies d \equiv \text{div}(z) + d.$$

Conversely, $d = d' + \text{div}(z)$ for some effective divisor $d' \in \mathbb{Z}[X]$ and some $z \in K(X)$. Since d' is effective $\text{div}(z) \leq d$, hence $\text{div}(z) \leq d$, meaning $\text{div}(z^{-1}) \geq d$, implying $0 \neq z^{-1} \in l_d$. □

Lemma 5.5.31. *Consider a divisor $d \in \mathbb{Z}[X]$ with $l_d > 0$. Let $z \in L_d \setminus 0$. $z \notin L_{d-P}$ for all but finitely many $P \in X$. It follows that $l_{d-P} = l_d - 1$ for all but finitely many $P \in X$.*

Proof. □

5.5.3 Riemann's Theorem

Definition 5.5.32. Let C be an irreducible projective plane curve with non-singular model X. The *genus* of C, denoted $g(C)$ or $g(X)$, is the smallest integer with

$$l_d \geq \deg\ d + 1 - g$$

for each $d \in \mathbb{Z}[X]$.

Lemma 5.5.33. *Suppose C is an irreducible projective plane curve. Define*

$$s(d) := \deg\ d + 1 - l_d \quad (d \in \mathbb{Z}[X])$$

Then

1. *If C has genus g, then $s(0) = 0$ and $g \geq 0$.*
2. *If $d \equiv d'$, then $s(d) = s(d')$.*
3. *If $d \leq d'$, then $s(d) \leq s(d')$.*

Proof. 1. is trivial.
2. If $d \equiv d'$, then $\deg\ d = \deg\ d'$ and $l_d = l_{d'}$, so $s(d) = s(d')$.
3. By Lemma 5.5.29

$$l_{d'} \leq l_d + \deg\ d' - \deg\ d,$$

hence

$$s(d) = \deg\ d + 1 - l_d \leq \deg\ d' + 1 - l_{d'} = s(d').$$

□

Lemma 5.5.34. *Consider a non-constant rational function z on X. For any divisor $d \in \mathbb{Z}[X]$ there is a divisor $d' \in \mathbb{Z}[X]$ with $d \equiv d'$ and an integer $r \geq 0$ such that*

$$d' \leq r(z)_0$$

Proof. Write $(z)_0 = \sum_{P \in X} n_P P$ and $d = \sum_{P \in X} m_P P$. Set $t := z^{-1}$ and consider the set

$$T := \{P \in X : m_P > 0 \text{ and } \mathrm{ord}_P(t) \geq 0\}.$$

Set $f := \prod_{P \in T}(t - t(P))^{m_P}$. For $P \in X$ with $\mathrm{ord}_P(y) \geq 0$ note that P is a zero of $(y - y(P))^{m_P}$, hence $\mathrm{ord}((y - y(P))^{m_P}) > m_P$. For such P we thus have $m_P - \mathrm{ord}_P(f) \leq 0 = n_P$. Set $d' = d - \mathrm{div}(f)$. When $P \notin T$, $m_P \leq 0$ or $\mathrm{ord}_P(y) < 0$. In the first case $m_P - \mathrm{ord}_P(f) \leq 0 \leq n_P$, obviously. In the second case $n_P > 0$, hence for a sufficiently large r_P, $m_P - \mathrm{ord}_P(f) \leq r_p n_P$. Since there are only finitely many P adhering to the second cases, it follows that for a sufficiently large r, $m_P - \mathrm{ord}_P(f) \leq r n_P$ for each $P \in X$, hence $d \equiv d'$ and

$$d' \leq r n_P.$$

□

Theorem 5.5.35. *(Riemann's theorem) Every irreducible plane curve, C, has a genus. Moreover,*

$$g(C) \geq 0.$$

Proof. Let $z \in K(X)$ be non-constant, set $n := [K(X) : K(z)]$ and let τ be the smallest integer such that

$$l_{r(z)_0} \geq rn - \tau$$

for every $r \geq 0$ (cf. Proposition 5.5.27). Then

$$s(r(z)_0) = rn + 1 - l_{r(z_0)} \leq \tau + 1$$

Since $(s(r(z)_0))_{r \geq 0}$ is increasing, it follows that $s(r(z)_0) = \tau + 1$ for a sufficiently large r. Set $g(C) = \tau + 1$. Let $d \in \mathbb{Z}[X]$. There is a $d' \equiv d$ such that for some $r' \geq r$,

$$d' \leq r'(z)_0.$$

It thus follows that

$$s(d) = s(d') \leq s(r'(z)_0) = g,$$

hence

$$l_d \geq \deg\ d + 1 - g,$$

hence C has genus g by the minimality of τ. □

Corollary 5.5.36. *If d_0 is a divisor on X such that $l_{d_0} = \deg\ d_0 + 1 - g(X)$ and d, d' is a pair of divisors on X such that $d \equiv d' \geq d_0$, then*

$$l_d = \deg\ d + 1 - g(X).$$

Proof. It suffices to prove that $l_{d'} \leq \deg\ d' + 1 - g(X)$. Indeed,

$$\begin{aligned} l_{d'} &\leq l_{d_0} + \deg\ (d' - d_0) = \deg\ d_0 + 1 - g(X) + \deg\ d' - \deg\ d_0 \\ &= \deg\ d' + 1 - g(X). \end{aligned}$$

□

Corollary 5.5.37. *Let z be a non-constant rational function on X. Then*

$$g(X) = \deg\ r(x)_0 - l_{r(x)_0} + 1 = s(r(x)_0)$$

for all sufficiently large r.

Proof. This is by construction of genus as the smallest $\tau + 1$ where τ is an integer with $l_{r(x)_0} \geq r \deg\ r(x)_0 - \tau$ for all r, and indeed, we have also seen that $s(r(x)_0) = \tau + 1 = g(X)$ for all sufficiently large r. □

Corollary 5.5.38. *There is an integer N such that for all divisors $d \in \mathbb{Z}[X]$ with* $\deg\ d \geq N$*,*

$$l_d = \deg\ d + 1 - g(X).$$

Proof. Pick a divisor d_0 on X such that $l_{d_0} = \deg\ d_0 + 1 - g(X)$. Set $N := \deg\ d_0 + g(X)$. Let d be a divisor on X with $d \geq N$. Then

$$\deg\ (d - d_0) + 1 - g(X) \geq \deg\ d_0 + N - \deg\ d_0 + 1 - g(X) = 1 > 0.$$

It follows that $l_{d-d_0} > 0$, hence $d - d_0 + \mathrm{div}(f) \geq 0$ for some rational function f on X (cf. Lemma 5.5.30), hence $d \equiv d + \mathrm{div}(f) \geq d_0$, hence by the first of the two above corollaries,

$$l_d = \deg\ d + 1 - g(X).$$

□

We use the above corollary to give a method for calculating the genus of a given curve. Specifically, we use the functoriality of Set $\ni X \mapsto \mathbb{Z}[X] \in$ ab and the fact that every curve is birationally equivalent to a plane curve with only ordinary multiple points.

Proposition 5.5.39. *Let C be an irreducible projective plane curve of degree n with only ordinary multiple points. Then*

$$g(C) = g^*(C).$$

Proof. WLOG z intersects $C = V(f)$ in $n := \deg\ C$ distinct points $P_1, \ldots, P_n$. Define

$$E_m := m \sum_1^n P_i - E_C \quad (m \geq 1)$$

and note that

$$N := \deg\ E_m = mn - \sum_{P \in C} m_P(C)(m_P(C) - 1).$$

Define

$$A_m := \{g \in K[x, y, z] : g \text{ is a form of degree } m \text{ adjoint to } C\}.$$

We want to compute the dimension of A_m for large m. Consider a $g \in A_m$. Then $m_P(g) \geq m_P(C) - 1$ for each $P \in C$. Letting $Q_1, \ldots, Q_t$ be the multiple points of C. Then

$$\begin{aligned} &\dim V(m, (r_{Q_1}(C))Q_1, \ldots, r_{Q_t}(C)Q_t) \\ &= \frac{m(m+3)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C) - 1)}{2}. \end{aligned}$$

for large m (cf. Theorem 4.3.38). Note the $V(m, (r_{Q_1}(C))Q_1, \ldots, r_{Q_t}(C)Q_t) = (A_m \setminus 0)/\sim$, hence for large m

$$\begin{aligned} \dim\ A_m &= \frac{m(m+3)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C) - 1)}{2} + 1 \\ &= \frac{(m+1)(m+2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C) - 1)}{2}. \end{aligned}$$

Consider the K-linear function

$$\begin{aligned} \rho : A_m &\to L_{E_m} \\ g &\mapsto \frac{\overline{g}}{\overline{z^m}} \end{aligned}$$

Note that $\rho(g) = 0$ if and only if $f \mid g$. We prove that ρ is surjective. Let $t = \frac{a}{b} \in L_{E_m}$. Then

$$\operatorname{div}(\frac{a}{b}) \geq -E_m = E_C - m \sum_1^n P_i,$$

hence

$$\operatorname{div}(az^m) = \operatorname{div}(a) + m \sum_1^n P_i \geq E + \operatorname{div}(b).$$

We can then by Proposition 5.4.61 write

$$az^m = \lambda b + \mu f,$$

hence

$$t = \frac{a}{b} = \frac{\lambda}{z^m} = \rho(\lambda) \in K(C).$$

Define

$$\psi : V(m-n,3) \to A_m$$
$$h \mapsto hf$$

Then we get a short exact sequence

$$0 \longrightarrow V(m-n,3) \xrightarrow{\psi} A_m \xrightarrow{\rho} L_{E_m} \longrightarrow 0$$

For sufficiently large m, we thus have

$$\begin{aligned} l_{E_m} &= \frac{(m+1)(m+2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C)-1)}{2} - \dim\ V(m-n,3) \\ &= \frac{(m+1)(m+2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C)-1)}{2} - \frac{(m-n+2)(m-n+1)}{2} \\ &= \deg\ E_m + 1 + \left(\frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C)-1)}{2} \right) \end{aligned}$$

and since $(\deg\ E_m)_{m \leq 1}$ is a strictly increasing sequence it follows that for sufficiently large m, by Corollary 5.5.38,

$$g(C) = \frac{(n-1)(n-2)}{2} - \sum_{P} \frac{m_P(m_P-1)}{2}.$$

□

Lemma 5.5.40. *Consider* $t := \frac{x}{y} K(\mathbb{P}^1)$. *Then*

$$l_{r(t)_0} = r + 1.$$

Proof. Set $P := [0,1]$. Then $r(t)_0 = rP$, hence $l_{r(t)_0} \leq \deg\ r(t)_0 + 1 = r + 1$. Note that

$$\left\{ t^{-j} : j \in \{0, \ldots, r\} \right\} \subset L_{r(t)_0}$$

is linearly independent over K, hence $l_{r(t)_0} = r + 1$. □

Proposition 5.5.41.

$$g(C) = 0 \iff C \cong \mathbb{P}^1,$$

i.e. a curve has genus 0 *if and only if it is rational.*

Proof. " $\Longrightarrow$ ": If $g(C) = 0$, then

$$l(P) \geq \deg\ P + 1 - g(C) = 1 + 1 - g(C) = 2 > 0$$

for each $P \in X$, hence C is rational by Corollary 5.5.28.
" $\Longleftarrow$ ": Set $t := \frac{x}{y} \in K(\mathbb{P}^1)$. Then for large r,

$$g(C) = \deg\ r(t)_0 - l_{r(t)_0} + 1 = r - (r+1) + 1 = 0$$

by Corollary 5.5.37. □

Corollary 5.5.42. *For any irreducible projective plane curve* C*,*

$$g(C) \leq g^*(C).$$

Proof. Let C' be a curve birationally equivalent to C whose multiple points are all ordinary. Then

$$g(C) = g(C') = g^*(C') \leq g^*(C).$$

□

Corollary 5.5.43. C *is rational if and only if*

$$\frac{(n-1)(n-2)}{2} = \sum_{P \in C} \frac{m_P(C)(m_P(C)-1)}{2}.$$

Corollary 5.5.44. *With the setup of Proposition 5.5.39 and its proof,*

1. *every* $h \in L_{E_m}$ *can be written as* $\frac{\overline{\lambda}}{\overline{z}^m}$ *where* h *is an adjoint of* C *of degree* m*.*

2. $$\deg\ E_{n-3} = 2g(C) - 2 \text{ and } l_{E_{n-3}} \geq g(C).$$

Proof. 1. This follows from ρ in Proposition 5.5.39 being surjective.
2. For the first identity we have

$$\begin{aligned}\deg\ E_{n-3} &= (n-3)(n) - \sum_P m_P(C)(m_P(C)-1) \\ &= (n-1)(n-2) - \left[\sum_P m_P(C)(m_P(C)-1)\right] - 2 = 2g(C) - 2.\end{aligned}$$

Note that when $m < n$, no form g of degree m but 0 is divisible by f, hence $L_{E_m} = A_m$, hence

$$l_{E_{n-3}} = \dim\ A_{n-3} \leq \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{m_P(C)(m_P(C)-1)}{2} = g(C).$$

□

Proposition 5.5.45. *C has genus 1 if (here we need* char $K \neq 2$*) and only if it is birationally equivalent to a non-singular cubic.*

Proof. Suppose C is birationally equivalent to a non-singular cubic X. Then

$$g(C) = g(X) = g^*(X) = \frac{(3-1)(3-2)}{2} - \sum_{P} \frac{m_P(X)(m_P(X)-1)}{2} = 1.$$

Suppose conversely that $g(C) = 1$. In particular, note that C is not rational, since this only happens when $g(C) = 0$. Then $l_P = 1$ for each $P \in C$. Then

$$l_{rP} = \deg\ rP + 1 - g(C) = r$$

for each $r \geq 1$. It thus follows that $L(2P)$ has a basis $\{1, v\}$ for some $v \in K(\mathbb{P}^1) \setminus K$. Write $\operatorname{div}(x) = \sum_Q n_Q Q$. Then $n_P \geq -2$. In particular, $n_P \in \{-1, -2\}$. If $n_P = -1$, then $(v^{-1})_0 = 1$, meaning C would be rational. We thus have $n_P = -2$. Then $[K(v) : K] = (v)_\infty = 2$. Moreover, there is a basis $\{1, v, w\}$ with $w \notin K(v)$, hence $K(C) = K(v, w)$. Note that there are basis elements $1, v, w, vw, v^2, w^2 \in L_{6P}$. We thus have

$$aw^2 + bvw + cw + \underbrace{\sum_{0}^{3} \alpha_i v^i}_{-Q(v)} = 0 \tag{11}$$

For suitable $a, b, c, \alpha_i \in K$ with at least one of these different from 0. Here $Q \in K[x]$ with $\deg\ Q \leq 3$. Note that since $\mathcal{O}_P(X)$ is a DVR for a suitable P,

$$\operatorname{ord}_P(aw^2 + bvw + cw) \in \{-6, -5, -3\} \text{ and } \operatorname{ord}_P(Q(v)) \in \{-6, -4, -2, 0\},$$

and $\operatorname{ord}_P(aw^2 + bvw + cw) = \operatorname{ord}_P(Q(v))$, we have WLOG $a = 1$ and that $\deg\ Q = 3$. Replacing w by $w - \frac{1}{2}(bv - c)$, we get that $w^2 = \prod_1^3 (v - \beta_i)$ for some $\beta_i \in K$. If $\beta_1 = \beta_2$, then $\left(\frac{w}{v-\beta_1}\right)^2 = v - \beta_3$, hence $v, w \in K\left(\frac{w}{v-\beta_1}\right)$, which would imply C is rational. We thus conclude that $\beta_1, \beta_2, \beta_3$ are distinct. We thus see that

$$K(C) = K(y^2 z - \prod_{1}^{3} (x - \beta_i z))$$

hence C is birationally equivalent to the non-singular curve $y^2 z - \prod_1^3 (x - \beta_i z)$. □

Corollary 5.5.46. *A non-singular plane curve of degree n has genus*

$$\frac{(n-1)(n-2)}{2}.$$

Example 5.5.47. The above corollary shows that there are plane curves that are not birationally equivalent to a non-singular plane curve. For instance $y^2xz = x^4 + y^4$ has a node, i.e. an ordinary double point, hence it has genus is

$$\frac{(4-1)(4-2)}{2} - \frac{2(2-1)}{2} = 3 - 1 = 2$$

and no m satisfies $\frac{(m-1)(m-2)}{2} = 2$, so no non-singular curve can be birationally equivalent to $y^2xz = x^4 + y^4$.

Corollary 5.5.48. *Lines an conics are rational. No other non-singular plane curves are rational.*

Corollary 5.5.49. *Singular irreducible cubics are rational.*

Proof. A singular irreducible cubic has one ordinary multiple point; either a node or a cusp. When it has a node, its genus is

$$\frac{(3-1)(3-2)}{2} - 2(2-1) = 1 - 1 = 0.$$

When it has a cusp its genus is smaller than 0 by the same argument, hence its genus is also equal to 0. □

Example 5.5.50. Do some genus computations

5.5.4 Differentials of a Curve

Lemma 5.5.51. *Let $f \in K[x, y]$ be an affine curve. We define*

$$D : \Gamma(f) \to K(f)$$
$$\overline{g} \mapsto g_x(\overline{x}, \overline{y}) - \frac{f_x(\overline{x}, \overline{y})}{f_y(\overline{x}, \overline{y})} g_y(\overline{x}, \overline{y})$$

This is a well-defined derivation.

Proof. Let $\overline{g} = \overline{h}$. Then

$$\begin{aligned}
&g_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}g_y(\overline{x},\overline{y}) - \left(h_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}h_y(\overline{x},\overline{y})\right)\\
&= (g-h)_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}(g-h)_y(\overline{x},\overline{y})\\
&= (af)_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}(af)_y(\overline{x},\overline{y})\\
&= a_x(\overline{x},\overline{y})f(\overline{x},\overline{y}) + a(\overline{x},\overline{y})f_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}(a_y(\overline{x},\overline{y})f(\overline{x},\overline{y}) + a(\overline{x},\overline{y})f_y(\overline{x},\overline{y}))\\
&= a(\overline{x},\overline{y})f_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}a(\overline{x},\overline{y})f_y(\overline{x},\overline{y}) = 0.
\end{aligned}$$

So D is well-defined. It is clearly K-linear. Let $\overline{g}, \overline{h} \in \Gamma(f)$. Then

$$\begin{aligned}
D\overline{g}\overline{h} &= (gh)_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}(gh)_y(\overline{x},\overline{y})\\
&= (\overline{g}h_x(\overline{x},\overline{y}) + g_x(\overline{x},\overline{y})\overline{h}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}(\overline{g}h_y(\overline{x},\overline{y}) + g_y(\overline{x},\overline{y})\overline{h})\\
&= \overline{g}\left(h_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}h_y(\overline{x},\overline{y})\right) + \overline{h}\left(g_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}g_y(\overline{x},\overline{y})\right)\\
&= \overline{g}D\overline{h} + \overline{h}D\overline{g}.
\end{aligned}$$

□

Proposition 5.5.52. *Let L be an algebraic function field in one variable over K.*

1. *$\Omega(L)$ is a one dimensional vector space over L.*
2. *(char $K = 0$). If $z \in L$ and $z \notin K$, then dz is a basis for $\Omega(L)$.*

Proof. 1. We may assume $L = K(f)$ for some affine plane curve $f \in K[x,y]$. Indeed, $L = K(C)$ for some curve C and any curve is birationally equivalent to an affine plane curve. We may assume $f_y \neq 0$, hence $f \nmid f_y$. Then $f_y(\overline{x},\overline{y}) \neq 0$. Note that $d\overline{x}$ and $d\overline{y}$ generate $\Omega(L)$ over L. Then

$$0 = df(\overline{x},\overline{y}) = f_x(\overline{x},\overline{y})d\overline{x} + f_y(\overline{x},\overline{y})d\overline{y},$$

hence

$$d\overline{y} = \frac{-f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}d\overline{x}.$$

Note that for $D : \Gamma(f) \to K$ of the prior lemma that

$$D\overline{x} = x_x(\overline{x},\overline{y}) - \frac{f_x(\overline{x},\overline{y})}{f_y(\overline{x},\overline{y})}x_y(\overline{x},\overline{y}) = 1.$$

By Lemma 1.9.82 and Lemma 1.9.85 it follows that $d \neq 0$, hence $\Omega(L)$ is a one dimensional vector space generated by $d\overline{x}$.
2. $\overline{x} \neq 0$, it is transcendental over K. Note then that $dz \neq 0$, since for some non-zero $g \in K[\overline{x}][t] \subset K(\overline{x})[t]$ of minimal degree (in t), $g(\overline{x}, z) = 0$, hence

$$0 = dg(\overline{x}, z) = g_x(\overline{x}, z)d\overline{x} + g_t(\overline{x}, z)dz.$$

Since $g \neq 0$, by the assumption that char $K = 0$, it follows that $g_x, g_t \neq 0$, hence $g_x(\overline{x}, z), g_t(\overline{x}, z) \neq 0$, implying

$$dz = \frac{-g_x(\overline{x}, z)}{g_y(\overline{x}, z)}d\overline{x} \neq 0.$$

It follows that dz is a basis for $\Omega(L)$. □

Definition 5.5.53. (char $K = 0$) Let L be an algebraic function field over K in one variable. Let $z, t \in L$ with t non-constant. There is a unique $v \in L$ such that $dz = vdt$. We call v the *derivate of z with respect to t*, and denote it by

$$\frac{dz}{dt} \text{ or when no confusion arises } z'.$$

Proposition 5.5.54. *Let L be an algebraic function field in one variable over K. Let $\mathcal{O}$ be a discrete valuation ring of L with uniformizing parameter t. If $f \in \mathcal{O}$, then so is $\frac{df}{dt}$.*

Proof. WLOG $L = K(C)$ for some affine curve $C \in K[x, y]$, hence we may assume $\mathcal{O} = \mathcal{O}_P(f)$ with $P = (0, 0)$ simple in C. Choose N such that $\text{ord}_P(\overline{x}'), \text{ord}_P(\overline{y}') \geq -N$. Let $f \in \mathcal{O}$. Suppose first $f \in K[\overline{x}, \overline{y}]$ Then

$$\text{ord}_P(f') = \text{ord}_P(f_x(\overline{x}, \overline{y})\overline{x}' + f_y(\overline{x}, \overline{y})\overline{y}') \geq -N.$$

Now in general we may write $f = \frac{g}{h}$, with $g, h \in K[\overline{x}, \overline{y}]$ and $h(P) \neq 0$. Then

$$\text{ord}_P(f') = \text{ord}_P\left(\frac{1}{h^2}(hg' - gh')\right) = \min(\text{ord}_P(g'), \text{ord}_P(g) + \text{ord}_P(h')) \geq -N$$

Write

$$f = \sum_{i<N} \lambda_i t^i + t^N z,$$

for suitably $\lambda_i \in K$ and $z \in \mathcal{O}$. Then

$$f' = \sum i\lambda_i t^{i-1} + zNt^{N-1} + t^N z'.$$

Since $\text{ord}_P(z') \geq -N$, each term in the above sum is an element of $\mathcal{O}$, hence $f' \in \mathcal{O}$. □

5.5.5 Canonical Divisors

Fix a projective plane curve C with nonsingular model X. Define $\Omega := \Omega_K(K(C))$. We call the elements of this vector space *the differentials on X or on C.*

Definition 5.5.55. Let $\omega \in \Omega$ and P a place on X. Let $t \in \mathcal{O}_P(X)$ be a uniformizing parameter and write $\omega = fdt$ for some $f \in K(X)$. We define the order of ω at P to be

$$\text{ord}_P(\omega) := \text{ord}_P(f).$$

Remark 5.5.56. One has to check that this is independent of choice of uniformizing parameter. Let s be another uniformizing parameter. Then $\omega = gdt$ for some $g \in K(X)$. Then

$$\frac{f}{g} = \frac{ds}{dt} \in \mathcal{O}_P(X) \text{ and } \frac{g}{f} = \frac{dt}{ds} \in \mathcal{O}_P(X),$$

hence $\text{ord}_P(f) = \text{ord}_P(g)$.

When C is a curve of degree 1 or 2, we know that Riemann's Theorem holds with equality for each divisor on C. The goal of the following lemma is to compute a divisor that will reveal the missing term that makes Riemann's Theorem hold true in the case of C having degree ≥ 3.

Lemma 5.5.57. *Suppose $C = V(f)$ is a plane curve of degree $n \geq 3$ with only ordinary multiple points. Let g be a plane curve of degree $n-3$. Then*

$$\text{div}(g) - E_C = \sum_{P \in X} \text{ord}_P(\omega)$$

for some $\omega \in \Omega$.

Proof. WLOG assume $z \bullet C = \sum_1^n P_i$ with distinct P_i, $[1,0,0] \notin C$ and no tangent at a multiple point passes through $[1,0,0]$. Set $t := \frac{x}{z}$ and $s := \frac{y}{z}$. Set $f_1 = f_x(t,s,1)$ and $f_2 = f_y(t,s,1)$. Define $E_m := m\sum_1^n \Phi^{-1}(P_i) - E_C$. Choose $\omega := dt$. Note that $\text{div}(g) - E_C \equiv \text{div}(h) - E_C$ for any non-zero rational function h on X. It thus suffices to prove that

$$\text{div}(\omega) = \text{div}(z^{n-3}) - E_C + Div(f_2) = E_{n-3} + \text{div}(f_2).$$

Since $f_2 = \frac{f_y}{z^{n-1}}$, this is equivalent to showing that

$$\text{div}(\omega) - \text{div}(f_y) = 2\sum_1^n \Phi^{-1}(P_i) - E_C.$$

We check that the coefficient on the LHS matches those on the RHS locally (i.e. at each $Q \in X$). Before doing this, note that

$$0 = df(t,s,1) = f_1 dt + f_2 ds + f_z(t,s,1)d1 = f_1 dt + f_2 ds$$

which implies

$$\omega = \frac{-f_2}{f_1} ds = \frac{-f_y}{f_x} ds.$$

Then for each $Q \in X$,

$$\mathrm{ord}_Q(\omega) - \mathrm{ord}_Q(f_y) = \mathrm{ord}_Q(ds) - \mathrm{ord}_Q(f_x).$$

We split the comparison of the LHS and RHS into two cases:

1. Suppose Q is a place centered at a P_i. Note that y is a line not through P_i and z is line through P_i not tangent to C at P_i. Then $s^{-1} = \frac{z}{y}$ is a uniformizing parameter in $\mathcal{O}_Q(X)$. Moreover,

   $$0 = d1 = dss^{-1} = sds^{-1} + s^{-1}ds \implies ds = -s^2 ds^{-1} \implies \mathrm{ord}_Q(ds) = -2.$$

   By Lemma 4.3.28, $f_x(P) \neq 0$, hence $\mathrm{ord}_Q(\omega) - \mathrm{ord}_Q(f_y) = -2$, and on the RHS at Q, we have $-2\Phi^{-1}(P_i) - (r_Q - 1) = -2\Phi^{-1}(P_i)$, hence LHS matches RHS.

2. Suppose Q is a place centered at $P = [a,b,1] \in C$, we may assume $a = b = 0$, since taking differential and derivative is invariant under translation.
   Suppose y as tangent to C at P: Since y passes through $[1,0,0]$, it follows that P is simple. Then t is a uniformizing parameter of $\mathcal{O}_P(C)$ and $f_y(P) \neq 0$, since its tangent a P is $y = f_x(P)x + f_y(P)y + f_z(P)z$. It thus follows that $\mathrm{ord}_Q(\omega) = \mathrm{ord}_Q(f_y) = 0$ which matches the coefficient at P on the RHS.
   Suppose y is not tangent to C to at P. Then s is a uniformizing parameter (cf. Lemma 5.4.16) in $\mathcal{O}_Q(X)$, hence $\mathrm{ord}_Q(ds) = 0$ and $\mathrm{ord}_Q(f_x) = r_Q - 1$ (cf. Proposition 5.4.20), which matches the RHS.

□

Lemma 5.5.58. *Let $\omega, \omega' \in \Omega \setminus 0$, then*

$$\mathrm{ord}_P(\omega') = \mathrm{ord}_P(\omega) + \mathrm{ord}_P(f)$$

for some $f \in K(C) \setminus 0$ for each $P \in X$. Hence, $\mathrm{div}(\omega) \sum_P \mathrm{ord}_P(\omega)P$ *is a well-defined divisor.*

Proof. Indeed, $\omega' = f\omega$ for some f. Moreover, by the prior lemma there is a differential τ with $\sum_P \text{ord}_P(\tau)P$ is a well-defined divisor, hence $\text{div}(\omega) = \text{div}(\tau) + \text{div}(f)$ for some f. □

Remark 5.5.59. Note that if ω and ω' are two non-zero differentials, we have $\text{div}(\omega) \equiv \text{div}(\omega)$. We call a divisor of the form $\text{div}(\omega)$ a *canonical divisor*.

Corollary 5.5.60. *Let W be a canonical divisor on X. Then* $\deg\ W = 2g(X) - 2$ *and* $l_W \geq g(X)$.

Proof. Note that $E_{n-3} = \text{div}\,(z^{n-3}) - E_C$, hence the result follows from Corollary 5.5.44 □

5.5.6 The Riemann-Roch Theorem for Irreducible Projective Plane Curves

For the following suppose C has degree greater than or equal to 3. Fix a canonical divisor of C, W say. We aim to find the missing term in Riemann's Theorem. Note that for large divisors this is just 0, i.e. for large d,

$$l_d = \deg\ d + 1 - g(C).$$

Lemma 5.5.61. *Let $P \in X$ and $d \in \mathbb{Z}[X]$. If $l_d > 0$, and $l_{W-d-P} \neq l_{W-d}$, then $l_{d+P} = l_d$.*

Proof. Again WLOG we may assume C has only ordinary multiple points, P is simple, and $C \cdot Z = \sum_1^n P_i$ with distinct P_i, $P \notin Z$. We may also assume $W = E_{n-3}$ and that d is effective (cf. Lemma 5.5.30). Then $L_{W-d} \subset L_W$. By assumption, we may pick an $h \in L_{W-d} \setminus L_{W-d-P}$. We may write $h = \frac{g}{z^{n-3}}$ where g is an adjoint of C of degree $n-3$. □

Theorem 5.5.62. *(Riemann-Roch) For any divisor d on X,*

$$l_d = \deg\ d + 1 - g(X) + l_{W-d}.$$

Proof. □

6 Algebraic Geometry using Schemes

6.1 Sheaves & Presheaves

6.1.1 Presheaves

Definition 6.1.1. Consider a category $\mathcal{C}$. A *presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C} \to \text{Set}$.

For a subcategory $\mathcal{C}'$ of Set, a *$\mathcal{C}'$-valued presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$. We denote the image of a morphism $\varphi : V \to U$ under $\mathcal{F}$ by φ_{UV}.

Remark 6.1.2. We will mainly be interested in the case of $\mathcal{C}$ being obtained from a topological space: Consider a topological space (X, τ_X). We consider the category (X, τ_X) induced by the preorder on $(\tau_X, \subset)$. This will be called a *presheaf on X* and denoted $X \to$ Set. Moreover we will mainly be interested in presheaves valued in algebraic categories such as Group, AbGroup, Ring, etc.

Example 6.1.3. Let X be a variety. For an open subvariety $U \subset X$, set

$$\mathcal{O}(U) := \{f \in K(X) : f \text{ is regular on some open subset } U' \subset U\},$$

which is readily verified to be a subring of $K(X)$. Consider open subsets $V \subset U \subset X$, we define a ring homomorphism (in fact a K-algebra homomorphism)

$$\begin{aligned} \mathcal{O}(U) &\to \mathcal{O}(V) \\ f &\mapsto f|_V \end{aligned}$$

Indeed, if f is regular on some $U' \subset U$, then $f|_V$ is regular on $U' \cap V$. Then

$$\begin{aligned} \mathcal{O} : X &\to \text{CRing}^{\text{op}} \\ U &\mapsto \mathcal{O}(U) \\ V \subset U &\mapsto \mathcal{O}(U) \to \mathcal{O}(V) \end{aligned}$$

defines a presheaf on X valued in the category of commutative rings.

Definition 6.1.4. For objects U, V in $\mathcal{C}$ and a morphism $\varphi : V \to U$, for an $s \in \mathcal{F}(U)$, we define *restriction of s to V* to be

$$s|_V := \varphi_{UV}.$$

Lemma 6.1.5. *Consider a presheaf on a topological space $\mathcal{F} : X \to$ grp. Consider moreover a collection of ideals $\{I_U \trianglelefteq \mathcal{F}(U) : U \in \tau_X\}$ with $\varphi_{UV}(I_U) \subset I_V$ for each $U \supset V$. For each $U \subset X$ open, set $\mathcal{F}'(U) := \mathcal{F}(U)/I_U$. We have a well-defined map of groups*

$$\begin{aligned} \varphi'_{UV} : \mathcal{F}'(U) &\to \mathcal{F}'(V) \\ s + I_U &\mapsto s|_V + I_V \end{aligned}$$

for $U \supset V$. Then

$$\mathcal{F}' : X \to \text{grp}$$
$$U \mapsto \mathcal{F}'(U)$$
$$U \supset V \mapsto \varphi'_{UV}$$

Proof. Indeed, for $U \supset V \supset W$,

$$\varphi'_{VW}\varphi'_{UV}(s + I_U) = \varphi_{VW}\varphi_{UV}(s) + I_W = \varphi_{UW}(s) + I_W = \varphi'_{UW}(s + I_U),$$

for each $s + I_U \in \mathcal{F}'(U)$. Moreover,

$$\varphi'_{UU}(s + I_U) = \varphi_{UU}(s) + I_U = s + I_U.$$

□

6.1.2 Sheaves

Definition 6.1.6. Consider a topological space X. consider a presheaf $\mathcal{F} : X \to \text{Set}$ on X. $\mathcal{F}$ is called a *sheaf on* X if

1. For every open $U \subset X$ and every open cover $\{U_i\}$ of U and every $\phi, \varphi \in \mathcal{F}(U)$, if $\varphi|_{U_i} = \phi|_{U_i}$ for every i, then $\varphi = \phi$.
2. For every open $U \subset X$ and every open covering $\{U_i\}$ and for every collection of elements (functions)

$$\left\{ s_i \in \mathcal{F}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for every } j \right\}_i$$

there is an element $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

A sheaf valued in a subcategory of Set is defined in a similar way for sheaves to how it was defined for presheaves.

Remark 6.1.7. By condition 1. an element $s \in \mathcal{F}(U)$ obtained as described in 2. is unique.

Lemma 6.1.8. *Consider a topological space X and a presheaf $\mathcal{F}$ on X valued in the category of abelian groups that satisfies the second condition of being a sheaf. Then $\mathcal{F}$ is a sheaf on X if and only if for every open $U \subset X$ every open covering $\{U_i\}$ and every $s \in \mathcal{F}(U)$, if $s|_{U_i} = 0$ for every i, then $s = 0$.*

Proof. " $\Longrightarrow$ ": If $\mathcal{F}$ is a sheaf on X, then since $s|_{U_i} = 0 = 0|_0$ for every i, then $s = 0$.
" $\Longleftarrow$ ": If $s|_{U_i} = t|_{U_i}$ for every i, hence

$$0 = s|_{U_i} - t|_{U_i} = s - t|_{U_i}$$

for every i, which by assumption means $s - t = 0$. □

Lemma 6.1.9. *Let X be a topological space with a presheaf $\mathcal{O}$ of abelian groups. Then $\mathcal{O}$ is a sheaf if and only if for each open set $U \subset X$ and each open covering $\{U_i\}$ of U we may identify $\mathcal{O}(U)$ with the subring of $\prod_i \mathcal{O}(U_i)$,*

$$\mathcal{O}^{\{U_i\}}(U) := \left\{ (s_i) \in \prod_i \mathcal{O}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \right\}$$

Proof. Let an open set $U \subset X$ and open covering $\{U_i\}$ of U be given. Then

$$\mathcal{O}(U) \to \mathcal{O}^{\{U_i\}}(U)$$
$$s \mapsto \left(s|_{U_i}\right)$$

is injective if and only if the first sheaf property is fulfilled and Surjective if and only if the second sheaf property is fulfilled. □

Corollary 6.1.10. *If X is a topological space, $\mathcal{O}$ is a sheaf valued in the category of abelian groups and $\{U_i\}$ is an open covering of affine, disjoint subsets of X. It follows that*

$$\mathcal{O}(U) \simeq \prod_i \mathcal{O}(U \cap U_i)$$

for each open subset $U \subset X$.

Example 6.1.11. The presheaf $\mathcal{O}$ on a variety X is a sheaf: Consider an open subvariety $U \subset X$ and an open covering $\{U_i\}$ of U and suppose there is a rational function $f \in \mathcal{O}(U)$ such that $f|_{U_i} = 0$ for every i. Viewing f as a partial function on U it follows that given a point $x \in \text{dom } f$, since $x \in \text{dom } f \cap U_i$ for some i,

$$f(x) = f|_{U_i}(x) = 0 \implies f = 0.$$

Consider an open covering, $\{U_i\}$ of a subvariety $U \subset X$ and a collection of rational functions

$$\left\{ f_i \in \mathcal{O}(U_i) : f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \right\}_i,$$

then the function

$$f : U \to K$$
$$x \mapsto f_i(x) \quad \text{if } x \in U_i$$

is a rational function on U such that $f|_{U_i} = f_i$. Indeed, restriction in the sense of Definition 6.1.4 is in this instance literally restriction of partial functions in the usual sence, so the definition of f is indenpendent of the choice of U_i.

Lemma 6.1.12. *Let $\mathcal{F} : X \to$ Set be a sheaf. Let $U \subset X$ be an open subset. Then we define*

$$\mathcal{F}|_U : U \to \text{ab}$$

to be the presheaf given by $\mathcal{F}$ restricted to the subspace topology on U of open sets in X. This is a sheaf on U.

Proof. Let $V \subset U$ be open and $\{V_i\}$ an open cover of V. Consider $\varphi, \phi \in \mathcal{F}|_U(V) = \mathcal{F}(V)$ with

$$\varphi|\, V_i = \mathcal{F}|_U(V_i \subset V)(\varphi) = \mathcal{F}|_U(V_i \subset V)(\phi) = \phi|_{V_i}$$

for each i. Then since $\mathcal{F}$ is a sheaf, we get $\varphi = \phi$, hence $\mathcal{F}|_V$ satisfies the first condition for being a sheaf.
Let $V \subset U$ be open with an open covering $\{V_i\}$ and consider a collection of elements $\{s_i \in \mathcal{F}|_U(V_i)\}$ with

$$s_i|_{V_i \cap V_j} = \mathcal{F}|_U(V_i \cap V_j \subset V)(s_i) = \mathcal{F}|_U(V_i \cap V_j \subset V)(s_j) = s_j|_{V_i \cap V_j}$$

for each i and j. Then there is an $s \in \mathcal{F}(V) = \mathcal{F}|_U(V)$ such that

$$\mathcal{F}|_U(V_i \subset V)(s) = s|_{V_i} = s_i.$$

We thus conclude that $\mathcal{F}|_U$ is a sheaf. □

Recall from D.3.5 the definition of a direct limit. We shall explicitly construct the direct limit in the category of groups

Definition 6.1.13. Let $G_\bullet : (I, \leq) \to$ grp be a direct system with associated collection of objects and morphisms $(\{G_i\}, \{f_{ij}\})$. Given elements $(x_i, i), (x_j, j)$ in

$$\bigsqcup_{i \in I} G_i$$

we define an equivalence relation $\sim$, by postulating $x_i \sim x_j$ if there exists $k \in I$ with $i, j \leq k$ and

$$f_{ik}(x_i) = f_{jk}(x_j).$$

The *direct limit of* $G_\bullet$ is defined to be the set

$$\varinjlim G_i := \bigsqcup_{i \in I} G_i / \sim .$$

Given a pair of elements $[x_i], [x_j] \in \varinjlim G_i$ we define

$$[x_i][x_j] := [f_{ik}(x_i) f_{jk}(x_j)]$$

where $k \in I$ is an element such that $i, j \leq k$. There is a canonical natural transformation $\phi : G_\bullet \Rightarrow \varinjlim G_i$ given by

$$\{G_i \to \varinjlim G_i, x_i \mapsto [x_i]\}$$

Remark 6.1.14. By functioriality the definition of the binary operation of the direct limit of $\{G_i\}$ is independent of the choice of k. Since for $i \leq l$,

$$f_{il}(x_i) = f_{ll} f_{il}(x_i) \implies [x_i] = [f_{il}(x_i)].$$

This means that for every $h \geq k$,

$$[f_{ik}(x_i) f_{jk}(x_j)] = [f_{kh}(f_{ik}(x_i) f_{jk}(x_j))] = [(f_{kh} f_{ik})(x_i)(f_{kh} f_{ik})(x_j)] = [f_{ih}(x_i) f_{jh}(x_j)].$$

Lemma 6.1.15. *Keeping the notation from the prior definition, the set* $\varinjlim G_i$ *with the described binary operation is a group. When the direct system is valued in* ab*, the direct limit is an abelian group*

Proof. first we check that the operation is well defined. Let $([x_i], [x_j]), ([x_p], [x_q])$ in the direct limit of $\{G_i\}$. Then $f_{ik}(x_i) = f_{pk}(x_p)$ and $f_{jh}(x_j) = f_{qh}(x_q)$ for suitable $k, h \in I$ with $i, p \leq k$ and $j, q \leq h$. Pick an $l \in I$ with $h, k \leq l$. Then

$$[x_i][x_j] = [f_{il}(x_i) f_{jl}(x_j)] = [f_{pl}(x_p) f_{ql}(x_q)] = [x_p][x_q].$$

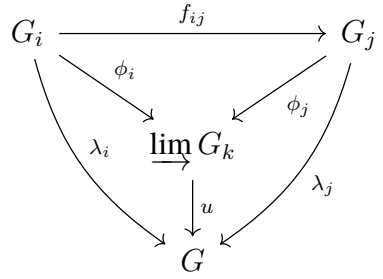
One readily verifies that this operation is associative. Note that since each f_{ij} is a group homomorphism, the identity of each G_i is in the same equivalence class and one sees that $e := [e_i]$ is neutral with respect to the binary operation in $\varinjlim G_i$. Lastly defining $[x_i]^{-1} = [x_i^{-1}]$, one readily verifies that this is independent of the choice of representative and that it is inverse element of $[x_i]$. □

Lemma 6.1.16. *Consider a direct system* $(\{G_i\}, \{f_{ij}\})$ *of groups. Then* $\varinjlim G_i$ *with* $\phi : G_\bullet \Rightarrow \varinjlim G_i$ *is the direct limit in the category of groups.*

Proof. Let a cocone (G, λ) of the direct system be given. We define

$$u : \varinjlim G_k \to G$$
$$[x_i] \mapsto \lambda_i(x_i)$$

This is well defined by the naturality of $\{\lambda_i\}$. Let $i, j \in I$ with $i \leq j$ be given. Clearly, u is the unique homomorphism making



commute. We thus conclude that our construction is indeed the direct limit in the categorical sense. □

Remark 6.1.17. One can extend this result to direct systems in subcategories of grp such as ab, CRing and Mod_R.

Definition 6.1.18. Consider a topological space X and a presheaf $\mathcal{F}$ on X valued in ab. Let $P \in X$. We define the *stalk of* $\mathcal{F}$ *at* P to be

$$\mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F} \in \text{ab}.$$

To be precise, $N_P(X)$ denote the directed set (wrt. $\supset$) of open neighborhoods of P. $\mathcal{F}_P$ is then the direct limit of the direct system given by $\mathcal{F}$ restricted to $N_P(X)$ Given an element $s \in \mathcal{F}(U)$ we call $[s]_P := [s]$ a *germ* of f at P.

Remark 6.1.19. Note that by construction $[s_U], [t_V] \in \mathcal{F}_P$ are equal if and only if there is some open neighborhood of P, $W \subset U \cap V$ say, with

$$s|_W = t|_W .$$

There is a natural transformation

$$\phi : \mathcal{F}(U) \Rightarrow \mathcal{F}_p$$
$$s \mapsto [(s, U)]_P$$

Example 6.1.20. Given a variety X over a field K, $\mathcal{O}_P$ consists exactly of rational functions that are regular on some open neighborhood of P, i.e.

$$\mathcal{O}_P = \mathcal{O}_P(X).$$

Definition 6.1.21. Given a category $\mathcal{C}$ *a morphism of presheaves* is just a morphism in the functor category $\text{Set}^{\mathcal{C}^{\text{op}}}$, i.e. a morphism $\Phi : \mathcal{F} \to \mathcal{G}$ of parallel presheaves on $\mathcal{C}$ is just a natural transformation $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$.
An isomorphism of presheaves is therefor just a natural isomorphism of presheaves.
A morphism of sheaves is just a morphism of presheaves in the category of sheaves, $\text{Sh}(X)$, which is a subcategory of $\text{Set}^{\mathcal{T}(X)^{\text{op}}}$.

Definition 6.1.22. Consider a parallel pair of presheaves $\mathcal{F}, \mathcal{G} : X \to \text{ab}$ and fix a point $P \in X$. For a morphism of presheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$, there is a unique induced morphism of stalks at P given by

$$\Phi_P : \mathcal{F}_P \to \mathcal{G}_P$$
$$(U, s) \mapsto (U, \Phi_U(s))$$

such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow{\Phi_P} & \mathcal{G}_P \end{array}$$

for each open neighborhood U of P.

Remark 6.1.23. Explicitly, we can describe the above map as taking a germ $[s_U]_P$ to $[\Phi_U(s)]_P$ where U is any open neighborhood of P.

Proposition 6.1.24. *Consider a parallel pair of sheaves $\mathcal{F}, \mathcal{G} : X \to \text{ab}$ and morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$. Then Φ is an isomorphism if and only if $\Phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for each $P \in X$.*

Proof. Suppose Φ is an isomorphism. Then Φ_P is an isomorphism for each P by functoriality of the assignment $\mathcal{D} \mapsto \lim \mathcal{D}$ for diagrams indexed by some category $\mathcal{I}$ in a category that admits limits of such diagrams.
Conversely suppose Φ_P is an isomorphism for each P. It suffices to show that each component of Φ is an isomorphism of abelian groups. Fix an open set U in X.
Injectivity: Suppose $s \in \mathcal{F}(U)$ is a section with $\Phi_U(s) = 0$. Then in particular for each $P \in U$,

$$\Phi_P([(s, U)]_P) = [(\Phi_U(s), U)]_P = [0]_P.$$

Then on some open neighborhood U_P of P, $s|_{U_p} = 0$. This with the fact that $\{U_P\}$ defines an open cover of U, implies $s = 0$ by the first sheaf property.
Surjectivity: Consider a section $t \in \mathcal{G}(U)$. Let $P \in U$. Then

$$[\Phi_{W_P}(s_{W_P})]_P = \Phi_P([s_{W_P}]_P) = [(t, U)]_P$$

for a suitable section $s_{W_P} \in \mathcal{F}(W_P)$ for some open neighborhood $W_P \subset U$ of P. We may assume that $\Phi_{W_P}(s_{W_P}) = t|_{W_P}$. For pairs of points $P, Q \in U$, by naturality,

$$\Phi_{W_P \cap W_Q}\left(s_{W_P}|_{W_P \cap W_Q}\right) = t|_{W_P \cap W_Q} = \Phi_{W_P \cap W_Q}\left(s_{W_Q}\big|_{W_P \cap W_Q}\right),$$

and since $\Phi_{W_P \cap W_Q}$ is injective,

$$s_{W_P}|_{W_P \cap W_Q} = s_{W_Q}\big|_{W_P \cap W_Q}.$$

The second sheaf property implies the existence of a section $s \in \mathcal{F}(U)$ with $s|_{W_P} = s_{W_P}$ for each $P \in U$. We aim to show that $\Phi_U(s) = t$. Again by naturality,

$$\Phi_U(s)|_{W_P} = \Phi_{W_P}(s_{W_P}) = t|_{W_P} \implies (\Phi_U(s) - t)|_{W_P} = 0$$

for each $P \in U$. Then by the first sheaf property $\Phi_U(s) - t = 0$. □

Definition 6.1.25. Consider a parallel pair of presheaves $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \text{ab}$ and morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$.

- We define *the presheaf kernel of* Φ to be the presheaf

$$\ker\ \Phi : \mathcal{C} \to \text{ab}$$
$$U \mapsto \ker\ \Phi_U$$
$$U \to V \mapsto \varphi_{UV}|_{\ker\ \Phi_V} : \ker\ \Phi_V \to \ker\ \Phi_U$$

- We define *the presheaf cokernel of* Φ to be the presheaf

$$\text{pscoker}\ \Phi : \mathcal{C} \to \text{ab}$$
$$U \mapsto \text{coker}\ \Phi_U$$
$$U \to V \mapsto \text{coker}\ \Phi_V \to \text{coker}\ \Phi_U, x + \text{im}\ \Phi_V \mapsto \pi_{UV}(x) + \text{im}\ \Phi_U$$

- We define *the presheaf image of* Φ to be the presheaf

$$\text{psim}\ \Phi : \mathcal{C} \to \text{ab}$$
$$U \mapsto \text{im}\ \Phi_U$$
$$U \to V \mapsto \pi_{UV}|_{\text{im}\ \Phi_V}\ \text{im}\ \Phi_V \to \text{im}\ \Phi_U, x \mapsto \pi_{UV}(x)$$

Remark 6.1.26. Unsurprisingly, the welldefinedness of the above presheaves are ensured by naturality. Functorialty follows from the functioriality of $\mathcal{F}$ and $\mathcal{G}$.

Lemma 6.1.27. *The presheaf kernel of a morphism of sheaves is a sheaf.*

Proof. Consider presheaves $\mathcal{F}, \mathcal{G} : X \to \text{ab}$ and a morphism of presheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$. Let $U \subset X$ be an open subset and let $s \in \ker\ \Phi(U)$ be given with an open covering $\{U_i\}$ of U such that $s|_{U_i} = 0$ for each i, then since $s \in \mathcal{F}(U)$ and $s|_{U_i} \in \mathcal{F}(U_i)$, it follows that $s = 0$. Suppose $\{s_i \in \ker\ \Phi(U_i)\}$ is a family of sections with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there is an $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for each i. We claim that $s \in \ker\ \Phi(U)$. Indeed, note that

$$\Phi_U(s)|_{U_i} = \Phi_{U_i}(s_i) = 0$$

for each i, hence $\Phi_U(s) = 0$ by the first sheaf property. □

Remark 6.1.28. The same result cannot be attained. The failure of the image presheaf to be a sheaf comes from the fact that while we can paste local sections $\Phi_{U_i}(s)$ together to form some global section t in $\mathcal{G}(U)$, it may be that the image of Φ_U does not contain t

Example? Why does cokernel presheaf fail?

Lemma 6.1.29. *Consider a presheaf $\mathcal{F} : X \to \text{ab}$ on topological space X. There is a sheaf $\mathcal{F}^+ : X \to \text{ab}$ and morphism of presheaves $\Theta : \mathcal{F} \Rightarrow \mathcal{F}^+$ such that for any sheaf $\mathcal{G} : X \to \text{ab}$ and any morphism of presheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ there is a unique morphism of sheaves $\Psi : \mathcal{F}^+ \Rightarrow \mathcal{G}$ such that $\Phi = \Psi\Theta$. Furthermore $(\mathcal{F}^+, \Theta)$ is unique up to unique isomorphism.*

Proof. We construct a sheaf

$$X \to \text{ab}$$

Given an open subset U in X, we define $\mathcal{F}^+(U)$ to be the group of elements $([s_{W_P}]_P)$ in

$$\prod_{P \in U} \mathcal{F}_P$$

such that for each $P \in U$ there is an open neighborhood of P, $V_P \subset W_P \subset U$ with an element $t \in \mathcal{F}(V_P)$ such that for each $Q \in V_P$ the germ $[t]_Q \in \mathcal{F}_Q$ of t at Q satisfies

$$[s_{W_Q}] = [t]_Q.$$

Given $U \supset V$ we define $\varphi_{UV} : \mathcal{F}^+(U) \to \mathcal{F}^+(V)$ to be given by restriction. It thus readily follows that $\mathcal{F}^+$ is a sheaf. We have a natural transformation

$$\Theta : \mathcal{F} \Rightarrow \mathcal{F}^+$$
$$\{\Theta_U : \mathcal{F}(U) \to \mathcal{F}^+(U), s \mapsto ([s]_P)\}.$$

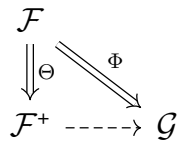
Let a natural transformation of presheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ be given. We define a natural transformation of sheaves

$$\Psi : \mathcal{F}^+ \Rightarrow \mathcal{G}$$

defined by the collection of morphisms

$$\{\Psi_U : \mathcal{F}^+(U) \to \mathcal{G}(U), ([s]_P) \mapsto \Theta_U(s) \mid U \subset X \text{ open}\}$$

This is well-defined since $\mathcal{G} \cong \mathcal{G}^+$ or in other words since $\mathcal{G}(U) \simeq \mathcal{G}^+(U)$ for each U. Ψ is the unique morphism of sheaves making the diagram



commute. Note that $(\mathcal{F}^+, \Theta)$ is thus the coequalizer of two copies of the identity transformation of $\mathcal{F}$ (cf. Example D.3.9) which means it is unique up to unique isomorphism. $\square$

Remark 6.1.30. We call $\mathcal{F}^+$ *the sheaf associated with* $\mathcal{F}$

Definition 6.1.31. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set U in X, $\mathcal{F}'(U) \leq \mathcal{F}(U)$ and, letting $\varphi_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$ denote the image of $U \subset V$ under $\mathcal{F}$, the image of $U \subset V$ under $\mathcal{F}'$ is the restriction of φ_{UV} to $\mathcal{F}(V)$.

Remark 6.1.32. Note that given $(U, s) \in \mathcal{F}'_P$, we have that $s \in \mathcal{F}'(U) \subset \mathcal{F}(U)$, hence $(U, s) \in \mathcal{F}_P$.

Example 6.1.33. The presheaf kernel of morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ is sheaf called the *sheaf kernel*. By construction $(\ker\ \Phi)(U) = \ker\ \Phi_U \leq \mathcal{F}(U)$ and

$$(\ker\ \Phi)(U \subset V) = \varphi_{UV}|_{\ker\ \Phi_U},$$

hence $\ker\ \Phi$ is a subsheaf of $\mathcal{F}$.

Definition 6.1.34. A morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ is *injective* if $\ker\ \Phi = 0$, i.e. the sheaf that takes every open subset of X to the trivial group.

Remark 6.1.35. Φ being injective is equivalent to Φ_U being injective for each open set U.

Definition 6.1.36. For a morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$, we define the *image of* Φ to be $\text{im}\ \Phi := (\text{psim}\ \Phi)^+$

Proposition 6.1.37. *Given a morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$, the canonical morphism* $\text{im}\ \Phi \Rightarrow \mathcal{G}$ *is injective, hence it can be indentified with a subsheaf of* $\mathcal{G}$.

Proof. □

Definition 6.1.38. A morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ is *surjective* if $\text{im}\ \Phi = \mathcal{G}$.

Definition 6.1.39. A sequence

$$\ldots \xrightarrow{\Phi^{i-1}} \mathcal{F}^{i-1} \xrightarrow{\Phi^{i}} \mathcal{F}^{i} \xrightarrow{\Phi^{i+1}} \mathcal{F}^{i+1} \longrightarrow \ldots$$

is *exact*, if $\ker\ \Phi^i = \text{im}\ \Phi^{i-1}$ for each $i \in \mathbb{Z}$.

Definition 6.1.40. Consider a sheaf $\mathcal{F}$ and subsheaf $\mathcal{F}' \subset \mathcal{F}$. We define the *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ to be the sheaf associated with

$$X \to \text{ab}$$
$$U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$$
$$U \subset V \mapsto \mathcal{F}(V)/\mathcal{F}'(V) \to \mathcal{F}(U)/\mathcal{F}'(U), s + \mathcal{F}'(V) \mapsto s|_U + \mathcal{F}'(U)$$

Remark 6.1.41. Fix a point $P \in X$ and consider $(U, s) \in (\mathcal{F}/\mathcal{F}')_P$. Note that then $s \in \mathcal{F}(U)/\mathcal{F}'(U)$, hence $(U, s) \in \mathcal{F}_P/\mathcal{F}'_P$.

With the above in mind we can make the following definition

Definition 6.1.42. For a morphism of sheaves $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$, we define the *cokernel of* Φ to be $\text{coker}\ \Phi := (\text{pscoker}\ \Phi)^+$.

Definition 6.1.43. For a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, we define the *direct image sheaf*, $f_\star\mathcal{F}$, on Y, to be the sheaf

$$Y \to \text{ab}$$
$$U \mapsto \mathcal{F}(f^{-1}(U))$$
$$U \subset V \mapsto \varphi_{f^{-1}(U)f^{-1}(V)}$$

Remark 6.1.44. We check that this indeed a sheaf Check!

Definition 6.1.45. For a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y, we define the *inverse image sheaf*, $f^{-1}\mathcal{G}$ on X to be the sheaf associated with the presheaf

$$X \to \text{ab}$$
$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$
$$U \subset V \mapsto \varphi_{\lim_{W \supset f(U)} \mathcal{G}(W) \lim_{W \supset f(V)} \mathcal{G}(W)}$$

Definition 6.1.46. Consider a topological space X, a subspace $Z \subset X$ and a sheaf $\mathcal{F}$ on X. Let ι denote the inclusion map $Z \hookrightarrow X$. We define *the restriction of $\mathcal{F}$ to Z* to be the sheaf $\mathcal{F}|_Z := \iota^{-1}\mathcal{F}$.

Remark 6.1.47. Fix a point $P \in Z$. Let $(U, s) \in (\mathcal{F}|_Z)_P$... Do!

6.2 Schemes

We adopt the convention that all rings are commutative.

6.2.1 Affine Schemes

Definition 6.2.1. For a ring R, we define *the spectrum of R* to be the set

$$\text{Spec } R := \{I \subset R : I \text{ is a prime ideal}\}$$

For an ideal $I \subset R$, we define

$$V(I) := \{J \in \text{Spec } R : I \subset J\}$$

Remark 6.2.2. In the special case of $R = K[x_1, \ldots, x_n]$ for a field K, we denote the vanishing set of an ideal $I \subset K[x_1, \ldots, x_n]$ by $Z(I)$ instead of $V(I)$ to avoid confusion.

Lemma 6.2.3. *If $I \subset J \subset R$ are ideals, then $V(I) \supset V(J)$.*

Proof. This is trivial. □

Lemma 6.2.4. *For ideals $I, J \subset R$*

$$V(IJ) = V(I) \cup V(J).$$

Proof. Consider a prime ideal $\mathfrak{p} \in \text{Spec } R$ containing IJ. If $\mathfrak{p} \supset J$ we are done. Otherwise given any $a \in I$ and choosing $b \in J \setminus \mathfrak{p}$ we have that $a \in \mathfrak{p}$ hence $\mathfrak{p} \in V(I)$. The converse inclusion follows from $I, J \supset IJ$ by the prior lemma. □

Lemma 6.2.5. *For a family of ideals $\{I_\mu\}$ in R,*

$$V\left(\sum_\mu I_\mu\right) = \bigcap_\mu V(I_\mu).$$

Proof. Again one inclusion follows from $I_\nu \subset \sum_\mu I_\mu$. Suppose conversely that $\mathfrak{P} \in \text{Spec } R$ is given such that it contains every I_μ, then since ideals are closed under addition it contains $\sum_\mu I_\mu$. □

Proposition 6.2.6. *For a ring R, the family of sets*

$$\{V(I) : I \text{ is an ideal in } R\}$$

are the closed set in a topology on Spec R.

Proof. Indeed, note that

$$\bigcap_{I \subset R} V(I) = V\left(\sum_{I \subset R} I\right) = V(R) = \emptyset.$$

and that Spec $R = V(0)$. The two prior lemmas show that this family is closed under countable union and intersection over arbitrary index sets. □

Remark 6.2.7. The topology in question is unsurprisingly called the *Zariski topology on* Spec R. For an algebraically closed field K and an affine variety $V \subset \mathbb{A}^n$, the points of V correspond to the maximal ideals of $\Gamma(V)$. In this way we can identify V with the subspace of maximal ideals of $\Gamma(V)$ in Spec $\Gamma(V)$. In this sense Spec $\Gamma(V)$ is a refinement of the notion of point in a variety. Namely, *point* in V have by definition come to mean the *subvariety* of V, when we take the view that our vantage point lies in Spec $\Gamma(V)$ when studying V.

Lemma 6.2.8. *If $I, J \subset R$ are ideals, then $V(I) \subset V(J)$ if and only if* $\text{rad}(I) \supset \text{rad}(J)$.

Proof. Note that $V(I) \subset V(J)$ if and only if every prime ideal that contains I also contains J which again is equivalent to

$$\bigcap_{\mathfrak{p} \supset I \text{ prime}} \mathfrak{p} \supset \bigcap_{\mathfrak{p} \supset J \text{ prime}} \mathfrak{p},$$

which by lemma 1.6.38 is equivalent to

$$\text{rad}(I) \supset \text{rad}(J).$$

□

Given a ring R and prime ideal $\mathfrak{p}$ in R. We denote the localization of R with respect to $\mathfrak{p}$ by $R_{\mathfrak{p}}$.

Definition 6.2.9. Given a ring R, we define a presheaf on Spec R by,

$$\mathcal{O} : (\text{Spec } R, \tau_{\text{Spec } R})^{\text{op}} \to \text{Ring}$$

that takes an open subset U of Spec R to the subring of $\prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$,

$$\mathcal{O}(U) := \left\{ s \in \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}} : \forall \mathfrak{p} \in U \exists V \text{ op. ngh. of } \mathfrak{p} \text{ s.t. } s|_V = \frac{a}{f} \text{ w/ } f \notin \mathfrak{q} \text{ for each } \mathfrak{q} \in V \right\}.$$

Moreover, $V \subset U$ is mapped to the restriction map $\mathcal{O}(U) \to \mathcal{O}(V), s \mapsto s|_V$.

Remark 6.2.10. Clearly this presheaf is a sheaf, since the elements of $\mathcal{O}(U)$ are functions and restriction is just set function restriction in the same we saw in the example discussing the presheaf Γ.

Definition 6.2.11. For $f \in R$, we define

$$D(f) := (\text{Spec } R) \setminus V(\langle f \rangle)$$

Lemma 6.2.12. *For a ring R,*

$$\{D(f) \subset \text{Spec } R : f \in A\}$$

forms a basis of the Zariski topology on Spec R.

Proof. Indeed, for $\mathfrak{p} \notin V(I)$, there is an $f \in I$ not in $\mathfrak{p}$, hence $\mathfrak{p} \notin V(\langle f \rangle)$, meaning $\mathfrak{p} \in D(f)$. Then $V(I) \cap D(f) = \emptyset$. This means that every point in an open subset U in Spec R has an open neighborhood contained in U. It follows that the collection in question forms an open basis for the Zariski topology. □

Lemma 6.2.13. *Given a ring R and an element $f \in R$, $D(f)$ is quasi-compact with respect to the Zariski subspace topology.*

Proof. Let $\{V_\alpha\}$ be an open cover of Spec R. WLOG $V_\alpha = D(h_\alpha)$ for some $h_\alpha \in R$ for each α. Then

$$V(\langle f \rangle) = \bigcap_\alpha V(\langle h_\alpha \rangle) = V\left(\sum_\alpha \langle h_\alpha \rangle\right),$$

which implies $\text{rad}(\langle f \rangle) = \text{rad}\left(\sum_\alpha \langle h_\alpha \rangle\right)$. Then $f^n = \sum_1^m h_i$ for some $n \geq 1$ and $\{h_1, \ldots, h_m\} \subset \{h_\alpha\}$, hence

$$D(f) = D(f^n) = \bigcup_1^m D(h_i).$$

□

Corollary 6.2.14. Spec R *is quasi-compact.*

Lemma 6.2.15. *Consider a ring R, an element $f \in R$ and a section $s \in \mathcal{O}(D(f))$. There are $a_1, \ldots, a_m, h_1, \ldots, h_m \in R$ such that*

$$D(f) = \bigcup_1^m D(h_i)$$

and

$$s(\mathfrak{p}) = \frac{a_i}{h_i} \in R_\mathfrak{p}$$

for every $\mathfrak{p} \in D(h_i)$ for each i.

Proof. First of all, there are $\{\lambda_\alpha\}$ in R such that

$$D(f) = \bigcup_\alpha D(\lambda_\alpha)$$

and $s(\mathfrak{p}) = \frac{a'_\alpha}{g_\alpha}$ for some $a'_\alpha, g_\alpha \in R$ with $g_\alpha \notin \mathfrak{p}$ for each $\mathfrak{p} \in D(\lambda_\alpha)$. Since $D(\lambda_\alpha) \subset D(g_\alpha)$, it follows that $\text{rad}(\langle \lambda_\alpha \rangle) \subset \text{rad}(\langle g_\alpha \rangle)$, hence for some $k_\alpha \geq 1$,

$$\lambda_\alpha^{k_\alpha} = l_\alpha g_\alpha \implies \frac{l_\alpha a'_\alpha}{\lambda_\alpha^{k_\alpha}} = \frac{a'_\alpha}{g_\alpha}.$$

Setting $h_\alpha := \lambda_\alpha^{k_\alpha}$ and $a_\alpha := l_\alpha a'_\alpha$. We get, upon noting that $D\left(h_\alpha^{k_\alpha}\right) = D(\lambda_\alpha)$, a cover $\{h_\alpha\}$ of $D(f)$ with

$$s(\mathfrak{p}) = \frac{a_\alpha}{h_\alpha}$$

with $h_\alpha \notin \mathfrak{p}$. Since $D(f)$ is quasi-compact we obtain a finite cover of such $D(h_\alpha)$. □

Lemma 6.2.16. *Consider a ring R. Then*

1. *For every point $\mathfrak{p} \in \text{Spec } R$*
$$\mathcal{O}_\mathfrak{p} \simeq R_\mathfrak{p}.$$
2. *For any $f \in R$,*
$$\mathcal{O}(D(f)) \simeq R_f.$$
3. *In particular,*
$$\Gamma(\text{Spec } R, \mathcal{O}) := \mathcal{O}(\text{Spec } R) = \mathcal{O}(D(1)) \simeq R_1 = R.$$

Proof. 1. Consider the ring map

$$\sigma : \mathcal{O}_\mathfrak{p} \to R_\mathfrak{p}$$
$$[s_U] \mapsto s_U(\mathfrak{p})$$

Which exists upon applying the universal property for $\mathcal{O}_\mathfrak{p}$ on $\text{ev}_\mathfrak{p} : \mathcal{O} \Rightarrow R_\mathfrak{p}$. To check surjectivity, let $\frac{a}{f} \in R_\mathfrak{p}$ be given. By definition $f \notin \mathfrak{p}$, hence $\mathfrak{p} \not\supset \langle f \rangle$. We thus get that $D(f)$ is an open neighborhood of $\mathfrak{p}$. Let $\frac{a}{f} : D(f) \to \bigcup_{\mathfrak{q} \in D(f)} R_\mathfrak{q}$ denote the constant function on $D(f)$ taking $\mathfrak{q}$ to $\frac{a}{f}$. Note that $\mathfrak{q} \not\supset \langle f \rangle$ for each $\mathfrak{q} \in D(f)$, so $\frac{a}{f}$ is a section in $\mathcal{O}(D(f))$. In particular this is an element of $\mathcal{O}_\mathfrak{p}$ and

$$\sigma\left(\frac{a}{f}\right) = \frac{a}{f}.$$

To check injectivity, consider $s, t \in \mathcal{O}_\mathfrak{p}$ with $s(\mathfrak{p}) = r(\mathfrak{p})$. On a sufficiently small neighborhood U of $\mathfrak{p}$, we have that $s|_U = \frac{a}{f} \in R_\mathfrak{q}$ and $t|_U = \frac{b}{g} \in R_\mathfrak{q}$ for each $q \in U$. There is

some $h \in R \setminus \mathfrak{p}$ such that $h(ag - bf) = 0$, hence on $D(f) \cap D(g) \cap D(h)$, s and t agree. This implies $s = t$. We conclude that σ is an isomorphism of rings.

2. We define a homomorphism

$$\tau : R_f \to \mathcal{O}_P$$

That takes $\frac{a}{f^n} \in R_f$ to the function $s \in \prod_{\mathfrak{p} \in D(f)}$ that takes $\mathfrak{p} \in D(f)$ to $\frac{a}{f^n}$, which is well-defined since $\mathfrak{p} \not\supset \langle f^n \rangle$ for every $n \geq 0$. We claim that this is an isomorphism.

To show injectivity, suppose

$$\tau\left(\frac{a}{f^n}\right) = \tau\left(\frac{b}{f^m}\right)$$

for a pair $\frac{a}{f^n}, \frac{b}{f^m} \in R_f$. Then for every $\mathfrak{p} \in D(f)$

$$h_{\mathfrak{p}}\left(af^m - bf^n\right) = 0$$

for some $h_{\mathfrak{p}} \in R \setminus \mathfrak{p}$. Set $\mathfrak{a} := \text{Ann}(af^m - bf^n)$. Note then that $\mathfrak{p} \not\supset \mathfrak{a}$, since $h_{\mathfrak{p}} \in \mathfrak{a}$. We get that $V(\mathfrak{a}) \cap D(f) = \emptyset$, hence $V(\mathfrak{a}) \subset V(\langle f \rangle)$, which is equivalent to $\langle f \rangle \subset \text{rad}(\mathfrak{a})$. Then $f^l(af^m - bf^n) = 0$ for some $l \geq 0$. It thus follows that $\frac{a}{f^n} = \frac{b}{f^m}$ in R_f.

To show surjectivity, note first that $D(f)$ has a finite cover $\{D(h_i)\}_1^m$ with $s(\mathfrak{p}) = \frac{a_i}{h_i} \in R_{\mathfrak{p}}$ for some $a_i \in R$ for each $\mathfrak{p} \in D(h_i)$ for each i. Consider a pair of indeces i and j and note that under

$$\tau' : R_{h_i h_j} \to \mathcal{O}(D(h_i h_j)) = \mathcal{O}(D(h_i) \cap D(h_j)),$$

which takes a fraction to its associated constant section on $D(h_i h_j)$, $\frac{a_i}{h_i}$ and $\frac{a_j}{h_j}$ agree, then $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ in $R_{h_i h_j}$. Then for some $k \geq 0$

$$(h_i h_j)^k (a_i h_j - a_j h_i) = 0 \iff h_j^{k+1}\left(h_i^k a_i\right) = h_i^{k+1}\left(h_j^k a_j\right).$$

Note that $D(f) = \bigcup_1^m D(h_i^{k+1})$ and $s(\mathfrak{p}) = \frac{a_i}{h_i} = \frac{h_i^k a_i}{h_i^{k+1}}$ for each $\mathfrak{p} \in D(h_i) = D(h_i^{k+1})$. For the sake of simplifying notation set $\lambda_i := h_i^{k+1}$ and $c_i := h_i^k a_i$. With this notation we find $\lambda_j c_i = \lambda_i c_j$ for each i and j. Since $f \in \text{rad}\left(\sum_1^m \lambda_i\right)$, there is an $n \geq 1$ and $b_1, \ldots, b_m \in R$ with

$$f^n = \sum_1^m b_i \lambda_i.$$

Set

$$a := \sum_1^m b_i c_i.$$

Then we find

$$\lambda_j a = \sum_1^m \lambda_j b_i c_i = c_j \sum_1^m \lambda_i b_i = c_j f$$

Then on each $D(h_i)$ for each $\mathfrak{p} \in D(h_i)$,

$$\tau\left(\frac{a}{f^n}\right)(\mathfrak{p}) = \frac{c_i}{\lambda_i} = s(\mathfrak{p}) \implies \tau\left(\frac{a}{f^n}\right) = s,$$

hence τ is also surjective. We thus conclude that τ is a ring isomorphism.
3. Follows immediately from 2. □

Definition 6.2.17. A *ringed space* is a topological space X together with a sheaf

$$\mathcal{O}_X : X \to \text{CRing}.$$

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_{X,P}$ for $P \in X$ is a local ring

Definition 6.2.18. A morphism of ringed spaces, $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ and a natural transformation $f^\# : \mathcal{O}_Y \Rightarrow f_\star \mathcal{O}_X$.

We wish to define a morphism between locally ringed spaces. To do this we first show how to construct a ring homomorphism between the stalks $\mathcal{O}_{Y,f(P)}$ and $\mathcal{O}_{X,P}$ for each $P \in X$. Indeed, $f^\#$ induces a ring homomorphism

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{U \ni f(P)} \mathcal{O}(U) \to \varinjlim_{U \ni f(P)} \mathcal{O}_X\left(f^{-1}(U)\right) = \varinjlim_{f^{-1}(U) \ni P} \mathcal{O}_X\left(f^{-1}(U)\right)$$

Composing this with

$$\varinjlim_{f^{-1}(U) \ni P} \mathcal{O}_X\left(f^{-1}(U)\right) \hookrightarrow \varinjlim_{W \ni P} \mathcal{O}_X(W) = \mathcal{O}_{X,P},$$

we obtain a map

$$f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

Definition 6.2.19. A morphism of locally ringed spaces is a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_P^\#$ is local ring homomorphism (cf. Definition 1.6.109) for each $P \in X$.

Lemma 6.2.20. *Let $(X, \mathcal{O})$ be a locally ringed space. Let U be an open set in X. Then $(U, \mathcal{O}|_U)$ is a locally ringed space.*

Proof. We already proved that $\mathcal{O}|_U$ is a sheaf in an earlier lemma. It thus suffices to prove that for each $P \in U$, $(\mathcal{O}|_U)_P$ is local. Note that the set of open neighborhoods of P contained in U is coinitial in the set of open neighborhoods of P. It thus follows that

$$(\mathcal{O}|_U)_P \simeq \mathcal{O}_P,$$

hence $(\mathcal{O}|_U)_P$ is local. □

Lemma 6.2.21. *Consider a ringed space $(X, \mathcal{O})$ and an open subset $U \subset X$. Denote the from X inherited sheaf by $\mathcal{O}_U$. The map $\iota : U \subset X$ together with*

$$\iota^{\#} : \mathcal{O} \to \iota_{\star}\mathcal{O}_U$$

given by

$$\{\varphi_{UU\cap V} : \mathcal{O}(V) \to \mathcal{O}_U(U \cap V) = \mathcal{O}(U \cap V) : V \subset Y \text{ open}\}.$$

defines a morphism of ringed spaces. If $(X, \mathcal{O})$ is a locally ringed space, then this morphism is a morphism of locally ringed spaces.

Proof. ι is a continuous map and the naturality of $\iota^{\#}$ follows immediately from the functorialty of $\mathcal{O}$. The map $\iota^{\#}_P$ is just the identity map for each $P \in U$, $(\iota, \iota^{\#})$ defines a morphism of locally ringed spaces □

Lemma 6.2.22. *Consider a morphism of ringed spaces $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Let $U \subset X$ and $U' \subset Y$ open be given with $\varphi(U) \subset U'$. Consider $\iota : U \hookrightarrow X$. Then $\varphi|_U$ together with*

$$\iota^{\#}\, \varphi^{\#}\big|_{\mathcal{O}_Y|_{U'}} : \mathcal{O}_Y|_{U'} \Rightarrow (\varphi|_U)_{\star}\, \mathcal{O}_X|_U$$

is a morphism of ringed spaces. If ringed spaces in question are locally ringed spaces, then this morphism is a morphism of locally ringed spaces.

Proof. We already know that $\varphi|_U$ is a continuous map. $\iota^{\#}\, \varphi^{\#}\big|_{\mathcal{O}_Y|_{U'}}$ is a natural transformation. With the assumption of locally ringed spaces, note that the square

$$\begin{array}{ccc} (\mathcal{O}_Y)_{\varphi(P)} & \xrightarrow{\varphi^{\#}_P} & (\mathcal{O}_X)_P \\ \downarrow & & \downarrow \\ (\mathcal{O}_Y|_{U'})_{\varphi|_U(P)} & \xrightarrow{\left(\iota^{\#}\varphi^{\#}\big|_{\mathcal{O}_Y|_{U'}}\right)_P} & (\mathcal{O}_X|_U)_P \end{array}$$

where the vertical arrows are isomorphisms, commutes. It thus follows that $\left(\iota^{\#}\, \varphi^{\#}\big|_{\mathcal{O}_Y|_{U'}}\right)_P$ is a local homomorphism for each $P \in U$. □

Lemma 6.2.23. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Let open coverings $\{U_i\}$ and $\{V_i\}$ of respectively X and Y be given. Consider a family of morphisms of ringed spaces $\{f_i : U_i \to V_i\}$ such that*

1. $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ *for each* i, j.
2. $f_i|^{\#}_{U_i\cap U_j} = f_j|^{\#}_{U_i\cap U_j}$ *for each* i, j.

Then there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} : U_i \to V_i$ agrees with f_i for each i.

Proof. It's a basic fact of topology that the map

$$f : X \to Y$$

defined by $f(P) = f_i(P)$ for $P \in U_i$, is the unique continuous map that agrees with f_i when restricted to U_i. Let $V \subset Y$ be an open set. We define

$$\begin{aligned} f_V^\# : \mathcal{O}_Y(V) &\to f_*\mathcal{O}_X(V) \\ (s|_{V_i \cap V}) &\mapsto \left((f_i^\#)_{V_i \cap V}(s|_{V_i \cap V})\right) \end{aligned}$$

Note that

$$\begin{aligned} (f_i^\#)_{V_i \cap V}(s|_{V_i \cap V})\big|_{f_i^{-1}(V_i \cap V_j \cap V)} &= \left(f_i^\#\big|_{\mathcal{O}_Y|_{V_i \cap V_j}} \right)_{V \cap V_i} (s|_{V_i \cap V_j \cap V}) \\ &= \left(f_j^\#\big|_{\mathcal{O}_Y|_{V_i \cap V_j}} \right)_{V \cap V_i} (s|_{V_i \cap V_j \cap V}) \\ &= (f_j^\#)_{V_i \cap V}(s|_{V_i \cap V})\big|_{f_i^{-1}(V_i \cap V_j \cap V)}, \end{aligned}$$

hence this is a well-defined ring map. Naturality follows from each $f_i^\#$ being a natural transformation for each i. We thus conclude that $(f, f^\#) : X \to Y$ defines a morphism of ringed spaces. Note that for $V \subset V_i$ and $s = (s|_{V \cap V_j}) \in \mathcal{O}_Y(V)$, we have

$$\begin{aligned} \left(f|_{U_i}^\# \right)_V (s) = \iota^\#_{f^{-1}(V) \cap U_i} f_V^\#\big|_{\mathcal{O}_Y|_{V_i}} (s) &= \varphi_{f^{-1}(V)U_i}((f_j^\#)_{V \cap V_j}(s|_{V \cap V_j})) \\ &= \left(\left(f_i^\#\right)_{V \cap V_i} (s|_{V \cap V_i})\Big|_{f_i^{-1}(V_i \cap V)} \right) \\ &= ((f_i^\#)_{V \cap V_i}(s|_{V \cap V_i})) = \left(f_i^\#\right)_V (s). \end{aligned}$$

If $(g, g^\#) : X \to Y$ is another morphism of ringed spaces with $(g, g^\#)|_{U_i} = f_i$ for each i, then $g = f$ and

$$\begin{aligned} g_V^\#(s) = (g^\#_{V \cap V_i}(s|_{V \cap V_i})\big|_{U_i}) &= (\varphi_{f^{-1}(V \cap V_i)U_i} g^\#_{V \cap V_i}(s|_{V \cap V_i})) = ((f_i^\#)_{V \cap V_i}(s|_{V_i \cap V})) \\ &= f_V^\#(s). \end{aligned}$$

for each $V \subset Y$ and $s \in \mathcal{O}_Y(V)$ □

Remark 6.2.24. When $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces and $\{f_i\}$ is a family of morphisms of locally ringed spaces, it follows readily that for $P \in U_i$,

$$f_P^\# = (f_i^\#)_P,$$

hence the prior upgrades to one about locally ringed spaces.

Lemma 6.2.25. *1.* Spec R *is a locally ringed space for each ring* R*.*

2. If $\sigma : A \to B$ *is a map of rings, then it induces a morphism of locally ringed spaces*

$$(f, f^{\#})_{\sigma} : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$$

3. The functor

$$\begin{aligned} \text{Spec} : \text{CRing} &\to \text{LRS} \\ R &\mapsto \text{Spec } R \\ \sigma : A \to B &\mapsto (f, f^{\#})_{\sigma} \end{aligned}$$

is fully faithful. Here LRS *denotes the category of locally ringed spaces.*

Proof. 1. Follows from Lemma 6.2.16 1.

2. We define

$$\begin{aligned} f : \text{Spec } B &\to \text{Spec } A \\ \mathfrak{p} &\mapsto \sigma^{-1}(\mathfrak{p}) \end{aligned}$$

which is well-defined by Proposition 1.6.39. For an ideal $I \subset A$ one readily verifies that $f^{-1}(V(I)) = V(\sigma(I))$ I haven't actually defined what this means: DO!, which shows that f is continuous. For an open set $U \subset \text{Spec } A$, we take

$$f^{\#}_U : \mathcal{O}_{\text{Spec } A}(U) \to \left(f_{\star}\mathcal{O}_{\text{Spec } B}\right)(U) = \mathcal{O}_{\text{Spec } B}\left(f^{-1}(U)\right)$$

to be ring map taking a section $s \in \mathcal{O}_{\text{Spec } A}$ to the section

$$\begin{aligned} f^{-1}(U) &\to \bigcup_{\mathfrak{p} \in f^{-1}(U)} B_{\mathfrak{p}} \\ \mathfrak{p} &\mapsto \sigma_{\mathfrak{p}}(s \circ f)(\mathfrak{p}) \end{aligned}$$

where $\sigma_{\mathfrak{p}} : A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$ is the localization of σ with respect to $\mathfrak{p}$. We check the naturality condition: Suppose $U \subset V$. Then for $s \in \mathcal{O}_{\text{Spec } A}(U)$ and $\mathfrak{p} \in f^{-1}(V)$,

$$f^{\#}_U\big|\,(s)_{f^{-1}(V)}(\mathfrak{p}) = \sigma_{\mathfrak{p}}((s \circ f)(\mathfrak{p})) = \sigma_{\mathfrak{p}}\left(\left(s|_V \circ f\right)(\mathfrak{p})\right) = f^{\#}_V\left(s|_V\right)(\mathfrak{p}).$$

Let $\mathfrak{p} \in \text{Spec } B$ and $(U, s) \in \mathcal{O}_{\text{Spec}A,\mathfrak{p}}$. Then

$$\left(f^{\#}_{f(\mathfrak{p})}(s)\right)(\mathfrak{p}) = f^{\#}_U(s)(\mathfrak{p}) = \sigma_{\mathfrak{p}}(s(f(\mathfrak{p}))) = \sigma\left(s\left(\sigma^{-1}(\mathfrak{p})\right)\right),$$

hence the square

$$\begin{array}{ccc} \mathcal{O}_{\text{Spec } A, f(\mathfrak{p})} & \xrightarrow{f^{\#}_{\mathfrak{p}}} & \mathcal{O}_{\text{Spec } B, \mathfrak{p}} \\ \downarrow{\scriptstyle \text{ev}_{f(\mathfrak{p})}} & & \downarrow{\scriptstyle \text{ev}_{\mathfrak{p}}} \\ A_{\sigma^{-1}(\mathfrak{p})} & \xrightarrow{\sigma_{\mathfrak{p}}} & B_{\mathfrak{p}} \end{array}$$

commutes. Since $\text{ev}_{f(\mathfrak{p})}$ and $\text{ev}_{\mathfrak{p}}$ are isomorphisms, it follows that $f^{\#}_{\mathfrak{p}}$ is a local ring homomorphism. We thus conclude that $(f, f^{\#})_{\sigma}$ is a morphism of locally ringed spaces.
3. Let a morphism of locally ringed spaces $(f, f^{\#}) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$. Then there is a ring map

$$\sigma := f^{\#}_{\text{Spec } A} : A \simeq \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \to \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } B}) \simeq B$$

We check that $(g, g^{\#})_{\sigma} = (f, f^{\#})$. For a $\mathfrak{p} \in \text{Spec } B$, consider the local ring homomorphism (cf. Definition MorphismOfRingedSpaces)

$$\sigma_{\mathfrak{p}} : A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$$

Note that

$$\sigma^{-1}(\mathfrak{p}) = \sigma_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{p}} \cap B) = \mathfrak{m}_{f(\mathfrak{p})} \cap A = f(\mathfrak{p})$$

hence g and f agree. Now note that given a global section s on Spec A and a point $\mathfrak{p} \in \text{Spec } B$,

$$g^{\#}_{\text{Spec} A}(s)(p) = \sigma_{\mathfrak{p}}(s \circ f(\mathfrak{p})) = f^{\#}_{\mathfrak{p}}(s)(\mathfrak{p}) = f^{\#}_{\mathfrak{p}}(s)(\mathfrak{p})$$

hence by naturality $f^{\#}_U = g^{\#}_U$ for every open subset U of Spec A. We thus conclude

$$(f, f^{\#}) = (g, g^{\#})_{\sigma}.$$

□

Remark 6.2.26. Note that the construction of $f^{\#}$ only relies on the existence of local ring maps $A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$

Example 6.2.27. Consider a field K. The spectrum contains only one point, namely 0. The sections over Spec $K = \{*\}$ are thus the constant functions $\{*\} \to K$, hence the structure sheaf of Spec K is K itself.

Example 6.2.28. Consider a DVR R. This two prime ideals: 0 and the maximal ideal $\langle t \rangle \subset R$, where t is the uniformizing parameter of R. Note that $0 \notin V(\langle t \rangle)$, which implies $D(t) = \{0\}$. It follows that 0 is an *open point* (i.e. a point that is Zariski open), hence $\langle t \rangle$ is a closed point in Spec R. Moreover 0 is a dense point, since $V(0) = \text{Spec } R$ is the only

smallest closed set containing 0, since any other closed set is of the form $V(\langle t^k \rangle)$ and no such set contains 0. The local ring of 0 is $Q(R)$ and the local ring of $\langle t \rangle$ is just R, since points away from $\langle t \rangle$ are just the units of R.
The map

$$\iota : R \hookrightarrow Q(R)$$

induces a morphism of ringed spaces $(f, f^{\#})_\iota$ with

$$f : \text{Spec } Q(R) \to \text{Spec } R$$
$$0 \mapsto 0$$

Note that $(g, f_\iota^{\#})$ with

$$g : \text{Spec } Q(R) \to \text{Spec } R$$
$$0 \mapsto \langle t \rangle$$

is another morphism of ringed spaces. Since $f_0^{\#} : R \to Q(R)$ corresponds to ι and $\iota^{-1}(0) = 0 \neq \langle t \rangle = g(0)$, it follows that $(g, f_\iota^{\#})$ is not a morphism of locally ringed spaces. It is thus not induced by any ring map $R \to Q(R)$, since any ring map would induce a morphism of locally ringed spaces.

The above example shows the necessity of morphisms of locally ringed spaces having induced maps on stalks that are local ring homomorphisms.

Example 6.2.29. For a field K, we define *the affine line over* K, denoted $\mathbb{A}^1_K$ to be Spec $K[x]$. We denote the zero ideal in $\mathbb{A}^1_K$ by ξ. This (as with DVR's) is a dense point, since $\xi \notin V(f)$ for any $f \in K[x]$ of degree> 0, which is equivalent to $\{x\} \cap D(f) \neq \emptyset$ for any $f \in K[x]$ with deg $f > 0$. In particular, ξ is not a closed point. All other points of $\mathbb{A}^1_K$ are the maximal ideals of $K[x]$ which are in one-to-one correspondence with monic irreducible polynomials in $K[x]$. The points in $\mathfrak{p} \in \mathbb{A}^1_K \setminus \{\xi\}$ are thus all closed since $\mathfrak{p} = \langle f \rangle$ is a maximal ideal for some monic irreducible $f \in K[x]$, which implies $\{p\} = V(\langle f \rangle)$.

Example 6.2.30. When K is algebraically closed, closed points in $\mathbb{A}^1_K$ correspond to points in K. It is then apparent that $\mathbb{A}^1(K)$ and $\mathbb{A}^1_K$ are different constructions, since all points $\mathbb{A}^1(K)$ are closed, which means there is no dense point in $A^1(K)$.

Definition 6.2.31. In general, we define *affine space over* K to be

$$\mathbb{A}^n_K := \text{Spec } K[x_1, \ldots, x_n]$$

Definition 6.2.32. For a topological space X, *a generic point* is point x with

$$\overline{(\{x\})} = X.$$

I.e. a point whose singleton set is dense in X.

Lemma 6.2.33. *Let X be a topological and $x \in X$. Then $\overline{\{x\}}$ is irreducible.*

Proof. Suppose $\overline{\{x\}} = Y \cup Z$ for closed sets Y, Z in X. Then $x \in Y$ or $x \in Z$, hence $\overline{\{x\}} = Y$ or $\overline{\{x\}} = Z$. □

Lemma 6.2.34. *Let A be a ring. Then $Y \subset$ Spec A is irreducible if and only if Y contains a unique point that is generic in Y.*

Proof. If Y has a generic point, then it is clearly irreducible by the prior lemma. Suppose Y is irreducible. Set

$$\mathfrak{p} = \bigcap_{\mathfrak{q} \in Y} \mathfrak{q}.$$

We claim that this is a prime ideal. Indeed, if $xy \in \mathfrak{p}$, then $x \in \mathfrak{q}$ for every $\mathfrak{q} \in Y$ or $y \in \mathfrak{q}$ for every $\mathfrak{q} \in Y$, for if there were $\mathfrak{q}_1, \mathfrak{q}_2$ with $x \in \mathfrak{q}_1$, $y \notin \mathfrak{q}_1$, $y \in \mathfrak{q}_2$, $x \notin \mathfrak{q}_2$, then $Y \cap V(\langle x \rangle)$ and $Y \cap V(\langle y \rangle)$ would be proper closed components of Y. We now claim that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = Y$. Clearly $V(p) \supset Y$. Let $\mathfrak{q} \in Y$ and let $D(f)$ be an open neighborhood of $\mathfrak{q}$. Then $f \notin \mathfrak{q}$, hence clearly $f \notin \mathfrak{p}$. hence $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$.
Uniqueness of this $\mathfrak{p}$ follows from the fact that for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subset A$,

$$V(\mathfrak{p}_1) = V(\mathfrak{p}_2) \iff \mathfrak{p}_1 = \mathfrak{p}_2.$$

□

Lemma 6.2.35. *Consider a topological space X with an open covering $\{U_i\}$. If X is irreducible, then $X = U_i$ for some i.*

Proof. Indeed,

$$X = X \setminus U_i \cup X \setminus \left(\bigcup_{j \neq i} U_j \right).$$

Then $X = U_i$ or $X = X \setminus U_i$. Picking U_i with $X \cap U_i \neq \emptyset$ it follows that $X = U_i$. □

Proposition 6.2.36. *Let X be a scheme. $Y \subset X$ is irreducible if and only if it contains a unique generic point η.*

Proof. Again if it contains a generic point it is irreducible. Suppose Y is irreducible. By the prior lemma, Y must be contained in an open affine set in X. By Lemma 6.2.34 it follows that Y has a unique generic point. □

Example 6.2.37. Let K be an algebraically closed field. Then the maximal ideals of $K[x_1, x_2]$ correspond to points in $\mathbb{A}^2(K)$. The closed points of $\mathbb{A}^2_K$ thus correspond to the points of $\mathbb{A}^2(K)$ (cf. Hilbert's Nullstellensatz). Denote these by

$$\mathfrak{m}_{a,b} \quad \big((a,b) \in \mathbb{A}^2(K)\big)$$

Secondly, again the point $\xi \in \mathbb{A}^2_K$, the zero ideal in $K[x_1, x_2]$, is a generic point.
Thirdly, an irreducible polynomial $f \in K[x_1, x_2]$, corresponds to $\eta_f := \langle f \rangle \in \mathbb{A}^2_K$. Suppose $(a,b) \in f$ and let $D(g) \ni \mathfrak{m}_{a,b}$ be given. Then $g(a,b) \neq 0$ for otherwise $\mathfrak{a}, \mathfrak{b} \supset \langle g \rangle$. Then $Z(g) \not\supset Z(f)$ hence $\eta_f \not\supset \langle g \rangle$. We thus conclude that $\eta_f \in D(f)$, hence $\mathfrak{m}_{a,b}$ is a point of closure of $\{\eta_f\}$. On the other hand consider a point $\mathfrak{p} \in \mathbb{A}^2_K$ with $Z(\mathfrak{p}) \not\subset f$. Then $\eta_f \not\supset \mathfrak{p}$, hence there is a $g \in \eta_f \setminus \mathfrak{p}$. Then $\mathfrak{p} \in D(g)$ while $\eta_f \notin D(g)$, implying $\mathfrak{p}$ is not a point of closure of $\{\eta_f\}$. It follows that

$$\overline{\{\eta_f\}} = \{\eta_f\} \cup \{\mathfrak{m}_{a,b} : (a,b) \in f\}.$$

A point of $\mathbb{A}^2_K$ is thus the generic point ξ, a closed point $\mathfrak{m}_{a,b}$ or a generic point of an affine curve $f \subset \mathbb{A}^2(K)$. The map

$$\Phi : \mathbb{A}^2(K) \to \text{Specm } K[x_1, x_2] := \{m_{a,b} : (a,b) \in \mathbb{A}^2(K)\} \subset \mathbb{A}^2_K$$
$$(a,b) \mapsto \mathfrak{m}_{a,b}$$

thus defines a homeomorphism onto its image: bijection is ensured by points in $\mathbb{A}^2(K)$ being in bijection with maximal ideals in $K[x_1, x_2]$ (cf. HNS). To check continuity it is sufficient to check that $\Phi^{-1}(V(f))$ is closed in $\mathbb{A}^2$ when f is an affine plane curve over K. Note that

$$V(f) = \overline{\{\eta_f\}} = \{\eta_f\} \cup \{\mathfrak{m}_{a,b} : (a,b) \in Z(f)\},$$

hence $\Phi^{-1}(V(f)) = Z(f)$. Moreover Φ takes an affine plane curve f to its closed points in $\mathbb{A}^2_K$, i.e.

$$\Phi(Z(f)) = V(f) \cap \text{Specm } K[x_1, x_2],$$

which means Φ is bicontinuous. So while the notion of affine plane discussed here and the classical notion are not the same they are at least related. The relation between varieties V and Spec $\Gamma(V)$ will be made precise later.

We have now seen some examples of the geometric objects that we are interested in studying, namely schemes, which are locally ringed spaces that locally look the spectrum of a ring. To give a precise definition we will first give a definition of affine schemes.

Definition 6.2.38. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\text{Spec } R, \mathcal{O})$ for some ring R as locally ringed spaces.

Example 6.2.39. Spec R is trivially an example of an affine scheme, so $\mathbb{A}^n_K$ are all examples of affine schemes.

6.2.2 Schemes

Definition 6.2.40. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ where for each $P \in X$ there is an open neighborhood U of P such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. I.e. there is an open covering of X where each open set in the covering is an affine scheme.

Definition 6.2.41. A morphism of schemes is a morphism of local rings between schemes

Lemma 6.2.42. *Let R be a ring and consider $f \in R$. Then*

$$(D(f), \mathcal{O}_{\text{Spec } R}|_{D(f)}) \simeq \text{Spec } R_f,$$

Hence $D(f)$ is an affine scheme.

Proof. The canonical injection $R \hookrightarrow R_f$ induces a continuous map

$$\varphi : \text{Spec } R_f \to \text{Spec } R$$
$$\mathfrak{p} \mapsto \mathfrak{p} \cap R$$

Note that the image of φ is $D(f)$. Indeed, if I is an ideal containing f, then $\frac{f}{1} \in R_f I$, which is a unit, hence $R_f I$ is not prime. We thus conclude that if $\mathfrak{p} \subset R_f$ is prime, then $\mathfrak{p}$ does not contain f. Note that φ is injective by properties of localizations of ideals. Note then that

$$\phi : \text{Spec } R_f \to D(f)$$
$$\mathfrak{p} \mapsto \varphi(\mathfrak{p})$$

is a continuous, bijection that is closed. Indeed, $\phi(V(I)) = V(I \cap R)$. It follows that ϕ is a homeomorphism. Restricting $\varphi^\#$ to $\mathcal{O}_{\text{Spec } R}|_{D(f)}$ we obtain a natural transformation

$$\phi^\# : \mathcal{O}_{\text{Spec } R}|_{D(f)} \Rightarrow \phi_\star \mathcal{O}_{\text{Spec } R_f}$$

Let $\mathfrak{p} \in \text{Spec } R_f$. Note that

$$\left(\mathcal{O}_{\text{Spec } R}|_{D(f)}\right)_{\phi(\mathfrak{p})} = \left(\mathcal{O}_{\text{Spec } R}|_{D(f)}\right)_{\varphi(\mathfrak{p})} \simeq \mathcal{O}_{\text{Spec } R,\mathfrak{p}} \simeq R_{\varphi(\mathfrak{p})} = R_{\phi(\mathfrak{p})}.$$

Here the first isomorphism follows from the proof of Lemma D.3.25. The commutative diagram

$$\begin{array}{ccc} \left(\mathcal{O}_{\text{Spec } R}|_{D(f)}\right)_{\phi(\mathfrak{p})} & \xrightarrow{\phi^\#_{\mathfrak{p}}} & \mathcal{O}_{\text{Spec } R_f,\mathfrak{p}} \\ \downarrow{\scriptstyle \text{ev}_{\phi(\mathfrak{p})}} & & \downarrow{\scriptstyle \text{ev}_{\mathfrak{p}}} \\ R_{\phi(\mathfrak{p})} & \xrightarrow{\sigma_{\mathfrak{p}}} & (R_f)_{\mathfrak{p}} \end{array}$$

where $\sigma_{\mathfrak{p}}$ is the canonical isomorphism $R_{\phi(\mathfrak{p})} \simeq (R_f)_{\mathfrak{p}}$, shows that $\phi^{\#}_{\mathfrak{p}}$ is an isomorphism of local rings. Then $\phi^{\#}$ is a natural isomorphism (cf. Proposition 6.1.24). □

Lemma 6.2.43. *Let $(X, \mathcal{O})$ be a scheme and $U \subset X$ an open set. Then $(U, \mathcal{O}|_U)$ is a scheme.*

Proof. It remains to find an open cover of affine schemes on U. Let $P \in U$. There is an open neighborhood V of P in X with $\varphi : V \simeq \text{Spec} R$ for some ring R. Then $\varphi(U \cap V)$ is an open neighborhood of $\varphi(P)$ in Spec$\mathbb{R}$. We can find an open cover of $\varphi(U \cap V)$ consisting of sets $D(f)$. Pick an $f \in R$ with $\varphi(P) \in D(f) \subset \varphi(U \cap V)$. Then

$$(\varphi^{-1}(D(f)), \mathcal{O}|_{\varphi^{-1}(D(f))}) \simeq (D(f), \mathcal{O}_{\text{Spec } R}|_{D(f)}) \simeq (\text{Spec } R_f, \mathcal{O}_{\text{Spec } R_f}).$$

For the last isomorphism we use the prior lemma. □

Example 6.2.44. The scheme Spec $\mathbb{Z}$ consists of a generic point corresponding to the zero ideal in $\mathbb{Z}$ and the remaining point are closed points corresponding to ideals generated by a prime. Note that Spec $\mathbb{Z}$ has cofinite topology. Indeed, given an ideal $I = \langle p_1^{r_1} \cdots p_n^{r_n} \rangle \subset \mathbb{Z}$,

$$V(\langle I \rangle) = \{\langle p_1 \rangle, \ldots, \langle p_n \rangle\}$$

Lemma 6.2.45. Spec $\mathbb{Z}$ *is the terminal object in the category of affine schemes.*

Proof. Since $R \mapsto$ Spec R defines an equivalence of categories CRing $\to$ AffSch$^{\text{op}}$ and $\mathbb{Z}$ is the initial object in CRing, it follows that Spec $\mathbb{Z}$ is the terminal object in AffSch □

Remark 6.2.46. Note that Spec $R \to$ Spec $\mathbb{Z}$ is the morphism induced by $\mathbb{Z} \to R$.

Proposition 6.2.47. *The scheme* (Spec $\mathbb{Z}, \mathcal{O}_{\text{Spec } Z}$) *is the final object for the category of schemes.*

Proof. Let $(X, \mathcal{O})$ be a scheme. Let $\{U_i\}$ denote an affine cover of X. There are morphisms of affine schemes $\varphi_i : (U_i, \mathcal{O}|_{U_i}) \to$ Spec $\mathbb{Z}$. Now we check that φ_i and φ_j agree on the domains of there intersection (when these are non-empty). To check that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ it suffices to check that it agrees on each open affine set $W \subset U_i \cap U_j$. Now note that $\varphi_i|_W$, $\varphi_j|_W$ is a pair of morphisms of affine schemes, hence they must be equal since Spec Z is the terminal object in the category of affine schemes. It follows that

$$\varphi : X \to \text{Spec } Z$$

where $\varphi(P) = \varphi_i(P)$ if $P \in U_i$ is a well-defined morphism of schemes.
Now consider a morphism of schemes $\phi : X \to$ Spec $\mathbb{Z}$. Since for each i, $\phi|_{U_i} : U_i \to$ Spec $\mathbb{Z}$ defines a morphism of affine schemes it must be equal to φ_i, hence $\phi = \varphi$. □

Example 6.2.48. The spectrum of the zero ring is the empty set since any prime ideal must be proper and 0 has no proper ideals.

Proposition 6.2.49. 0 *is the initial object in the category of schemes.*

Proof. Indeed, 0 is the terminal object in the category of commutative rings and thus the initial object in the category of affine schemes. We extend the result to an arbitrary scheme, as in the proof of Proposition 6.2.47, by constructing a unique family of compatible local maps on an affine cover of said scheme. □

Example 6.2.50. Consider schemes X_1 and X_2 with open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$ and an isomorphism of schemes

$$\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \Rightarrow (U_2, \mathcal{O}_{X_2}|_{U_2})$$

A homeomorphism

$$\varphi : U_1 \to U_2$$

gives rise to an equivalent relation on the disjoint union of U_1 with U_2: For $x, y \in X_1 \sqcup X_2$, $x \sim_\varphi y$ if $x = y$ or $x \in U_1$ and $y = \varphi(x)$. We define the scheme obtained by *gluing* X_1 and X_2 along U_1 and U_2 via φ to be

$$(X_1 \sqcup X_2)/\sim .$$

For the purposes of this example denote this space by X. Consider the maps

$$\iota_k : X_k \to X$$
$$x \mapsto [x]_{\sim_\varphi}$$

for $k = 1, 2$. Let $U \subset X$. Then U is open if and only if $\iota_k^{-1}(U)$ is open for each k. Indeed if U is open if and only if $q^{-1}(U) \subset X_1 \sqcup X_2$ is open which is the case if and only if

$$\iota_k^{-1}(U) = (q \circ i_k)^{-1}(U) = i_k^{-1}\left(q^{-1}(U)\right)$$

is open. Here q denotes the quotient map with respect to $\sim_\varphi$ and i_k is the canonical embedding of X_k in $X_1 \sqcup X_2$ for $k = 1, 2$. Recall that $X_1 \sqcup X_2$ is equipped with the final topology of $\{i_k : k = 1, 2\}$. We define a sheaf

$$\mathcal{O}_X : X \to \text{CRing}$$

that takes an open set $U \subset X$ to

$$\left\{(s_1, s_2) \in \mathcal{O}_{X_1}\left(\iota_1^{-1}(U)\right) \times \mathcal{O}_{X_2}\left(\iota_2^{-1}(U)\right) \;\middle|\; \varphi^{\#}_{\iota_2^{-1}(U)\cap U_2}\left(s_2|_{\iota_2^{-1}(U)\cap U_2}\right) = s_1|_{\iota_1^{-1}(U)\cap U_1}\right\}.$$

For $U \subset V$, we define restriction of a pair of sections on $\iota_1^{-1(V)}$ and $\iota_2^{-1}(U)$ to a pair of sections on $\iota_1^{-1}(U)$ respectively $\iota_2^{-1}(U)$ by

$$\bullet|_U : \mathcal{O}_{X_1}(\iota_1^{-1}(V)) \times \mathcal{O}_{X_2}(\iota_2^{-1}(V)) \to \mathcal{O}_{X_1}(\iota_1^{-1}(U)) \times \mathcal{O}_{X_2}(\iota_2^{-1}(U))$$
$$(s_1, s_2) \mapsto \left(s_1|_{\iota_1^{-1}(U)\cap U_1} , s_2|_{\iota_2^{-1}(U)\cap U_2} \right).$$

Suppose $(s_1, s_2) \in \mathcal{O}_X(V)$. Set $t_1 := s_1|_{\iota_1^{-1}(V)\cap U_1}$ and $t_2 := s_2|_{\iota_2^{-1}(V)\cap U_2}$. Then

$$s_1|_{\iota_1^{-1}(U)\cap U} = t_1|_{\iota_1^{-1}(U)\cap U_1} = \left(\varphi^{\#}_{\iota_2^{-1}(V)\cap U_2}(t_2)\right)\Big|_{\iota_1^{-1}(U)\cap U_1}$$
$$= \varphi^{\#}_{\iota_2^{-1}(U)\cap U_2}\left(t_2|_{\iota_2^{-1}(U)\cap U_2}\right) = \varphi^{\#}_{\iota_2^{-1}(U)\cap U_2}\left(s_2|_{\iota_2^{-1}(U)\cap U_2}\right),$$

hence $(s_1, s_2)|_U \in \mathcal{O}_X(U)$. This assignment is obviously functorial. One also readily verifies that $\mathcal{O}_X$ is a sheaf using the that an open cover of X or more generally an open subset of X gives rise to open covers on X_k or $\iota_k^{-1}(U)$, together with the structure sheaf on X_k for $k = 1, 2$.

X **is a scheme:** Let $[x] \in X$. Then $x = \iota_k^{-1}([x])$ for some $k = 1, 2$. There is an open neighborhood $U \ni x$ in X_k with $\phi : U \simeq \text{Spec } A$ for some ring A. Note that $\iota_k : X_k \to X$ defines an injective morphism of locally ringed spaces with

$$(\iota_k)^{\#}_V : \mathcal{O}_X(V) \to \mathcal{O}_{X_1}\left(\iota_k^{-1}(V)\right)$$
$$(s_1, s_2) \mapsto s_k$$

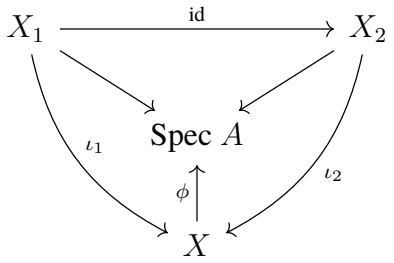
$\iota_k|_U : U \to \iota_k(U)$ defines an isomorphism of locally ringed spaces. Hence there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \text{Spec } A \\ {\scriptstyle \iota_k|_U}\downarrow & \nearrow & \\ \iota_k(U) & & \end{array}$$

hence $\iota_k(U) \simeq \text{Spec } A$.

We use the above example to construct a scheme that is not affine

Example 6.2.51. We use the prior example to construct a scheme that is not affine. Set $X_1 = X_2 = \mathbb{A}^1_K$ and $U_1 = U_2 = \mathbb{A}^1_K \setminus \{\mathfrak{p}\}$ with $\mathfrak{p} = \langle x \rangle$, $\varphi = \text{id}_{\mathbb{A}^1_K \setminus \{\mathfrak{p}\}}$. Consider a morphism $\phi : X \to \text{Spec } A$. Then we have a commutative diagram



Denote $\mathfrak{p}_1 := \iota_1(\mathfrak{p})$ and $\mathfrak{p}_2 := \iota_2(\mathfrak{p})$. Then

$$\phi(\mathfrak{p}_1) = \phi(\iota_1(\mathfrak{p})) = \phi(\iota_2(\mathfrak{p})) = \phi(\mathfrak{p}_2)$$

and since $\mathfrak{p}_1 \neq \mathfrak{p}_2$ it follows that ϕ cannot be an isomorphism of locally ringed spaces.

Proposition 6.2.52. *Let $(X, \mathcal{O})$ be a scheme and A a ring. The map*

$$\alpha : \mathrm{Hom}(X, \mathrm{Spec}\ A) \to \mathrm{Hom}(A, \Gamma(X, \mathcal{O}))$$
$$(f, f^\#) \mapsto f^\#_{\mathrm{Spec}\ A}$$

is bijective.

Proof. Let $\sigma : A \to \Gamma(X, \mathcal{O})$ be a ring map. Let $U \subset X$ be any open subset of X. Then we define $\sigma_U : A \to \mathcal{O}(U)$ to be given by $\varphi_{XU}\sigma$. When U is affine this induces a unique morphism of locally ringed spaces $(f_U, f_U^\#) : U \to \mathrm{Spec}\ A$. Let $W \subset U$ be an open affine subset and note that

$$\sigma_W = \varphi_{XW}\sigma = \varphi_{UW}\varphi_{XU}\sigma = (\sigma_U)_W .$$

Note also that $f_U|_W$ is a morphism of affine schemes induced by

$$(f_U|_W)^\#_W = \iota^\#_W \left(f_U^\#\right)_U = \varphi_{UW}\sigma_U = (\sigma_U)_W$$

hence $f_U|_W = f_W$. Thus if V is another open affine subset of X, then f_U and f_V will agree on $U \cap V$. Given an open affine cover $\{U_i\}$ of X, we thus obtain a morphism of schemes $f_\sigma : X \to \mathrm{Spec}\ A$ where

$$(f_\sigma)^\#_U : \mathcal{O}_{\mathrm{Spec}\ A}(U) \to (f_\sigma)_*\mathcal{O}(U)$$
$$s \mapsto \left((f_i^\#)_U(s)\right)$$

Here f_i is equal to f_U. The claim now is that $(f_\sigma)^\#_{\mathrm{Spec}\ A} = \sigma$. Indeed, for each $s \in A$

$$(f_\sigma)^\#_{\mathrm{Spec}\ A}(s) = \left((f_i^\#)_{\mathrm{Spec}\ A}(s)\right) = (\sigma_{U_i}(s)) = (\sigma(s)|_{U_i}) = \sigma(s).$$

We thus have a well-defined map

$$\alpha^{-1} : \mathrm{Hom}(X, \mathrm{Spec}\ A) \to \mathrm{Hom}(A, \Gamma(X, \mathcal{O}))$$
$$\sigma \mapsto f_\sigma$$

with left inverse α. Consider a morphism of schemes $f : X \to \mathrm{Spec}\ A$. Let an open affine cover $\{U_i\}$ of X be given. Note that $f|_{U_i}$ is induced by $\left(f|^\#_{U_i}\right)_{\mathrm{Spec}\ A} = (f^\#_{\mathrm{Spec}\ A})_{U_i}$, hence $f|_{U_i} = f_{(f^\#_{\mathrm{Spec}\ A})_{U_i}}$, so

$$f = f_{f^\#_{\mathrm{Spec}\ A}}.$$

It remains to prove that $f^\# = f^\#_{f^\#_{\text{Spec } A}}$. Note that

$$f^\#_{f^\#_{\text{Spec } A},i} = f_{f^\#_{\text{Spec } A}}\Big|^\#_{U_i} = f|^\#_{U_i}.$$

Let $s \in \mathcal{O}_{\text{Spec } A}(U)$, then

$$\left(f^\#_{f^\#_{\text{Spec } A}}\right)_U(s) = \left(\left(f^\#_{f^\#_{\text{Spec } A},i}\right)_U(s)\right) = \left(\left(f|^\#_{U_i}\right)_U(s)\right) = f^\#_U(s).$$

□

6.2.3 Projective Schemes

Definition 6.2.53. Let R be a graded (commutative) ring. We define the *projective spectrum of* R to be

$$\text{Proj } R := \{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \text{ and homogeneous}, R_+ \not\subset \mathfrak{p}\}.$$

Lemma 6.2.54. *Let R be a graded ring. Then*

$$\text{Proj } R = \emptyset \iff \text{rad}(0) \supset R_+.$$

Note that the second condition is equivalent to every element in R_+ being nilpotent

Proof. " $\Longrightarrow$ ": Let $\mathfrak{p} \subset R$ be a homogeneous prime ideal. Then $\mathfrak{p} \supset R_+$. It follows that

$$R_+ \subset \bigcap_{\mathfrak{p} \text{ homogeneous, prime }, \mathfrak{p} \supset R_+} \mathfrak{p} = \text{rad}(0).$$

Here we use a strengthened version of Lemma 1.6.38 for homogenous ideal in graded rings. The proof of this version is the same except for the fact that we consider

$$X := \{J \subset R : J \supset I \text{ is a homogeneous ideal}, a^n \notin J \text{ for all } n \geq 0\}$$

for $a \notin \text{rad}(I)$, to find a homogeneous prime $\mathfrak{p}$ not containing a.
" $\Longleftarrow$ ": Suppose all elements of R_+ are nilpotent. Consider an ideal $\mathfrak{a}$ not containing all of R_+. Then there is an element $a \in R_+ \setminus \mathfrak{a}$. hence $aa^{n-1} = a^n = 0 \in \mathfrak{a}$ for some $n > 0$. It follows that, $\mathfrak{a}$ is not prime, hence Proj $R = \emptyset$. □

Definition 6.2.55. Given a homogeneous ideal I in a graded ring R, we define

$$V(I) := V^{\mathbb{P}}(I) := \{\mathfrak{p} \in \text{Proj } R : \mathfrak{p} \supset I\}.$$

Lemma 6.2.56. *Let R be a graded ring, then for each pair of homogeneous ideals I, J in R*

$$V^{\mathbb{P}}(IJ) = V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J)$$

Proof. The proof is the exact same as for Spec (keeping in mind that the product of homogeneous ideals is homogeneous) □

Lemma 6.2.57. *Let R be a graded ring and $\{I_\mu\}$ a family of homogeneous ideals then*

$$V\left(\sum_\mu I_\mu\right) = \bigcap_\mu V(I_\mu)$$

Proof. Again, the proof is the same as for the affine case. □

Corollary 6.2.58. *For a graded ring R, the family of sets*

$$\{V(I) : I \subset R, \text{ homogeneous}\}$$

defines a closed basis for a topology on Proj R.

There are "strengthenings" of these two statements

Lemma 6.2.59. *Let $\mathfrak{p} \in$ Proj R and homogeneous ideals I and J in R be given. The following are equivalent*

1. $\mathfrak{p} \in V(IJ)$.
2. *For every pair of homogeneous elements $a, b \in R$, if $a \in I$ and $b \in J$ then $ab \in \mathfrak{p}$.*
3. $\mathfrak{p} \in V(I) \cup V(J)$.

Proof. obviously 1. is equivalent to 3. and clearly 2. implies 3. We also have that 2. implies 3. by Lemma 1.6.161 □

Lemma 6.2.60. *Let $\mathfrak{p} \in$ Proj R and a family of homogeneous ideals $\{I_\mu\}$ be given. The following are equivalent*

1. $\mathfrak{p} \in V\left(\sum_\mu I_\mu\right)$.
2. *For every $(a_\mu) \in \bigoplus_\mu I_\mu$ with a_μ homogeneous for each μ, we have $\sum_\mu a_\mu \in \mathfrak{p}$.*
3. $\mathfrak{p} \in \bigcap_\mu V(I_\mu)$.

Proof. The argument is parallel to the one given in the prior lemma. □

Fix a graded ring R. Let a prime ideal $\mathfrak{p} \in$ Proj R

$$H_\mathfrak{p} := \{x \in R : x \text{ homogeneous}, x \notin \mathfrak{p}\}.$$

This is a multiplicative set in R. Let $R_{(\mathfrak{p})} \subset H_{\mathfrak{p}}^{-1}R$ denote the ring of degree zero elements (cf. Lemma 1.6.168). For an open subset $U \subset \text{Proj } R$ we define $\mathcal{O}_{\text{Proj } R}(U)$ to be the set of function $s \in \prod_{\mathfrak{p}\in U} R_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ there is an open neighborhood V of $\mathfrak{p}$ such that for some homogeneous $a, f \in R$ for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and

$$s(\mathfrak{q}) = \frac{f}{a} \in S_{(\mathfrak{q})}.$$

Then we get a sheaf

$$\mathcal{O}_{\text{Proj } R} : \text{Proj } R \to \text{CRing}$$

where $\mathcal{O}_{\text{Proj } R}(U) \to \mathcal{O}_{\text{Proj } R}(V)$ induced by $U \supset V$ is given by function restriction to V.

Lemma 6.2.61. *For a graded ring R,* Proj R *is a locally ringed space*

Proof. Consider a point $\mathfrak{p} \in \text{Proj } R$. We prove that

$$\mathcal{O}_{\mathfrak{p}} \simeq R_{(p)}.$$

We check that

$$\begin{aligned} \text{ev}_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} &\to R_{(\mathfrak{p})} \\ s &\mapsto s(\mathfrak{p}) \end{aligned}$$

defines an isomorphism. The prove of this fact is almost identical to the affine case. One just replace $R_{\mathfrak{p}}$ with $R_{(\mathfrak{p})}$ in the proof. □

Definition 6.2.62. Let R be a graded ring and $f \in R_+$ a homogeneous element. Let $D_+(f)$ denote Proj $R \setminus V(\langle f \rangle)$.

Lemma 6.2.63. *Let R be a graded ring and $f \in R_+$ a homogeneous element. $D_+(f)$ is an open set in* Proj R *and*

$$\{D_+(f) : f \in R_+\}$$

is a cover of Proj R*. Furthermore, $(D_+(f), \mathcal{O}|_{D_+(f)})$ is isomorphic to* Spec $R_{(f)}$ *as locally ringed spaces. Here $R_{(f)}$ is the subring of degree 0 elements in R_f. It follows that* $(\text{Proj } R, \mathcal{O}_{\text{Proj } R})$ *is a scheme.*

Proof. By definition $D_+(f)$ is open since it is the complement of a closed set in Proj R. Let $\mathfrak{p} \in \text{Proj } R$. Then, since $\mathfrak{p} \not\supset R_+$, we can pick an $g \in R_+$ not in $\mathfrak{p}$, hence $\mathfrak{p} \in D_+(g)$,

hence the sets $D_+(f)$, for $f \in R_+$ homogeneous, define an open cover of Proj R.
Fix an $f \in R_+$. We define

$$\varphi : D_+(f) \to \text{Spec } R_{(f)}$$
$$\mathfrak{p} \mapsto R_f\mathfrak{p} \cap R_{(f)}$$

This is trivially well-defined, once one recognizes that the operations we apply to $\mathfrak{p}$ all preserve being a prime (see e.g. Corollary 1.6.104).This map has an inverse given by

$$\varphi^{-1} : \text{Spec } R_{(f)} \to D_+(f)$$
$$\mathfrak{p} \mapsto R_f\mathfrak{p} \cap R$$

We can extend these maps to maps from all homogeneous ideals in R and all (homogeneous) ideals in $R_{(f)}$ and vice versa. Note that both these maps are order preserving with respect to inclusion hence

$$\varphi(V(I)) = V(\varphi(I)) \text{ and } \varphi^{-1}(V(J)) = V(\varphi^{-1}(J))$$

for homogeneous ideals I in R and ideals J in $R_{(f)}$, which proves that φ is a homeomorphism. Canonically, $R_{(f)} \subset R_{(\mathfrak{p})}$ hence $\sigma_\mathfrak{p} : \left(R_{(f)}\right)_{\varphi(\mathfrak{p})} \simeq R_{(\mathfrak{p})}$, since any element in $\left(R_{(f)}\right)_{\varphi(\mathfrak{p})}$ are of the form

$$\frac{\frac{a}{f^n}}{\frac{x}{f^m}}$$

with appropriate restrictions on degrees (i.e. that deg $a -$ deg $f^n =$ deg $x -$ deg f^m) and in $R_{(\mathfrak{p})}$ this is equal to

$$\frac{af^m}{xf^n},$$

and trivially any element in $R_{(\mathfrak{p})}$ can be written on this form. We can then define a natural isomorphism $\varphi^\# : \mathcal{O}_{\text{Spec } R_{(f)}} \Rightarrow \varphi_\star \mathcal{O}_{\text{Proj } R}|_{D_+(f)}$. Indeed, for $U \subset \text{Spec } R(f)$ open we let the component $\varphi^\#_U$ be given by

$$\varphi^\#_U(s) : \mathfrak{p} \mapsto \sigma_\mathfrak{p}(s \circ \varphi)(\mathfrak{p}) \in R_{(p)},$$

whose inverse is given by

$$\left(\varphi^\#_U\right)^{-1}(s) : \mathfrak{p} \mapsto \sigma_\mathfrak{p}^{-1}(s \circ \varphi^{-1})(\mathfrak{p}).$$

Note that this construction is same as mentioned in Remark 6.2.26 so naturality follows. We thus conclude that $D_+(f) \simeq \text{Spec } R_{(f)}$. □

Lemma 6.2.64. *Consider a map of graded rings $\sigma : R \to S$. Then*

$$U = \{\mathfrak{p} \in \text{Proj } S : \mathfrak{p} \not\supset \sigma(R_+)\}$$

is an open set.

Proof. Let $\mathfrak{p} \in U$. Then there is an $f \in R_+$ not in $\sigma^{-1}(\mathfrak{p})$. Then $\sigma(f)$ is not in $\mathfrak{p}$. Let $\mathfrak{q} \in D_+(\sigma(f))$. Then $\sigma(f) \notin \mathfrak{q}$, hence $\mathfrak{q} \not\supset \sigma(R_+)$. It follows that $D_+(\sigma(f)) \subset U$, hence U is open. □

The above gives rise to a morphism of schemes,

$$f : U \to \text{Spec } R$$

whose underlying continuous map is given by

$$\mathfrak{p} \mapsto \sigma^{-1}(\mathfrak{p})$$

and where a component of $f^\#$ takes a section $s \in \mathcal{O}_{\text{Proj } R}(V)$ and maps it to the section

$$\mathfrak{p} \mapsto \sigma_{(\mathfrak{p})}(s \circ f(\mathfrak{p}))$$

where $\sigma_{(\mathfrak{p})} : S_{(f(\mathfrak{p}))} \to R_{(\mathfrak{p})}$ is the canonical map induced by σ.

Proposition 6.2.65. *Consider a map of graded rings $\sigma : R \to S$ with $\sigma_d : R_d \to S_d$ being an isomorphism for each $d \geq d_0$ for some $d_0 \geq 0$. Then $U = \text{Proj } S$ and $f : \text{Proj } S \to \text{Proj } R$ is an isomorphism.*

Proof. Let $\mathfrak{p} \in \text{Proj } S$. Then there is a homogeneous $f \in S_d$ for $d > 1$ such that $f \notin \mathfrak{p}$. Then $f^{d'} \notin \mathfrak{p}$ for some $d' \geq d_0$. Note that $f^{d'} = \sigma(g)$ for some $g \in R_{dd'}$, hence $\sigma(g) \notin \mathfrak{p}$, which implies $\mathfrak{p} \in U$. □

6.2.4 Reduced Schemes

Definition 6.2.66. A scheme $(X, \mathcal{O})$ is *reduced* if for each $U \subset X$ open, $\mathcal{O}(U)$ is reduced.

Definition 6.2.67. Given a scheme $(X, \mathcal{O})$, we can form the *reduced scheme induced by* X, to be the pair $X_{\text{red}} := (X, \mathcal{O}_{\text{red}})$ where (cf. Lemma 6.1.5)

$$\mathcal{O}_{\text{red}} : X \to \text{CRing}$$

is the presheaf induced by $\mathcal{O}$ and $\{\text{nil}(\mathcal{O}(U)) : U \subset X\}$.

Remark 6.2.68. Note that $\mathcal{O}_{\text{red}}$ defines a sheaf of rings, since it is just the quotient sheaf of $\mathcal{O}$, with $U \mapsto \text{nil}(\mathcal{O}(U))$, hence X_{red} is a ringed space. It is a locally ringed space since for each P by more general lemma about locally ringed spaces.

Lemma 6.2.69. *A scheme $(X, \mathcal{O})$ is reduced if and only if $\mathcal{O}_P$ is reduced for each $P \in X$.*

Proof. Suppose $(X, \mathcal{O})$ is reduced. Let $P \in X$ and $[s_U] \in \mathcal{O}_P \setminus 0$. For any open neighborhood V of P, $s_U|^n_{U\cap V} \neq 0$ for any $n \geq 1$, hence $[s_U] \neq 0 \in \mathcal{O}_p$. It thus follows that $\mathcal{O}_P$ is reduced.
Suppose conversely that $(X, \mathcal{O})$ is not reduced. Then there is a point P in an open set U with $\mathcal{O}(U)$ not reduced. I.e. there is a section $s \in \mathcal{O}(U) \setminus 0$ and an $n \geq 1$ with $s^n = 0$. Then $[s]^n = 0 \in \mathcal{O}_P$, hence $\mathcal{O}_P$ is not reduced. □

Lemma 6.2.70. *Let R be a ring. Then*

1. Spec R *is reduced if and only if* R *is reduced.*
2. Spec $R_{\text{red}} \simeq (\text{Spec } R)_{\text{red}}$

Proof. 1. If Spec R is reduced, then $\mathcal{O}(\text{Spec } R) \simeq R$ is reduced. Conversely, suppose R is reduced. Then $\mathcal{O}_\mathfrak{p} \simeq R_\mathfrak{p}$ is reduced for each $\mathfrak{p} \in \text{Spec } R$.
2. Consider the map $R \to R_{\text{red}}$. It induces a homeomorphism $\varphi : \text{Spec } R_{\text{red}} \to \text{Spec } R$ taking prime ideal $\mathfrak{p}/\text{nil}(R)$ to its corresponding prime ideal $\mathfrak{p}$ in Spec R and a natural transformation $\varphi^\# : \mathcal{O}_{\text{Spec } R} \Rightarrow \varphi_\star \mathcal{O}_{\text{Spec } R_{\text{red}}}$. Consider the canonical natural transformation $\pi : \mathcal{O}_{\text{Spec } R} \Rightarrow (\mathcal{O}_{\text{Spec } R})_{red}$ defined by $\mathcal{O}_{\text{Spec } R}(U) \twoheadrightarrow (\mathcal{O}_{\text{Spec } R})_{\text{red}}(U)$. Then we get a natural transformation

$$\overline{\varphi^\#} : (\mathcal{O}_{\text{Spec } R})_{\text{red}} \Rightarrow \varphi_\star \mathcal{O}_{\text{Spec } R_{\text{red}}}$$

Explicitly for $U \subset \text{Spec } R$, $s + \text{nil}(\mathcal{O}_{\text{Spec } R}(U)) \mapsto \varphi^\#_U(s)$. under $\overline{\varphi^\#}_U$. From $\sigma_\mathfrak{p} : R_{\varphi(\mathfrak{p})} \twoheadrightarrow (R_{\text{red}})_\mathfrak{p}$ we get the isomorphism $\widetilde{\sigma_\mathfrak{p}} : R_{\varphi(\mathfrak{p})}/\text{nil}(R_{\varphi(\mathfrak{p})}) \to (R_{\text{red}})_\mathfrak{p}$. Consider the map

$$\begin{aligned} \overline{\text{ev}_\mathfrak{q}} : \mathcal{O}_{\text{Spec } R,\mathfrak{q}} &\to (R_\mathfrak{q})_{\text{red}} \\ [s_U] &\mapsto s_U(\mathfrak{q}) + \text{nil}(R_\mathfrak{q}) \end{aligned}$$

Note that if $s_U(\mathfrak{q}) = \frac{a}{f} \in \text{nil}(R_\mathfrak{q})$, then since s is equal to the constant section $\frac{a}{f}$ in a sufficiently small neighborhood around $\mathfrak{q}$, it follows that $[s_U] \in \text{nil}(\mathcal{O}_{\text{Spec } R,\mathfrak{q}})$, hence

$$\begin{aligned} \widetilde{\text{ev}_\mathfrak{q}} : ((\mathcal{O}_{\text{Spec } R})_{\text{red}})_\mathfrak{q} \simeq \mathcal{O}_{\text{Spec } R,\mathfrak{q}}/\text{nil}(\mathcal{O}_{\text{Spec } R,\mathfrak{q}}) &\to (R_\mathfrak{q})_{\text{red}} \\ [s_U] + \text{nil}(\mathcal{O}_{\text{Spec } R,\mathfrak{q}}) &\mapsto s_U(\mathfrak{q}) + \text{nil}(R_\mathfrak{q}) \end{aligned}$$

The implicit isomorphism given in the domain of this map is readily verified. Indeed, we know it to be a fact in general that $(\mathcal{F}/\mathcal{G})_P \simeq \mathcal{F}_P/\mathcal{G}_P$. It is also clear that $\text{nil}(\mathcal{O}_{\text{Spec } R})_{\mathfrak{q}} = \text{nil}(\mathcal{O}_{\text{Spec } R,\mathfrak{q}})$. One then readily verfies that the square

$$\begin{array}{ccc} \mathcal{O}_{\text{Spec } R,\varphi(\mathfrak{p})}/\text{nil}(\mathcal{O}_{\text{Spec } R,\varphi(\mathfrak{p})}) & \xrightarrow{\overline{\varphi^{\#}}_{\mathfrak{p}}} & \mathcal{O}_{\text{Spec } R_{\text{red}},\mathfrak{p}} \\ \downarrow{\scriptstyle \widetilde{\text{ev}_{\varphi(\mathfrak{p})}}} & & \downarrow{\scriptstyle \text{ev}_{\mathfrak{p}}} \\ (R_{\varphi(\mathfrak{p})})_{\text{red}} & \xrightarrow{\widetilde{\sigma_{\mathfrak{p}}}} & (R_{\text{red}})_{\mathfrak{p}} \end{array}$$

commutes. Then $\overline{\varphi^{\#}}_{\mathfrak{p}}$ is an isomorphism for each $\mathfrak{p} \in \text{Spec } R_{\text{red}}$, hence $\overline{\varphi^{\#}}$ is a natural isomorphism, hence $(\varphi, \overline{\varphi^{\#}}) : (\text{Spec } R)_{\text{red}} \to \text{Spec } R_{\text{red}}$ defines an isomorphism of locally ringed spaces. □

Lemma 6.2.71. *The assignment*

$$(X, \mathcal{O}) \mapsto (X, \mathcal{O}_{\text{red}})$$

is functorial in the category of ringed spaces, hence if $(X, \mathcal{O}_X) \simeq (Y, \mathcal{O}_Y)$*, then*

$$X_{\text{red}} \simeq Y_{\text{red}}.$$

Proof. Let a morphism of locally ringed spaces $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be given. Since the assignment $R \mapsto R_{\text{red}}$ is functorial it follows readily that the assignment,

$$\varphi_U^{\#} \mapsto (s + \text{nil}(\mathcal{O}_Y(U)) \mapsto \varphi_U^{\#}(s) + \text{nil}(\varphi_\star \mathcal{O}_X(U)))$$

defines a natural transformation $(\mathcal{O}_Y)_{\text{red}} \Rightarrow \varphi_\star(\mathcal{O}_X)\text{red}$ and this assignment is functorial. □

Lemma 6.2.72. *Given a scheme* $(X, \mathcal{O})$*,* X_{red} *is a reduced scheme.*

Proof. It remains to check that X_{red} admits a cover of affine schemes. $(X, \mathcal{O})$ admits a cover of affine schemes. Let U be such an open set. Then $(U, \mathcal{O}|_U) \simeq \text{Spec } R$ for some ring R. Hence

$$(U, \mathcal{O}_{\text{red}}|_U) = (U, \mathcal{O}|_U)_{\text{red}} \simeq (\text{Spec } R)_{\text{red}} = \text{Spec } R_{\text{red}}.$$

Thus X_{red} is a scheme. □

Consider a scheme $(X, \mathcal{O})$. Let $\pi_{X_{\text{red}}} : X_{\text{red}} \to (X, \mathcal{O})$ be the morphism of schemes, whose underlying continuous function is the identity map and whose natural transformation has components

$$\begin{aligned} (\pi_{X_{\text{red}}})_U^{\#} : \mathcal{O}(U) &\twoheadrightarrow \mathcal{O}_{\text{red}}(U) \\ s &\mapsto s + \text{nil}(\mathcal{O}(U)) \end{aligned}$$

Proposition 6.2.73. *Consider a morphism of schemes $\varphi : X \to Y$ and suppose X is reduced. Then there is a unique morphism of schemes $\phi : X \to Y_{\text{red}}$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y_{\text{red}} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \pi_{Y_{\text{red}}}} \\ & & Y \end{array}$$

Proof. Let $U \subset Y$ be open, then, using that $\text{nil}(\mathcal{O}_X(U)) = 0$, we get a unique $\phi_U^{\#} : \mathcal{O}_{Y,\text{red}}(U) \to \varphi_\star \mathcal{O}_X(U)$ such that

$$\begin{array}{ccc} \mathcal{O}_{Y,\text{red}}(U) & \xleftarrow{\phi_U^{\#}} & \varphi_\star \mathcal{O}_X(U) \\ & \nwarrow^{\varphi_U^{\#}} & \uparrow{\scriptstyle (\pi_{Y_{\text{red}}})_U^{\#}} \\ & & \mathcal{O}_Y(U) \end{array}$$

commutes. One easily checks that these form the components of a natural transformation satisfying the condition. Uniqueness follows from using the uniqueness on the components of $\phi^{\#}$. □

6.2.5 The Residue Field at a Point

Definition 6.2.74. Let $(X, \mathcal{O})$ be a scheme and x a point in X. Then *the residue field at* x is the field

$$K(x) := \mathcal{O}_x / \mathfrak{m}_x,$$

where $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_x$.

Proposition 6.2.75. *Let $(X, \mathcal{O})$ be a scheme and x a point in X. Consider a field L. There is a 1-1 correspondence of morphisms of schemes* Spec $L \to X$ *and pairs (x, ι_x) where $x \in X$ and ι_x is an embedding $K(x) \hookrightarrow L$.*

Proof. Suppose $\varphi :$ Spec $L \to X$ is a morphism of schemes. Note that Spec $L = \{*\}$, hence φ is given by mapping the one point of Spec L to some $x \in X$. Since φ is a morphism of locally ringed spaces, it follows that $\varphi_P : \mathcal{O}_x \to L$ is a local ring map, meaning ker $\varphi_P = \varphi_P^{-1}(0) = \mathfrak{m}_x$. We thus get an embedding $K(x) = \mathcal{O}_x/\mathfrak{m}_x \hookrightarrow L$. Note that if $U \subset X$ is an open neighborhood around x, then the diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & L \\ & \searrow & \uparrow \\ & & K(x) \end{array}$$

commutes.

Conversely, suppose we are given a point x and an embedding $\iota_x : K(x) \hookrightarrow L$. Let $\varphi : \text{Spec } L \to X$, be given by $* \mapsto x$. The structure sheaf of Spec$L$ consists of L itself. We define a natural transformation

$$\mathcal{O} \Rightarrow L$$

in the following way: When $x \in U \subset X$, define $\mathcal{O}(U) \to \mathcal{O}_{\text{Spec } L}(\varphi^{-1}(U)) = L$ to be the composite of $\mathcal{O}(U) \to K(x)$ and $K(x) \hookrightarrow L$. When $x \notin U$, note that $\mathcal{O}_{\text{Spec } L}(\varphi^{-1}(U)) = \mathcal{O}_{\text{Spec } L}(\emptyset) = 0$, which is the terminal object in the category of rings. The map $\mathcal{O}(U) \to \varphi_* \mathcal{O}_{\text{Spec } L}(U)$ is then obvious. It is clear that this assignment is natural. □

6.2.6 Varieties & Schemes

Definition 6.2.76. Given a scheme S, *a scheme over* S is a scheme X together with a morphism $X \to S$. A morphism of schemes over S from X to Y is a morphism of schemes $X \to Y$ such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

commutes. We denote the category of schemes over S by $\mathfrak{Sch}(S)$. And the category of schemes over the spectrum of a ring R by $\mathfrak{Sch}(R)$.

Theorem 6.2.77. *Fix an algebraically closed field K. Then there is fully faithful functor*

$$t : \text{Var}_K \to \mathfrak{Sch}(K)$$

such that each variety V is homeomorphic to the closed points of the underlying space of $t(V)$, $\text{sp}(t(V))$.

Proof. We first construct an endofunctor t on Top and that this restricts to a fully faithful functor $t : \text{Var}_K \to \mathfrak{Sch}(K)$. For a space X, let $t(X)$ denote the irreducible closed subsets of X. Note that if $Y \subset X$ is closed then $t(Y) \subset t(X)$, if $Y_1, Y_2 \subset X$ are closed, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ and if $\{Y_\alpha\}$ is a family of closed subsets of X, then $t\left(\bigcap_\alpha Y_\alpha\right) = \bigcap_\alpha t(Y_\alpha)$. It follows that the sets of the form $t(Y)$ form the closed sets of a topology on $t(X)$. Consider a continuous map $f : X_1 \to X_2$. Then we define

$$\begin{aligned} t(f) : t(X_1) &\to t(X_2) \\ Y &\mapsto \text{cl}(f(Y)) \end{aligned}$$

Let $t(Z) \subset t(X_2)$. Then $t(f)^{-1}(t(Z)) = t(f^{-1}(Z))$, which implies $t(f)$ is continuous. Indeed, if $\text{cl}(f(V)) \in t(Z)$, then $f(V) \subset \text{cl}(f(V)) \subset Z$, hence $V \subset f^{-1}(f(V)) \subset f^{-1}(Z)$, hence $V \in t(f^{-1})$. Conversely, if $V \subset f^{-1}(Z)$ is irreducible, then $V \subset X$ is irreducible and $f(V) \subset Z$, hence $t(V) = \text{cl}(f(V)) \subset Z$, so $V \in t(f)^{-1}(t(Z))$. The assignment, $f \mapsto t(f)$ is clearly functorial, since for a continuous function g, $\text{cl}(g(A)) = \text{cl}(g(\text{cl}(A)))$ for any subset A of dom g.
Consider the map

$$\alpha : X \to t(X)$$
$$P \mapsto \text{cl}(\{P\})$$

This is clearly continuous, since for $t(Y) \subset t(X)$ closed,

$$\alpha^{-1}(t(Y)) = Y.$$

There is a bijection between the closed sets in X and the closed sets in $t(X)$, given by

$$Y \subset X \mapsto t(Y)$$

with inverse

$$t(Y) \mapsto \alpha^{-1}(t(Y)) = Y.$$

Thus there is a bijection between open sets of X and open sets of $t(X)$.
Consider a variety V over K as a locally ringed space with the usual sheaf of regular functions $\mathcal{O}$. We claim that $(t(V), \alpha_\star\Gamma)$ is a schemes. Since any variety has an open cover of affine subvarieties, it suffices to prove the claim for V affine cf. trivial lemma not yet written. Now not that the maximal ideals of $\Gamma(V)$ are in one-to-one correspondence with the points V. Note moreover that $I_P(V) := \{f \in \Gamma(V) : f(P) = 0\} \subset \Gamma(V)$ is a maximal ideal in $\Gamma(V)$ distinct from every other maximal ideal in $\Gamma(V)$ (reference to some lemmas). It thus follows that

$$\beta : V \to \text{Specm } \Gamma(V)$$
$$P \mapsto I_P(V)$$

is a bijection of V with the maximal ideals of $\Gamma(V)$, which we denote Specm $\Gamma(V)$. Now note that $\beta^{-1}(V(I) \cap \text{Specm } \Gamma(V)) = Z(I) \subset V$ for each closed set $V(I) \cap \text{Specm } \Gamma(V) \subset \text{Specm } \Gamma(V)$. One also easily sees that β is a closed map, hence it follows that β is a homeomorphism. Now note that stalk of Spec $\Gamma(V)$ of one these maximal ideals $I_P(V)$ is isomorphic to the local ring at P, $\mathcal{O}_P(V)$. Let $U \subset \text{Spec } \Gamma(V)$ be open. Consider a

section $s \in \mathcal{O}_{\text{Spec } \Gamma(V)}(U)$. Let $P \in \beta^{-1}(U)$. Then we can map s to some regular function $\overline{s} \in \mathcal{O}_P(V)$. It follows that

$$\begin{aligned} \beta_U^{\#} : \mathcal{O}_{\text{Spec } \Gamma(V)}(U) &\to \beta_\star \mathcal{O}(U) \\ s &\mapsto \overline{s} \end{aligned}$$

Consider some regular function z in $\mathcal{O}(\beta^{-1}(U))$. On some open set W. This can be represented as some rational function $\frac{a}{f}$ on V defined on W, i.e. with $f \notin I_Q(V)$ for each $Q \in W$. Then $\beta(W) \subset U$ is an open set with $\frac{a}{f} \in \mathcal{O}_{\text{Spec } \Gamma(V)}(U)$ (i.e. the constant map $\mathfrak{p} \mapsto \frac{a}{f}$) being a section that maps to z, hence $\beta_U^{\#}$ is surjective. Suppose $\overline{s} = 0$. For some open subset $W \subset U$, s is represented by a rational function $\frac{a}{f}$ defined at each $Q \in W$. Shrinking W sufficiently, we see that $a \in \Gamma(V)$ agrees with $0 \in \Gamma(V)$ on W. Since W is dense in V it follows that $a = 0$. We thus conclude that $s = 0$, hence $\beta_U^{\#}$ is injective. We thus get a natural isomorphism $\mathcal{O}_X \cong \beta_\star \mathcal{O}$. Consider the bijective map

$$\begin{aligned} \gamma : t(V) &\to \text{Spec } \Gamma(V) \\ W &\mapsto I_V(W) \end{aligned}$$

one readily verifies that this is in fact a homeomorphism. Note that

$$\begin{array}{ccc} V & \xrightarrow{\beta} & \text{Spec } \Gamma(V) \\ \downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \gamma} & \\ t(V) & & \end{array}$$

commutes. Hence $\mathcal{O}_{\text{Spec } \Gamma(V)} \simeq \beta_\star \mathcal{O} = (\gamma\alpha)_\star \mathcal{O} = \alpha_\star \gamma_\star \mathcal{O}$. Thus setting $\mathcal{O}_{t(V)} := \alpha_\star \mathcal{O}$, we thus get an isomorphism of ringed spaces $\text{Spec } \Gamma(V) \simeq (t(V), \mathcal{O}_{t(V)})$. It follows that $(t(V), \mathcal{O}_{t(V)})$ is an affine scheme. There is a canonical ring map $K \to \Gamma(V)$, hence we get a morphism of schemes $(t(V), \mathcal{O}_{t(V)}) \cong \text{Spec } \Gamma(V) \to \text{Spec } K$, hence $(t(V), \mathcal{O}_{t(v)})$ is a scheme over K.

Let V and W be varieties over K. We show that

$$\begin{aligned} t : \text{Var}_K(V, W) &\to \mathfrak{Sch}(K)(t(V), t(W)) \\ \varphi &\mapsto t(\varphi) \end{aligned}$$

is bijective □

Proposition 6.2.78. *Consider a projective varierty V with homogeneous coordinate ring $\Gamma^{\mathbb{P}}(V)$. Then*

$$t(V) \simeq \Gamma^{\mathbb{P}}(V).$$

Proof. □

Definition 6.2.79. Given a field K, *the dual ring of* K is the ring

$$K[\epsilon]/\langle\epsilon^2\rangle.$$

Remark 6.2.80. The dual ring is a finitely generated K-algebra generated by $\epsilon + \langle\epsilon^2\rangle$. We denote this generator by ϵ for simplicity (when this does not cause confusion). The dual ring is also a two dimensional vector space over K. There is one prime ideal in the dual numbers over K containing $\langle\epsilon\rangle^2$, namely the one generated by ϵ. It follows that Spec $K[\epsilon]/\langle\epsilon^2\rangle = \{\langle\epsilon\rangle\}$.

Definition 6.2.81. Given a scheme X, *Zariski tangent space at a point* $x \in X$, is the vector space over $K(x)$,

$$T_x := \text{Dual}_{K(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2),$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

Lemma 6.2.82. *Let x be an element of a scheme X over a field K such that $K(x) \simeq K$. Then*

$$\mathcal{O}_{X,x} \simeq K \oplus \mathfrak{m}_x$$

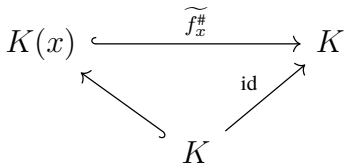
Proof. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O}_{X,x} \longrightarrow K(x) = K \longrightarrow 0$$

since X is a scheme over K there is a morphism $K \hookrightarrow \mathcal{O}_{X,x}$, so we can apply the splitting lemma to obtain an isomorphism $\mathcal{O}_{X,x} \simeq K \oplus \mathfrak{m}_x$ □

Proposition 6.2.83. *Let K be a ring and X a scheme over K and denote the dual numbers of K by A. There is a one-to-one correspondence of K-morphisms from* Spec A *to X and points in X whose residue field is isomorphic to K together with an element of T_x.*

Proof. Let f : Spec $A \to X$ be a K-morphism. Note that Spec $A = \{\langle\epsilon\rangle\}$. Then the map f takes $\langle\epsilon\rangle$ to some point $x \in X$. Since $A_{\langle\epsilon\rangle}/\mathfrak{m} \simeq K$, in addition, we have a commutative diagram



Since id is surjective, it follows that $K(x) \simeq K$. Note that $f_x^\#$ induces a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \langle\epsilon\rangle/\langle\epsilon\rangle^2 \simeq K$ which is an element of T_x.

Conversely, consider a point $x \in X$ such that $K(x) \simeq K$ and a linear map $l : \mathfrak{m}_x/\mathfrak{m}_x^2 \to K(x) = K$. We define

$$f : \text{Spec } A \to X$$
$$\epsilon \mapsto x$$

the structure sheaf of Spec A consists of A itself. When $U \subset X$ is an open set not containing x, we see that $f^{-1}(U) = \emptyset$, hence $\mathcal{O}_X(U) \to f_\star \mathcal{O}_{\text{Spec } A}(U)$ has to be the zero map. Suppose U is an open neighborhood of x. We define K linear map

$$\beta : \mathcal{O}_{X,x} = K \oplus \mathfrak{m}_x \to A = K \oplus \langle \epsilon \rangle$$
$$a + b \mapsto a + l(b + \mathfrak{m}_x^2)\epsilon$$

One readily verfies by direct computation that this is a multiplicative map and therefor a ring map. Thus upon precomposing this map with $\mathcal{O}(U) \to \mathcal{O}_{X,x}$ we obtain a natural transformation $\mathcal{O}_X \Rightarrow \mathcal{O}_{\text{Spec } A}$.

It remains to check that the above assignments are mutual inverses: Consider a morphim $f : \text{Spec } A \to X$. Denote the point in the image of f by x and the associated K-linear map by l. Denote the morphism of schemes associated with this pair by $f_{x,l}$. Clearly the underlying continuous maps agree. Consider $a + b \in \mathcal{O}_{X,x} = K \oplus \mathfrak{m}_x$. Note that $f_x^\#$ is a K-algebra map, hence it fixes K. Note also that if $b \in \mathfrak{m}_x$, then $f_x^\#(b) = \beta\epsilon$ for some $\beta \in K$, hence $l(b + \mathfrak{m}_x^2) = \beta$. Then

$$f_x^\#(a + b) = a + f_x^\#(b) = a + l(b + \mathfrak{m}_x)\epsilon.$$

Let U be an open neighborhood of x. It readily follows that $f_U^\# = \left(f_{x,l}^\#\right)_U$. Checking that $f_{x,l}$ induces x and l, for a point $x \in X$ rational over K and an element $l \in T_x$, is a trivial task. $\square$

A Logic

B First Order Predicate Logic

The purpose of this section is to introduce a framework for doing mathematics. To be precise we want to define a formal language in which, we in principle could formalise the math described in the main sections of these note. We will choose first order logic `FOL`.

B.1 The Metatheory of First Order Logic

B.1.1 Classical Metatheory

We assume a notion of *finiteness*, we assume the existence of finite sets or lists of element. With this notion consider symbols **0** and **s**, from which we build a *potentially infinite* list **N** of *natural numbers* of finite strings formed by appending **s** to the left to the prior string in the list, where the list starts with **0**. In other words the have a list

$$\mathbf{N} = [\mathbf{0}, \mathbf{s0}, \mathbf{ss0}, \dots]$$

where $\dots$ signify that to look up the next string in the list, we concatenate from the right **s** the last formed string in the list. Each string is either the string **0** or $\sigma\mathbf{0}$, where σ is some non-empty finite string of **s**'s. We thus have the following for possibly empty strings of **s**'s, σ, π, ρ

$$\sigma\pi\mathbf{0} = \pi\sigma\mathbf{0},$$
$$\mathbf{s}\sigma\pi\mathbf{0} \equiv \mathbf{s}\pi\sigma\mathbf{0},$$
$$\mathbf{s}\sigma\pi\mathbf{0} \equiv \sigma\mathbf{s}\pi\mathbf{0},$$
$$\sigma\mathbf{0} \equiv \pi\mathbf{0} \overset{\text{meta}}{\iff} \mathbf{s}\sigma\mathbf{0} \equiv \mathbf{s}\pi\mathbf{0},$$
$$\sigma\mathbf{0} \equiv \pi\mathbf{0} \overset{\text{meta}}{\iff} \sigma\rho\mathbf{0} \equiv \sigma\rho\mathbf{0},$$

where "$\equiv$" means "identical to". For two elements in **N**, n, m we write $n < m$ if n is formed earlier in the list than m and $n > m$ if n is formed later in the list than m. By $n+1$ we mean the next element to be formed in the list. And if n is not **0** we write $n - 1$ to be the string formed immediately before n. We define

$$\sigma\mathbf{0} + \mathbf{0} :\equiv \sigma\mathbf{0} \text{ and } 0 + \pi\mathbf{0} :\equiv \pi\mathbf{0}$$

and

$$\sigma\mathbf{0} + \pi\mathbf{0} :\equiv \sigma\pi\mathbf{0}.$$

We also have a principle of induction. I.e. if some statement P holds for **0** and whenever P holds for n, then it holds for n + 1, then P holds for all strings in **N**.
We also have a principle of recursion. I.e. If we define X_0 and whenever X_n is defined, so is X_{n+1}, then X_m is defined for every string m in **N**.
Note here that we have implicitly introduced the notion of indexing. Given a finite list of objects, initialize a point on **N** at **0**, pick out an element in the list of objects and assign the value point at in **N**, move the point to the next element in **N** and move pick out the next element in the list of objects. This lets us consider a length n list of object, e.g. $t_1, \ldots, t_n$.
These informal metamathematical concepts form a sufficient *classical metatheory* for the syntax of first order logic.

B.1.2 Strong Metatheory

Sometimes we need to strengthen our metatheory with *the law of excluded middle* and *naive set theory*. The first of these notions is that a statement **in the metatheory** is either true or false. The last of which is the notion that we can consider a sufficiently small collection of objects and ask whether a given object is a member of such a collection. We will need these two notions to define the semantics of first order logic.

B.2 The Alphabet of First Order Logic

The Logical Symbols

`FOL` is a framework for formalizing different theories. It therefore has to flexible enough to be augmented to work in a large variety of contexts and domains. We therefor make the following distinction between the role symbols that comprising `FOL` play: The *logical symbols* in the language of `FOL` are those symbols that are not context specific. These are

1. **Variables**: These are play the role of placeholders for objects in the domain to which we want to apply `FOL`. Eg. x might be a placeholder for a set in Zermol-Fraenkel set theory or n might be a placeholder for a natural number in Peano arithemetic or $\mathcal{C}$ could be a placeholder for a category in Category Theory. We presuppose a potentially infinite list of such symbols. That is we may always generate sufficiently large number of distinct variables.

2. **Logical operators:** $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies).

3. **Logical quantifiers:** $\forall$ (universal quantifier, to be read as "for all" or "for every") and $\exists$ (existential quantifier, to be read as "exists" or "there is").

4. **Equality symbol:** = a *relational symbol* (scroll down a little for a definition) which is not context specific.

B.2.1 The Non-logical Symbols

Those symbols that are introduced in some context to use FOL to talk about some domain. these are

1. **Constant symbols:** These are symbols that signify a specific object in a specific domain. Examples are $\emptyset$ in for example Zermelo-Fraenkel set theory or 0 in Peano arithmetic.

2. **Function symbols:** A symbol that is a placeholder for an object that depends on finite list of objects in a domain called arguments. For instance if F is a function symbol that takes n arguments, then given variables $x_1, \ldots, x_n$, $F(x_1, \ldots, x_n)$ is to be thought of as a placeholder of some object. A function symbol has a natural number attached to them called an *arity*. A function symbol that takes n arguments is an n-ary function. Examples are + in Peano Arithmetic or $\cup$ in Zermelo-fraenkel set theory. In particular, these are examples of *binary* functions.

3. **Relation symbols:** Symbol that signify relations of objects. Like function symbols, these have a natural number arity attached to them; and given a relation symbol R and objects $x_1, \ldots, x_n$, $R(x_1, \ldots, x_n)$ signifies that $x_1, \ldots, x_n$ are in relation by R.

To actually give these syntactical meaning, we need to define rules for building well-formed formulae in FOL. I.e. we need to define what strings of symbols, we are allowed to write in the language. A set of potentially infinite list of non-logical symbols is called a *signature* often denoted $\mathcal{L}$.

B.2.2 Terms and Formulae

For a signature $\mathcal{L}$, a string of symbols is an *$\mathcal{L}$-term* if it is a result of finitely applying these rules in some order

T0: Each variable is an $\mathcal{L}$-term.

T1: Each constant symbol in $\mathcal{L}$ is an $\mathcal{L}$-term.

T2: If $\tau_1, \ldots, \tau_n$ are any $\mathcal{L}$-terms and F is an n-ary function symbol in $\mathcal{L}$, then $F(\tau_1, \ldots, \tau_n)$ is an $\mathcal{L}$-term.

When the symbols are not involving any non-logical symbols, it is called a *term*. Those terms of the form T0 or T1 are called *atomic terms*. Note that in the above, the τ's are placeholders for objects in the language **not** for objects in a domain of interest. If we want to prove a property Φ of terms, then we do that by proving that each of the three categories of terms satisfy Φ, which is called the *induction on term construction*.
An $\mathcal{L}$*-formula* is a string of symbols resulting from finite application of the rules

F0: For $\mathcal{L}$-terms τ_1, τ_2 then $\tau_1 = \tau_2$ is an $\mathcal{L}$-formula.

F1: For $\mathcal{L}$-terms $\tau_1, \ldots, \tau_n$ and R a non-logical n-ary relation symbol in $\mathcal{L}$, then $R(\tau_1, \ldots, \tau_n)$ is an $\mathcal{L}$-formula.

F2: if φ is an $\mathcal{L}$-formula, then $\neg\varphi$ is an $\mathcal{L}$-formula.

F3: if φ and ψ are $\mathcal{L}$-formulae, then $\varphi \to \psi, \varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulae.

F4: if φ is an $\mathcal{L}$-formula, then given a variable x, $\exists x\varphi$ and $\forall x\varphi$ are $\mathcal{L}$-formulae.

When an $\mathcal{L}$-formula is build from terms, it is simply called a *formula*. F0- and F1-formulae are called *atomic formulae*. The *induction on formula construction* refers to the fact that proving a property Φ is satisfied by formulae, we can prove that it is satisfied for atomic formulae, and that for $\mathcal{L}$-formulae φ, ψ satisfying Φ, given a variable x, then $\neg\varphi, \varphi\wedge\psi, \varphi\vee\psi, \varphi \to \psi, \exists x\varphi, \forall x\varphi$ also satisfy Φ.
Consider a variable x and a formula φ of the form $\forall x\psi$ or $\exists x\psi$. If x is used to construct ψ and does not immediately appear after a logical quantifier at some position in ψ, it is said to be in the *range of* said logical quantifier. Such a variable is said to *bound at that position* by the last quantifier that it is in the range of. A variable that is not bound by a quantifier at a particular position is said to be *free* at position. A variable may appear in a formula as both bounded and free. For instance $(\forall x(x \neq x)) \to \exists z(x = x)$. The first two occurences of x are bound by the $\forall$ and the last two are bound by the $\exists$. The set of free variables of a formula (a rule given later will show that this set is uniquely defined) is denoted free(φ). A formula φ is a *sentence* if free(φ) = $\emptyset$, so $x \neq x \to x = x$ is a formula while $\forall x(x \neq x \to x = x)$ is a sentence. A term is *closed* if it contains no variables. For a formula ϕ and variables $x_1, \ldots, x_n$ with $\{x_1, \ldots, x_n\} \subset$ free(ϕ), we denote ϕ by $\phi(x_1, \ldots, x_n)$. In the metalanguage of FOL, we write $\varphi \equiv \psi$ for formulas, if the strings comprising these are identical.
When τ is a term and x is a variable in τ, for a term ω, we get a new term by *substituting* x with ω, i.e. by replacing every instance of x by ω, which we may do considering τ as unary function symbol in $\mathcal{L}$. We denote this new term by $\tau(x/\omega)$.
For a formula φ and a variable x and τ a term, $\varphi(x/\tau)$ is the formula obtained by replacing

every free instance of x in φ by τ. So we obtain a notion of substitution for formulas as well. A substition of x in a formula φ with a term τ is called *admissible* if it is not in the range of a quantifier that binds a variable in τ. If x does not occur as a free variable in φ, then trivially x is not bound hence, $\varphi(x/\tau)$ is admissible. In this case the string φ is unchanged, since no instance x is replaced by τ. So $\varphi \equiv \varphi(x/\tau)$. In general, for an admissible substitution, we write $\varphi(\tau)$ instead of $\varphi(x/\tau)$. When we declare that a symbol, ϕ, in the metalanguage denotes another, ψ, we write $\phi :\equiv \psi$. For instance, $\varphi(\tau) := \varphi(x/\tau)$.

B.3 Axioms and Inference Rules

An *axiom* is a special formula, which will be one ingredient in producing new formulas. In a looser sense (for now), think of axioms as statements that are valid in some context. Sometimes we also consider *axiom schemae*, defined to be a collection axioms taken to be valid for every instance of some fixed list of function symbols, a fixed list of relation symbols and a fixed list of formulae.

B.3.1 Logical Axioms

The *logical axioms* are those axioms that are valid in any context. These are given by the following axiom schemae. Consider $\varphi, \varphi_1, \varphi_2, \varphi_3, \psi$ arbitrary formulae. We have formulae

$$\begin{aligned}
&\text{L0}: && \varphi \vee \neg\varphi \\
&\text{L1}: && \varphi \to (\psi \to \varphi) \\
&\text{L2}: && (\psi \to (\varphi_1 \to \varphi_2)) \to ((\psi \to \varphi_1) \to (\psi \to \varphi_2)) \\
&\text{L3}: && (\varphi \wedge \psi) \to \varphi \\
&\text{L4}: && (\varphi \wedge \psi) \to \psi \\
&\text{L5}: && \varphi \to (\psi \to (\varphi \wedge \psi)) \\
&\text{L6}: && \varphi \to (\varphi \vee \psi) \\
&\text{L7}: && \psi \to (\varphi \vee \psi) \\
&\text{L8}: && (\varphi_1 \to \varphi_3) \to ((\varphi_2 \to \varphi_3) \to ((\varphi_1 \vee \varphi_2) \to \varphi_3)) \\
&\text{L9}: && \neg\varphi \to (\varphi \to \psi).
\end{aligned}$$

Moreover if x is a variable in φ and τ a term such that $\varphi(x/\tau)$ is admissible, then

$$\begin{aligned}
&\text{L10}: && \forall x\varphi(x) \to \varphi(\tau) \\
&\text{L11}: && \varphi(\tau) \to \exists x\varphi(x).
\end{aligned}$$

Suppose x is a variable of ψ with $x \notin \text{free}(\psi)$. Then

$$\begin{aligned} &\text{L12}: \quad \forall x(\psi \to \varphi(x)) \to (\psi \to \forall x\varphi(x)) \\ &\text{L13}: \quad \forall x(\varphi(x) \to \psi) \to (\exists\varphi(x) \to \psi). \end{aligned}$$

Lastly we have axioms for the binary relational symbol $=$, for general n-ary relational symbols and for n-ary functional symbols. Let $\tau, \tau_1, \ldots, \tau_n, \tau_1', \ldots, \tau_n'$ be terms and R an n-ary relation symbol and F an n-ary function symbol.

$$\begin{aligned} &\text{L14}: \quad \tau = \tau \\ &\text{L15}: \quad \tau_1 = \tau_1' \wedge \cdots \wedge \tau_n = \tau_n' \to (R(\tau_1, \ldots, \tau_n) \to R(\tau_1', \ldots, \tau_n')) \\ &\text{L16}: \quad \tau_1 = \tau_1' \wedge \cdots \wedge \tau_n = \tau_n' \to (F(\tau_1, \ldots, \tau_n) = F(\tau_1', \ldots, \tau_n')) \end{aligned}$$

The above axioms when L_0 is excluded is called the logical axioms of *intuitionistic logic*. The inclusion of L_0 comprises the logical axioms of *classical logic*. We use $:\Longleftrightarrow$ to define relations on symbols in metalanguage of `FOL`. For instance, we define for formulae φ, ψ, terms τ, τ' and a variable x,

$$\begin{aligned} \varphi \leftrightarrow \psi &:\overset{\text{meta}}{\Longleftrightarrow} (\varphi \to \psi \wedge \psi \to \varphi) \\ \exists! x\varphi &:\overset{\text{meta}}{\Longleftrightarrow} \exists x\varphi(x) \wedge \forall y(\varphi(y) \to x = y) \\ \tau \neq \tau' &:\overset{\text{meta}}{\Longleftrightarrow} \neg(\tau = \tau'). \end{aligned}$$

B.3.2 Non-logical Axioms: Defining a Theory

To the logical axioms we may add a set of formulae (non-logical axioms), formed from a signature $\mathcal{L}$, which we call an *$\mathcal{L}$-theory*. Examples of non-logical axioms are those of, `ZF`-axioms which forms Zermelo-Fraenkel set theory, the axioms of topology which forms the theory of topology, `PA`-axioms which forms number theory, etc. For mathematical theories such axioms are always sentences.

B.3.3 Proofs

To deduce new formulas from the axioms of some $\mathcal{L}$-theory, we introduce two *inference rules*:

$$\frac{\varphi \to \psi \quad \varphi}{\psi} \qquad \text{and} \qquad \frac{\varphi}{\forall x\varphi}.$$

These are called *modus ponens* and *generalization* respectively.
We want to use the notion of axiom and inference rule to prove statements in different $\mathcal{L}$-theories. Let Φ be a finite set of $\mathcal{L}$-formulae and ψ be an $\mathcal{L}$-formula. We say that ψ *is*

provable from Φ, denoted $\Phi \vdash \psi$ if there are $\mathcal{L}$-formulae $\varphi_1, \ldots, \varphi_n$, called *a formal proof of* ψ satisfying $\varphi_n \equiv \psi$ and for $i \leq n$ φ_i satisfies one of the following conditions

1. It is a logical axiom.
2. It is an element of Φ
3. There are $j, k < i$ such that $\varphi_j \equiv \varphi_k \to \varphi_i$.
4. There is a $j < i$ with $\varphi_i \equiv \forall x \varphi_j$, where $x \notin \text{free}(\phi)$ for any of the $\phi \in \Phi$.

We take $\vdash \psi$ to mean $\emptyset \vdash \psi$. In this case ψ is called a *tautology*. Φ is called a *context*. If ψ has no formal proof in a fixed context Φ, we write $\Phi \nvdash \psi$. Rather than writing down formal proofs down in, for instance, a box proof, we will just note that this is possible to do and refer to tools like Lean and Rocq which uses an embedding of `FOL` in type theory to enable formal proof writing that can be checked automatically. Instead, we will write proofs in natural language in such a way that it to a sufficiently knowledgeable reader is somewhat clear what a formal proof would look like. At a certain point we also formulate formulae in natural language. Additionally we take certain basic proofs in `FOL` for granted. This treatment of `FOL` only serves to provide a clear and precise language for building mathematical theory and we will quickly become satisfied that it exists and be happy that if we properly formulate theory in natural language it will be formalizable in `FOL`.

B.3.4 Tautology and Logical Equivalence

We say that formulae φ and ψ are *logically equivalent* if $\vdash \varphi \leftrightarrow \psi$. We write

$$\varphi \iff \psi \overset{\text{meta}}{\iff} \vdash \varphi \leftrightarrow \psi.$$

If $\Phi \vdash \varphi \leftrightarrow \psi$, we write $\varphi \iff_\Phi \psi$. In a natural language context, Φ will be a sequence of sentences in natural language and will not explicitly be written as a finite set, so therefore we usually just write $\varphi \iff \psi$ if we want to symbolically express that φ is equivalent to ψ in a given context (it should then be clear what the context is). In the same vein we write $\varphi \implies_\Phi \psi$ to mean $\Phi \vdash \varphi \to \psi$ and in a mathematical context we shorten it to $\varphi \implies \psi$.

B.3.5 Proofs in Natural Deduction

A different approach to building a system of deduction in `FOL` is to write inference rules which tells us how to introduce and eliminate certain formulae to obtain new formulae.

There is a different notion of proof in natural deduction. Luckily this approach is equivalent to that of introducing logical axioms, modus ponens and generalization. So when building proofs, we can also apply introduction and elimination rules.

B.3.6 The Deduction Theorem

Beyond the extra proof techniques that are provided by natural deduction which remain unmentioned we give a useful metatheoretic principical

Theorem B.3.1. *(Deduction Theorem) Let a signature $\mathcal{L}$ be given and Φ be a set of $\mathcal{L}$-formulae. Consider $\mathcal{L}$-formulae φ, ψ. Then*

$$\Phi + \varphi \vdash \psi \overset{\text{meta}}{\iff} \Phi \vdash \varphi \to \psi,$$

where $\Phi + \varphi$ is $\Phi \cup \{\varphi\}$.

Sketch! Convert to Proof

Proof sketch. " $\Longleftarrow$ ": Suppose $\Phi \vdash \varphi \to \psi$. With the assumption $\Phi + \varphi$, we then get $\varphi \to \psi$, by the metatheoretic assumption, there is a formal proof $\varphi_1, \ldots, \varphi_n :\equiv \varphi \to \psi$ from Φ. Then using modus ponens on φ_n and $\varphi \in \Phi + \varphi$ we get ψ. Then from $\Phi + \varphi$, $\varphi_1, \ldots, \varphi_n, \psi$ is a sequence of formulae such that $\Phi + \varphi \vdash \psi$.
" $\Longrightarrow$ ": Suppose $\Phi + \varphi \vdash \psi$. Then there is a formal proof $\varphi_1, \ldots, \varphi_n$ of ψ from $\Phi + \varphi$. We now aim to show $\Phi \vdash \varphi \to \varphi_n$ (note that $\varphi_n \equiv \psi$) for each $i \leq n$. Note that if $\varphi_i \in \Phi$ or is a logical axiom, we first using the logical axiom L_1 get $\sigma \equiv \varphi_i \to (\varphi \to \varphi_i)$ and using modus ponens on σ and φ_i we get $\varphi \to \varphi_i$. If $\varphi_i \equiv \varphi$, we get $\Phi \vdash \varphi \to \varphi$, since $\vdash \varphi \to \varphi$. From this we conclude $\Phi \vdash \varphi \to \varphi_0$, since φ_0 can only be obtained in the three ways described above. Suppose now that for $j < i$, $\Phi \vdash \varphi \to \varphi_j$. If we then can show that when φ_i is obtained from modus ponens or generalization, it will follow by (metatheoretical) induction on $i \leq n$ that $\Phi \vdash \varphi \to \varphi_n$. [...]

B.3.7 Consistency and Compactness

To discuss the notion of consistency we first introduce the notion of *ex falso quodlibet*. given formulae φ and ψ, this is $\{\varphi, \neg\varphi\} \vdash \psi$. It is proven in the following way: we get φ and $\neg\varphi$ from the context. From modus ponens applied to $\neg\varphi \to (\varphi \to \psi)$ (from L9) and $\neg\varphi$ we get $\varphi \to \psi$. Applying modus ponens to $\varphi \to \psi$ and φ we get ψ as desired. An *ex falso quadlibet introduction* is $\Phi \vdash \varphi \wedge \neg\varphi$ where φ is some $\mathcal{L}$-formula. We denote the existence of such an instance, $\varphi \wedge \neg\varphi$ by False
A set of formulae Φ is called *consistent* if $\Phi \nvdash$ False. We denote this by $\mathrm{Con}(\varphi)$. A set of

formulae that is not consistent is called *inconsistent*, denoted $\neg\text{Con}(\Phi)$. Note that if a set of formulae is not consistent then we may prove anything from that set of formulae. I.e. for a formula φ

$$\neg\text{Con}(\Phi) \overset{\text{meta}}{\Longrightarrow} \Phi \vdash \text{False} \overset{\text{meta}}{\Longrightarrow} \Phi \vdash \varphi.$$

If on the other hand Φ is a consistent set of formulae and we can prove a formula φ from Φ, then then there is no way to prove $\neg\varphi$ from Φ,

$$\text{Con}(\Phi) \text{ and } \Phi \vdash \varphi \overset{\text{meta}}{\Longrightarrow} \Phi \nvdash \neg\varphi.$$

Indeed if $\Phi \vdash \neg\varphi$, and we suppose $\Phi \vdash \varphi$, then $\Phi \vdash \text{False}$, hence $\neg\text{Con}(\Phi)$.

A use case for the principle of ex falso quodlibet is contraposition

Theorem B.3.2. *(Proof by Contraposition) Given $\mathcal{L}$-formulae φ and ψ,*

$$\vdash (\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi).$$

Proof. We aim to apply $\wedge$-introduction. To do this we need to show $\vdash (\varphi \to \psi) \to (\neg\psi \to \varphi)$ and $\vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$. First we show $\varphi \to \psi \vdash (\neg\psi \to \neg\varphi)$ which can be shown by $\varphi \to \psi + \neg\psi \vdash \neg\varphi$. Now note that $\{\varphi \to \psi, \neg\psi\} + \varphi \vdash \psi \wedge \neg\psi$, since by modus ponens on $\varphi \to \psi$ and φ results in ψ, hence upon applying $\wedge$-introduction we get $\psi \wedge \neg\psi$. We can therefor use ex falso to get $\neg\varphi$.
Next we show $\neg\psi \to \neg\varphi \vdash \varphi \to \psi$ which can be shown by $\{\neg\psi \to \neg\varphi, \varphi\} \vdash \psi$. Using L_0 we get $\psi \vee \neg\psi$. If ψ, then we are done. If $\neg\psi$, then using modus ponens on $\neg\psi \to \neg\varphi$ and $\neg\psi$ we get $\neg\varphi$, hence $\varphi \wedge \neg\varphi$. Then using ex falso we obtain ψ. □

Remark B.3.3. Note that $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ can be done fully in intuitionistic logic. A more general statement then the above is

$$\Phi + \varphi \vdash \psi \overset{\text{meta}}{\Longleftrightarrow} \Phi + \neg\psi \vdash \neg\varphi.$$

With the above in mind, it is easy to see how one proves this.

Theorem B.3.4. *(Proof by Contradiction) Let Φ be a set of $\mathcal{L}$-formulae and φ, ψ be $\mathcal{L}$-formulae. Then*

$$\Phi + \neg\varphi \vdash \text{False} \overset{\text{meta}}{\Longrightarrow} \Phi \vdash \varphi$$

Proof. There is some way to prove $\psi \wedge \neg\psi$ for some $\mathcal{L}$-formula from $\Phi + \neg\varphi$, hence $\Phi \vdash \neg\varphi \to \psi \wedge \neg\psi$. We then apply L_0 to get $\varphi \vee \neg\varphi$. If φ, we are done. If $\neg\varphi$ we get $\psi \wedge \neg\psi$ using modus ponens, hence using exfalso we get φ. □

Given some $\mathcal{L}$-formula φ and a set of $\mathcal{L}$-formulae Φ we also have that

$$\neg\text{Con}(\Phi + \varphi) \overset{\text{meta}}{\Longrightarrow} \Phi \vdash \neg\varphi,$$

since then $\Phi + \varphi \vdash$ False which is equivalent to $\Phi + \neg(\neg\varphi) \vdash$ False, so using contradiction we get $\Phi \vdash \neg\varphi$. Conversely if

$$(\text{there is an } \mathcal{L}\text{-formula } \varphi \text{ such that } \Phi \vdash \neg\varphi) \overset{\text{meta}}{\Longrightarrow} \neg\text{Con}(\Phi + \varphi),$$

since then get $\Phi + \varphi \vdash \varphi \wedge \neg\varphi$, hence by contradiction, $\Phi + \varphi \vdash$ False.

Lemma B.3.5. *Let Φ be a set of $\mathcal{L}$-formulae and $\Phi' \subset \Phi$. Then $\neg\text{Con}(\Phi')$ implies $\neg\text{Con}(\Phi)$*

Proof. Indeed, if there is a formula φ such that $\Phi' \vdash \varphi \wedge \neg\varphi$, then trivially $\Phi \vdash \varphi \wedge \neg\varphi$. □

Theorem B.3.6. *(Compactness Theorem) A set of $\mathcal{L}$-formulae Φ is consistent if and only if every finite subset $\Phi' \subset \Phi$ is consistent*

Proof. One implication follows immediately from the above lemma. Suppose Φ is not consistent. Then there is a formula φ and a sequence of formulae $\varphi_1, \ldots, \varphi_n$ with $\varphi_n = \varphi \wedge \neg\varphi$ that is a formal proof of $\varphi \wedge \neg\varphi$ from Φ. Now consider the finite subsequence $\varphi_{i_1}, \ldots, \varphi_{i_m} \in \Phi$, with $1 \leq i_1 < \ldots < i_m \leq n$. Then setting $\Phi' := \{\varphi_{i_1}, \ldots, \varphi_{i_m}\} \subset \Phi$ we get $\Phi' \vdash \varphi \wedge \neg\varphi$, hence $\neg\text{Con}(\Phi')$. □

B.3.8 Sentences in PNF and sPNF

Let $\exists\forall$ denote either $\exists$ or $\forall$.

Definition B.3.7. For an $\mathcal{L}$-sentence σ, it is in *Prenex Normal Form of σ*, denoted *PNF*, if it is on the form

$$\exists\forall_0 x_0 \exists\forall_1 x_1 \ldots \exists\forall_n x_n \sigma'$$

where σ' is quantifier free.
σ is in *special Prenex Normal Form*, denoted *sPNF*, if it is in PNF as above where $x_1, \ldots, x_n$ appear free in σ'.

Lemma B.3.8. *Given an $\mathcal{L}$-sentence σ there is a sentence τ in sPNF such that*

$$\sigma \iff \tau.$$

Proof. We don't prove this result. □

B.4 Semantics of First Order Logic

We are now going to assign truth values to formulae. That is we want for some signature $\mathcal{L}$ to assign either false or true to each $\mathcal{L}$-formula. In propositional logic this can be done by asserting truth tabels for logical connectives, $\wedge$, $\vee$, $\neg$, $\rightarrow$. This simple approach does not extend to FOL. Take for instance the formula, $\forall x \exists y \varphi(x, y)$. Could we possibly assign a truthtable to such a formula. Suppose for instance that $\varphi \equiv x < y$. Then there is a dependency on what x and y is and we are therefor at a loss in computing a truth table in terms of φ in general. Introducing an algorithm for determining truth for $\forall$ and $\exists$ is not terrible, but we need a way to map formulas onto some sort of context. To discuss how to do this we need the notion of an *interpretatation* for which we will assume some notion of naive set theory and a notion of law of excluded middle, i.e. we assume that a statement (to be interpreted informally) is either true or false. We are thus working in the strong metatheory.

B.4.1 Structures and Interpretations

given a signature $\mathcal{L}$ an $\mathcal{L}$-structure M is a pair of a non-empty set A (the domain of M) and a mapping from $\mathcal{L}$ to A taking constant symbols $c \in \mathcal{L}$ to an element $c^M \in A$, n-ary relation symbols $R \in \mathcal{L}$ to a set of n-tuples $R^M \subset A^n$ and n-ary function symbols F to a function $F^M : A^n \rightarrow A$. An *assignment* on an $\mathcal{L}$-structure M is a mapping, ι, taking variables x in $\mathcal{L}$ to A. An $\mathcal{L}$-interpretation I is then a pair consisting of an $\mathcal{L}$-structure M and an assignment ι on M where to a variable $x \in \mathcal{L}$ and an element $a \in A$, we define for variables y in $\mathcal{L}$

$$\iota[x \mapsto a](y) = \begin{cases} a & \text{if } x \equiv y \\ \iota(y) & \text{else} \end{cases}$$

$\iota[x \mapsto a]$ is thus itself an assigment on M. We define

$$I[x \mapsto a] := (M, \iota[x \mapsto a]).$$

For an $\mathcal{L}$-term τ we define $I(\tau) \in A$ by recursion on terms in the following way: On a variable x, $I(x) := \iota(x)$, on a constant c, $I(c) := c^M$, for an n-ary function symbol and terms $\tau_1, \ldots, \tau_n$, $I(F(\tau_1, \ldots, \tau_n)) := F^M(I(\tau_1), \ldots, I(\tau_n))$. We now define how to assign truth value to formulae under an interpretation. When a formula φ is true in an interpretatation I we write $I \vDash \varphi$. This mapping is constructed inductively on symbols that are used to construct formulae. So for instance given terms τ_1, τ_2 and variable x, $I \vDash \tau_1 = \tau_2 :\iff {}_{\text{meta}} I(\tau_1) = I(\tau_2)$ and $I \vDash \exists x \varphi :\iff {}_{\text{meta}}$ there exists a in A : $I[x \mapsto a] \vDash \varphi$ and $I \vDash \neg\varphi :\iff {}_{\text{meta}} I \vDash \varphi$is not true. I.e. we translate truth of a formula

in some interpretation to statements in the metatheory. I feel that there is only a need to illustrate the idea as I don't want to write all of the rules down. Under the assumption of classical logic in the meta theory, we now have that $I \vDash \varphi$ or $I \vDash \neg\varphi$. It is still necessarily NOT true that $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$. That is there may be statements in some theory that is true but not provable or a statement that is true but where $\Phi \vdash \neg\varphi$.
Given a set of $\mathcal{L}$-formulae Φ and an $\mathcal{L}$-structure M, we say that M *is a model of* Φ if for for every assignment ι and for each $\varphi \in \Phi$,

$$(M, j) \vDash \varphi.$$

If this is the case, we write

$$M \vDash \Phi.$$

We note that one can construct a signature $\mathcal{L}$ and a domain A for which there is a set of sentences Φ and two $\mathcal{L}$-structures M_1 and M_2 where M_1 is a model of Φ and M_2 is not. Take $\mathcal{L} = \{c, f\}$ where c is constant and f is a unary function symbol, $A = \{0, 1\}$ and $\Phi = \{\varphi_1, \varphi_2\}$ with $\varphi_1 :\equiv \forall x(x = c \vee x = f(c))$ and $\varphi_2 :\equiv \exists(x \neq c)$. Now we define $c^{M_1} := 0$, $f^{M_1}(0) = 1, f^{M_2}(1) = 0$ and $c^{M_2} := 0, f^{M_2}(0) = 0, f^{M_2}(1) = 1$. Fix an assignment and consider the associated interpretatations I_1 and I_2. It turns out that $I_i \vDash \varphi_2$ for $i = 1, 2$ while $I_1 \vDash \varphi_1$ and $I_2 \vDash \neg\varphi_1$. Indeed, $I_i \vdash \varphi_2$ is equivalent to $I_i[x \mapsto 0] \vDash x \neq c$ or $I_i[x \mapsto 1] \vDash x \neq c$. We check that $I_i[x \mapsto 1] \vDash x \neq c$. To do this we have to check that it isn't true that $I_i[x \mapsto 1](x)$ is equal to $I_i[x \mapsto 1](c)$. This is obvious since $I_i[x \mapsto 1](x) = 1$ and $I_i[x \mapsto 1](c) = c^{I_i} = 0$. To check that $M_1 \vDash \varphi_1$ it is sufficient to check that $I_1[x \mapsto 0] \vDash x = c$ and $I_1[x \mapsto 1] \vDash x = f(c)$. To check the first we simply note that $I_1[x \mapsto 0](x) = 0$ and $I[x \mapsto 0](c) = c^{M_1} = 0$. To check the second statement we again note that $I_1[x \mapsto 1](x) = 1$ and also note that $I_1[x \mapsto 1](f(c)) = f^{M_1}(I[x \mapsto 1](c)) = f^{M_1}(c^{M_1}) = f^{M_1}(0) = 1$. To see that $M_2 \vDash \neg\varphi_1$ (which is in fact equivalent to $M_2 \nvDash \varphi_1$, since φ_1 is a sentence), we need to check that $I_2 \vDash \exists x(x \neq c \wedge x \neq f(c))$ and for this it is sufficient to check that $I_2[x \mapsto 1] \vDash x \neq c \wedge x \neg f(c)$. Indeed, $I_2[x \mapsto 1](x) = 1$ and $I_2[x \mapsto 1](c) = 0$. Furthermore, $I_2[x \mapsto 1](f(c)) = f^{M_2}(I_2[x \mapsto 1](c)) = f^{M_2}(c^{M_2}) = f^{M_2}(0) = 0$.

B.4.2 Universal closure

Consider an $\mathcal{L}$-formula φ and a model M and some variable x. Then $M \vDash \varphi$ if and only if $M \vDash \forall x\varphi$. Upon defining $\overline{\varphi} :\equiv \forall x_1 \forall x_2 \forall \ldots \forall x_n \varphi$ where $x_1, \ldots, x_n$ are those variables that appear free in φ at some position, then we get that

$$M \vDash \varphi \iff {}_{\text{meta}} M \vDash \overline{\varphi}.$$

this $\forall$-construction is called the *universal closure of* φ.
We also introduce the following notation. Let φ be an $\mathcal{L}$-formula with free variables $x_1, \ldots, x_n$ and consider a sequence of elements in the domain A, $a_1, \ldots, a_n$. Then we define $M \vDash \varphi(a_1, \ldots, a_n)$ to mean that $(M, j[x \mapsto a_1][x \mapsto a_2] \ldots [x \mapsto a_n]) \vDash \varphi(a_1, \ldots, a_n)$ for every assignment j in M.

B.4.3 Isomorphisms of $\mathcal{L}$-structures

We now give following notion of checking that two models are essentially the same:

Definition B.4.1. Fix a signature $\mathcal{L}$. Consider two $\mathcal{L}$-structures M and N with domains A resp. B. We say that M and N are *isomorphic*, denoted $M \simeq N$, if there is a bijection $f : A \to B$ satisfying

1. For every constant symbol $c \in \mathcal{L}$

$$f(c^M) = f(c^N)$$

2. For n and every n-ary function symbol $F \in \mathcal{L}$ and all n-ary relation symbols $R \in \mathcal{L}$ and every $a_1, \ldots, a_n \in A$, we have that

$$f(F^M(a_1, \ldots, a_n)) = F^N(f(a_1), \ldots, f(a_n)),$$

and that

$$(a_1, \ldots, a_n) \in R^M \iff (f(a_1), \ldots, f(a_n)) \in R^N.$$

Two models are *isomorphic* if there respective underlying $\mathcal{L}$-structures are isomorphic.

Remark B.4.2. An immediate consequence is that if two models M and N are isomorphic, then for each $\mathcal{L}$-formula φ,

$$M \vDash \varphi \iff {}_{\text{meta}} N \vDash \varphi.$$

It may happen that there is a pair of models M' and N' for which

$$M' \vDash \varphi \iff {}_{\text{meta}} N' \vDash \varphi$$

for every $\mathcal{L}$-sentence φ where M' and N' are NOT isomorphic. This leads us to the give the following definition:
We say that an $\mathcal{L}$-structures M is *elementarily equivalent* to an $\mathcal{L}$-structure N, denoted $M \equiv_e N$ if

$$M \vDash \varphi \implies {}_{\text{meta}} N \vDash \varphi$$

It turns out that with this definition we also have that $N \equiv_e M$ when $M \equiv_e N$. Applying the fact about universal closure it sufficient to check $M \vDash \varphi \iff {}_{\text{meta}}N \vDash \varphi$ for every $\mathcal{L}$-sentence φ.

B.4.4 Soundness and the Soundness Theorem

Definition B.4.3. A logical calculus L is sound if every statement that on syntactical side can be proven is true on the semantical side

Theorem B.4.4. *Let Φ be a set of $\mathcal{L}$-formulae and M a model of Φ. Then for every $\mathcal{L}$-formula φ,*

$$\Phi \vdash \varphi \implies {}_{\text{meta}}\, M \vDash \varphi.$$

Sketch! Convert to Proof

Proof sketch. We first prove that given any model M, we have that $M \vDash \theta_i$ for $i \in \{0, \ldots, 16\}$, where θ_i is an instance of L_i, which are the logical axioms of FOL. For L_0 this follows from the assumption that we have classical logic in the metatheory. Given the logical symbols $\vee, \wedge, \neg, \to$, and formulae φ, ψ to for instance show that $M \vDash \varphi \vee \psi$ we have to show that $M \vDash \varphi$ or $M \vDash \psi$. Setting $\Theta :\equiv_{\text{meta}} M \vDash \varphi \vee \psi$, $\Phi :\equiv_{\text{meta}} M \vDash \varphi$ and $\Psi :\equiv_{\text{meta}} M \vDash \psi$, we note that the truth value of Θ is dependent on the truth values of Φ and Ψ, by definition, via a truth table. We can for L_i, $i = 1, \ldots, 9$, set $\Theta_i :\equiv_{\text{meta}} M \vDash \theta_i$ and see that the truth of this formula can be verified using such truth tables.
We now consider the domain of M, A and fix an assignment j, denote the associated interpretation (M, j) by I.
"L_{10} is valid in M": Fix an $\mathcal{L}$-formula φ, a term τ and a variable x such that $\varphi(x/\tau)$ is admissible. We need to prove that assuming $I \vDash \forall x \varphi(x)$ that $I \vdash \varphi(\tau)$. Our assumption is equivalent to for every $a \in A$, $I[x \mapsto a] \vDash \varphi(x)$. Then $I[x \mapsto I(\tau)] \vDash \varphi(x)$, which is equivalent to explain why $I \vdash \varphi(\tau)$. So $(M, j) \vDash \forall x \varphi(x) \to \varphi(\tau)$ for every assignment j, hence $M \vDash \forall x \varphi(x) \to \varphi(\tau)$.
"L_{11} is valid in M": Fix φ, x, τ as before. We need to show assuming $I \vDash \varphi(\tau)$ that there is an $a \in A$ such that $I[x \mapsto a] \vDash \varphi(x)$. Using the same fact as before, our assumption is equivalent to $I[x \mapsto I(\tau)] \vDash \varphi(x)$. Then $(M, j) \vDash \varphi(\tau) \to \exists x \varphi(x)$ for every interpretation j, hence $M \vDash \varphi(\tau) \to \exists x \varphi(x)$.
"L_{12} is valid in M": Fix a non-free variable x of a formula ψ and let $\varphi(x)$ be an arbitrary formula. Suppose that forall $a \in A$, if $I[x \mapsto a] \vDash \psi$, then $I[x \mapsto a] \vDash \varphi(x)$. Note that by insert fact later $I[x \mapsto a] \vDash \psi$ is equivalent to $I \vDash \psi$ and that by definition $I[x \mapsto a] \vDash \varphi(x)$ is equivalent to $I \vDash \forall x \varphi(x)$. Assume then $I \vDash \psi$. From what have just noted it follows then that $I \vDash \forall x \varphi(x)$. So we conclude that $I \vDash \forall x(\psi \to \varphi(x)) \to (\psi \to \forall x \varphi(x))$,

and since j was arbitrary, it follows that $M \vDash \forall x(\psi \to \varphi(x)) \to (\psi \to \forall x\varphi(x))$.

"L_{13} is valid in M": Assume that for every $a \in A$ if $I[x \mapsto a] \vDash \varphi(x)$, then $I[x \mapsto a] \vDash \psi$. Suppose there is a $b \in A$ such that $I[x \mapsto b] \vDash \varphi(x)$. Then by assumption $I[x \mapsto b] \vDash \psi$ which using the same fact as before is equivalent to $I \vDash \psi$. We thus conclude that $I \vDash \forall x(\varphi(x) \to \psi) \to (\exists x\varphi x \to \psi)$, and since j was arbitrarily chosen, $M \vDash \forall x(\varphi(x) \to \psi) \to (\exists x\varphi x \to \psi)$.

"L_{14} is valid in M": Let τ be a term. Clearly $I(\tau)$ is the same element as $I(\tau)$, $I \vDash \tau = \tau$ and it immediately follows that $M \vDash \tau = \tau$.

"L_{14} is valid in M": Let R be an n-ary relation symbol in $\mathcal{L}$ and $\tau_1, \ldots, \tau_n, \tau_1', \ldots, \tau_n'$ be $\mathcal{L}$-terms. Suppose $I(\tau_i)$ is the same object as $I(\tau_i')$ for each i. Suppose moreover that $I \vDash R(\tau_1, \ldots, \tau_i)$. Then $(I(\tau_1'), \ldots, I(\tau_n')) = (I(\tau_1), \ldots, I(\tau_n)) \in R^M \subset A$, hence by definition $I \vDash R(\tau_1', \ldots, \tau_n')$. This shows that

$$I \vDash \tau_1 = \tau_1' \wedge \ldots \wedge \tau_n = \tau_n' \to (R(\tau_1, \ldots, \tau_n) \to R(\tau_1', \ldots, \tau_n'))$$

which means

$$M \vDash \tau_1 = \tau_1' \wedge \ldots \wedge \tau_n = \tau_n' \to (R(\tau_1, \ldots, \tau_n) \to R(\tau_1', \ldots, \tau_n')).$$

"L_{15} is valid in M": Let F be an n-ary function symbol. Let $\tau_1, \ldots, \tau_n, \tau_1', \ldots, \tau_n'$ be given as before. Suppose $I(\tau_i)$ is the same object as $I(\tau_i')$ for each i. Then

$$F^M(I(\tau_1), \ldots, I(\tau_n)) = F^M(I(\tau_1'), \ldots, I(\tau_n'))$$

hence

$$I \vDash F(\tau_1, \ldots, \tau_n) = F(\tau_1', \ldots, \tau_n') \implies {}_{\text{meta}}M \vDash F(\tau_1, \ldots, \tau_n) = F(\tau_1', \ldots, \tau_n').$$

Now, let Φ be a set of $\mathcal{L}$-formulae and M a model of Φ. Let φ be an $\mathcal{L}$-formula. Suppose $\Phi \vdash \varphi$. Note now that every instance of modus ponens is obviously valid in M and that by insert fact every instance of generalization is valid in M. It thus follows that every application of these rules to conclude that $\Phi \vdash \varphi$ is valid in M. Since we know that the logical axioms of `FOL` are valid in M, it follows that if the formulae $\varphi_1, \ldots, \varphi_n$ that make up a formal proof of φ from Φ, are all valid. In particular, since $\varphi_n \equiv \varphi$, it follows that $M \vDash \varphi$, which finishes the proof (sketch) [...]

Corollary B.4.5. *1. For every tautology φ we get that $M \vDash \varphi$ for any model M.*

2. For a set of $\mathcal{L}$-formulae Φ, if there is a model M with $M \vDash \Phi$, then Con(Φ).

3. Every instance of a logical axiom is consistent.

4. For any sentence σ, model M and set of formulae Φ,

$$\textit{if } (M \nvDash \sigma \textit{ and } M \vDash \Phi) \textit{ then } \Phi \nvdash \sigma$$

Proof. 1. Indeed, M is a model of the empty set (of formulae), hence by an application of the above theorem the result follows.
2. Indeed, given a formula $\varphi \in \Phi$,

$$M \nvDash \varphi \wedge \neg\varphi,$$

means $\Phi \nvdash \varphi \wedge \neg\varphi$ by the soundness theorem.
3. Any model of a set of formulae is a model of any instance of logical axiom, hence by 2. such an instance is consistent.
4. This is just a special case of the contrapositive of the soundness theorem. □

B.5 Gödels Completeness Theorem

We now refine the definition of an $\mathcal{L}$-theory. A set T of $\mathcal{L}$-sentences is said to be an *$\mathcal{L}$-theory*. In mathematics a set of axioms will always consist of sentences. Note that as far as semantics is concerned we may replace every formula in a set of $\mathcal{L}$-formulae by the set of the universal closures of these formulae and nothing will change.

Definition B.5.1. An $\mathcal{L}$-theory T is *(syntactically) complete* if for every $\mathcal{L}$-sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. An $\mathcal{L}$-theory that is not complete is called *incomplete*

Remark B.5.2. Note that an inconsistent theory is incomplete.

Definition B.5.3. For an $\mathcal{L}$-theory T, let $\mathrm{Th}(T)$ denote the set of $\mathcal{L}$-sentences σ with $T \vdash \sigma$.

Remark B.5.4. Note that a consistent theory T is complete if and only if either $\sigma \in \mathrm{Th}(T)$ or $\neg\sigma \in \mathrm{Th}(T)$ for every $\mathcal{L}$-sentence σ.

Theorem B.5.5. *Let T be an $\mathcal{L}$-theory such that there is a model M of T. Then there is a complete $\mathcal{L}$-theory $\overline{T}$ with $\overline{T} \supset T$.*

Proof. Define $\overline{T}$ to be the sentences, σ, which satisfy $M \vDash \sigma$. Since for any sentence τ, either $M \vDash \tau$ or $M \vDash \neg\tau$, we get that either $\tau \in \overline{T}$ or $\neg\tau \in \overline{T}$. Let σ be any $\mathcal{L}$-sentence. If $\sigma \in \overline{T}$, then since $\sigma \vdash \sigma$, it follows that $\overline{T} \vdash \sigma$. Similarly, if $\neg\sigma \in \overline{T}$, then $\overline{T} \vdash \neg\sigma$. We thus conclude that $\overline{T}$ is complete. Since $M \vDash T$, it follows that $T \subset \overline{T}$. □

Definition B.5.6. Let M be a model of some theory T. Then $\mathrm{Th}(M) = \overline{T}$ is called the *model of M*

Remark B.5.7. Note that $\mathrm{Th}(M)$ is complete.

B.5.1 Maximally Consistent Theories

Definition B.5.8. An $\mathcal{L}$-theory T is *maximally consistent* if for $\text{Con}(T)$ and for every $\mathcal{L}$-sentence $\sigma \in T$ or $\neg\text{Con}(T + \sigma)$.

Remark B.5.9. From the last two facts gathered about consistency in the section on consistency and compactness we get that for a consistent theory T,

$$\neg\text{Con}(T + \sigma) \overset{\text{meta}}{\iff} T \vdash \neg\sigma,$$

so a consistent $\mathcal{L}$-theory T is maximally consistent if and only if $\sigma \in T$ or $T \vdash \neg\sigma$.

Lemma B.5.10. *For a consistent $\mathcal{L}$-theory T, T is maximally consistent if and only if $\sigma \in T$ or $\neg\sigma \in T$ for every $\mathcal{L}$-formula σ.*

Proof. Suppose first that T is maximally consistent. Let σ be an $\mathcal{L}$-sentence. By assumption $\sigma \in T$ or $T \vdash \neg\sigma$. Suppose $\sigma \notin T$. Then $T \vdash \neg\sigma$, and since $\text{Con}(T)$, $T \nvdash \sigma$. This means $T \nvdash \neg\neg\sigma$, hence $\neg\sigma \in T$.
Assume now $\sigma \in T$ or $\neg\sigma \in T$ for every $\mathcal{L}$-sentence σ. Let σ be an arbitrary $\mathcal{L}$-sentence. Suppose $\sigma \notin T$. Then $\neg\sigma \in T$. Hence, $\neg\sigma$ is by definition a formal proof of $T \vdash \neg\sigma$. □

Lemma B.5.11. *Let T be a consistent $\mathcal{L}$-theory. Then*

1. *If T is complete, then $\text{Th}(T)$ is maximally consistent.*
2. *If T is maximally consistent, then $T = \text{Th}(T)$.*

Proof. 1. Since T is complete, it is consistent. Let σ be an $\mathcal{L}$-sentence. Since T is completeness we also get $T \vdash \sigma$ or $T \vdash \neg\sigma$ which is equivalent to $\sigma \in \text{Th}(T)$ or $\neg\sigma \in \text{Th}(T)$ which by the prior lemma means that $\text{Th}(T)$ is maximally consistent.
2. Trivially $T \subset \text{Th}(T)$. Let $\sigma \in \text{Th}(T)$. Then $T \vdash \sigma$. By assumption $\sigma \in T$ or $\neg\sigma \in T$. For any there S where $\tau \in S$ or $\neg\tau \in S$ for a sentence $\tau \in \text{Th}(S)$, if $\neg\tau \in S$, then $\neg\text{Con}(T)$. So we see that $\neg\sigma \notin T$ which implies $\sigma \in T$. We thus conclude that $T = \text{Th}(T)$. □

Lemma B.5.12. *If an $\mathcal{L}$-theory has a model M, then $\text{Th}(M)$ is a maximally consistent extension of T.*

Proof. Let $\sigma \in T$. Then $T \vdash \sigma$, hence by soundness, $M \vDash \sigma$ which means $\sigma \in \text{Th}(M)$. $\text{Th}(M) = \overline{T}$ is readily seen to be maximally consistent. It is consistent since a theory with a model is automatically consistent (cf. Corollary B.4.5 2.) and for any $\mathcal{L}$-sentence σ, either $M \vDash \sigma$ or $M \vDash \neg\sigma$ which by definition is equivalent to either $\sigma \in \text{Th}(M)$ or $\neg\sigma \in \text{Th}(M)$ □

B.5.2 Universal List of Sentences

For this section and the one's on Lindenbaum's Lemma and Gödel's Completeness Theorem any signature $\mathcal{L}$ is countable. I.e. the symbols of $\mathcal{L}$ can be arranged in a list that is at most potentially infinite list $L_{\mathcal{L}}$. We now aim to encode every $\mathcal{L}$-sentence in an ordered list called the *universal list of* $\mathcal{L}$. Let ξ_i be the symbol at index i in $L_{\mathcal{L}}$. We encode this as

$$\#\xi_i := \underbrace{22\ldots2}_{i \text{ 2's}},$$

i.e. we set $\#\xi_0 := 2$ and recursively define $\#\xi_{i+1} := \#\xi_i 2$. For logical symbols we also define an encoding using 1's. First we define encodings for the logical connectives We can order

Symbol ξ	Encoding $\#\xi$
$=$	11
$\neg$	1111
$\wedge$	111111
$\vee$	11111111
$\rightarrow$	1111111111
$\exists$	111111111111
$\forall$	11111111111111

the variables in a potentially infinite list. For x_0 we define $\#x_0 := 1$ for the i'th variable x_i we define $\#x_{i+1} := \#x_i 11$. Now given any string of symbols in the language associated with $\mathcal{L}$, $\xi := \xi_0\xi_1\ldots\xi_n$ we define

$$\#\xi := \#\xi_0 0 \#\xi_1 0 \ldots 0 \#\xi_n,$$

i.e. we recursively define the encoding on strings through recursion in the length of the string. Now we define an order on strings $\#\xi$, where $\#\xi < \#\xi'$ if $|\#\xi| < |\#\xi'|$ ($|\bullet|$ is the length of a string) and when $|\#\xi| = |\#\xi'|$ if $\#\xi <_{\text{lex}} \#\xi'$, where $<_{\text{lex}}$ is the lexicographic ordering with $0 < 1 < 2$. We assume WLOG that every relational or functional symbol uses Polish notation. We now obtain the desired universal list of $\mathcal{L}$-sentences, which is the potentially infinite list

$$\Lambda_{\mathcal{L}} := [\sigma_0, \sigma_1, \ldots]$$

of $\mathcal{L}$-sentences where

$$\#\sigma_i < \#\sigma_j$$

for $i < j$. To see that is potentially infinite, note that there is a finite time algorithm for generating the next element in the list of possible strings,

1. Let $\#\sigma_n$ be some element generated in the list. Set $l := |\#\sigma_n|$. There are only finitely many possible strings of the form $\#\sigma$ of length l (since there are only finitely many strings consisting of 0's, 1's and 2's) where σ is a sentence.

2. If $\#\sigma_n$ is the largest of these strings compute the sentence σ with the smallest $\#\sigma$ of length $l + 1$ otherwise compute the σ with the smallest $\#\sigma > \#\sigma_n$.

B.5.3 Lindenbaum's Lemma

Theorem B.5.13. *(Lindenbaum's Lemma) Let $\mathcal{L}$ be a countable signature and T a consistent $\mathcal{L}$-theory. If there is a sentence σ_0 with $T \nvdash \sigma_0$. Then there is a maximally consistent theory $T^*_{\sigma_0} \supset T + \neg\sigma_0$.*

Proof. Consider the universal list of $\mathcal{L}$-sentences

$$\Lambda_{\mathcal{L}} = [\sigma_1, \sigma_2, \dots].$$

We place $\neg\sigma_0$ at the beginning of this list to obtain a list

$$\Lambda'_{\mathcal{L}} := [\neg\sigma_0, \sigma_1, \sigma_2, \dots].$$

Set $T_0 := [\neg\sigma_0]$ and recursively define

$$T_{i+1} := \begin{cases} T_n + [\sigma_{n+1}] & \text{if Con}(T + T_n + \sigma_{n+1}) \\ T_n & \text{otherwise} \end{cases}$$

As a result of this process we obtain a potentially infinite list

$$T^*_{\sigma_0} := [\neg\sigma_0, \sigma_{i_1}, \sigma_{i_2}, \dots],$$

Where we assume $\text{Con}(T + T_n + \sigma_{n+1})$ or $\neg\text{Con}(T + T_n + \sigma_{n+1})$. For this we could use law of excluded middle, which we already use in model theory. A weaker assumption that one can take is the weak König's Lemma whose statement is

An infinite 0-1-tree has an infinite branch

where a 0-1-tree is a certain type of subtree of a binary tree.
We now prove that $T^*_{\sigma_0}$ is a maximally consistent theory containing $T + \neg\sigma_0$. Clearly $\neg\sigma_0$ is in $T^*_{\sigma_0}$. Note that for every sentence σ there is an n such that $\sigma_n \equiv \sigma$. Hence it is sufficient to prove for each n if $\sigma_n \in T$, then $\sigma_n \in T^*_{\sigma_0}$. We prove this by induction in n. For $n = 1$, note that if $\sigma_1 \in T$, then $\text{Con}(T + \neg\sigma_0)$. Indeed, since $T \nvdash \sigma_0$, we have that $\text{Con}(T + \neg\sigma_0)$

by Remark B.5.9. It thus follows that Con($T + \sigma_1$), hence $T_1 = T_0 + \sigma = [\neg\sigma_0, \sigma_1]$, hence $\sigma_1 \in T^*_{\sigma_0}$. Suppose $\sigma_m \in T$ implies $\sigma \in T^*_{\sigma_0}$ for every $m \leq n$. Suppose $\sigma_{n+1} \in T$. Pick N to be the largest $N \leq n$ such that $\sigma_M \in T^*_{\sigma_0}$. Then $T_N = T_n$. Suppose for a contradiction that $\neg$Con($T+T_n+\sigma_{n+1}$). Then since $\sigma_{n+1} \in T$, $\neg$Con($T+T_n$), which means $\neg$Con($T+T_N$), but then $\sigma_N \notin T^*_{\sigma_0}$. Then Con($T + T_n + \sigma_{n+1}$), hence $T_{n+1} = T_n + \sigma_{n+1} \ni \sigma_{n+1}$. It thus follows that $\sigma_{n+1} \in T^*_{\sigma_0}$.

To show Con($T^*_{\sigma_0}$), by the Compactness Theorem it is sufficient to prove that every finite subset of $T^*_{\sigma_0}$ is consistent. Any such finite subset is contained in T_n for some n. So it is sufficient to prove Con(T_n). Note that since $T \nvdash \sigma_0$, we get Con($T + \neg\sigma_0$), hence using the Compactness Theorem, $T_0 = \{\neg\sigma_0\}$ is consistent. Suppose Con(T_m) for each $m \leq n$ for some n. If $\neg$Con($T + T_n + \sigma_{n+1}$), then $T_n = T_{n+1}$ and we are done. Suppose then that Con($T + T_n + \sigma_{n+1}$). Then since $T_{n+1} = T_n + \sigma_{n+1}$ is a finite subset $T + T_n + \sigma_{n+1}$, we get Con(T_{n+1}).

To conclude that $T^*_{\sigma_0}$ is maximally consistent, we prove that $\sigma \in T^*_{\sigma_0}$ or $\neg$Con($T^*_{\sigma_0} + \sigma$). Let an $\mathcal{L}$-sentence be given. Such a sentence is σ_n for some $n \geq 1$. Suppose Con($T + T_{n-1} + \sigma_n$), then $\sigma_n \in T_{n-1} + \sigma_n = T_n$. Suppose $\neg$Con($T + T_{n-1} + \sigma_n$). By Lemma B.3.5, $\neg$Con($T^*_{\sigma_0} + \sigma_n$). □

Remark B.5.14.

1. Note that if $T \vdash \sigma$, then $\neg$Con($T+\neg\sigma$). Note that if for a consistent theory Φ and a formula $\varphi \in \Phi$, then Con($\Phi + \varphi$). Then since $T^*_{\sigma_0}$ is maximally consistent, we first have that $\neg\sigma \notin T^*_{\sigma_0}$ and then that $\sigma \in T^*_{\sigma_0}$.

2. $T^*_{\sigma_0} \vdash \sigma \overset{\text{meta}}{\iff} \sigma \in T^*_{\sigma_0}$:
   " $\Longrightarrow$ ": Using maximal consistency, if $T^*_{\sigma_0} \vdash \sigma$, then Con($T^*_{\sigma_0} + \sigma$). Note that $\sigma \equiv \sigma_n$. Then every finite subtheory of $T^*_{\sigma_0} + \sigma$ is consistent, hence any finite subtheory of $T + T_{n-1} + \sigma_n$ is consistent, which means Con($T + T_{n-1} + \sigma_n$). By construction $\sigma \equiv \sigma_n \in T_n \subset T^*_{\sigma_0}$.
   " $\Longleftarrow$ ": This is obvious.

3. Note also $\sigma \iff \sigma'$ then $\sigma \in T^*_{\sigma_0} \overset{\text{meta}}{\iff} \sigma' \in T^*_{\sigma_0}$.

One readily verifies the following lemma

Lemma B.5.15. *Let T be an $\mathcal{L}$-theory where $\mathcal{L}$ is countable and σ_0 be given as in Lindenbaum's Lemma. For any $\mathcal{L}$-sentences $\sigma, \sigma_1, \sigma_2$,*

1. $\neg\sigma \in T^*_{\sigma_0} \overset{\text{meta}}{\iff}$ *not* $\sigma \in T^*_{\sigma_0}$.

2. $\wedge\sigma_1\sigma_2 \in T^*_{\sigma_0} \overset{\text{meta}}{\iff} \sigma_1 \in T^*_{\sigma_0}$ *and* $\sigma_2 \in T^*_{\sigma_0}$

3. $\vee\sigma_1\sigma_2 \in T^*_{\sigma_0} \overset{\text{meta}}{\iff} \sigma_1 \in T^*_{\sigma_0}$ *or* $\sigma_2 \in T^*_{\sigma_0}$.

4. $\to \sigma_1\sigma_2 \in T^*_{\sigma_0} \overset{\text{meta}}{\iff} \sigma_1 \in T^*_{\sigma_0}$ *then* $\sigma_2 \in T^*_{\sigma_0}$.

B.5.4 An Extension of a Signature & of a Theory

Definition B.5.16. Given a countable $\mathcal{L}$-signature a *term constant* is the term that results from applying one of these three rules recursively finitely many times.

C0: A variable-free $\mathcal{L}$-term is a term constant.

C1: if $\tau_0, \ldots, \tau_{n-1}$ are term constants and F is an n-ary function, then $F\tau_1 \cdots \tau_{n-1}$ is a term constant.

C2: If i, n are natural numbers and $\tau_0, \ldots, \tau_{n-1}$ are term constants, then $(i, \tau_0, \ldots, \tau_{n-1}, n)$ is a term constant.

The strings of the form $(i, \tau_0, \ldots, \tau_{n-1}, n)$ are called *special constants*. In particular $(i, 0)$ is a special constant.

Remark B.5.17. Note that term constants build with only C0 and C1 are in the language induced by $\mathcal{L}$ and are variable free, while those build with C2 may not be.

Definition B.5.18. Let a countable signature $\mathcal{L}$ be given. Then we define $\mathcal{L}_c$ to be $\mathcal{L}$ together with the special constants (of which there are countably many).

We now encode variable terms using the same encoding as we did to construct the universal list of sentences. Moreover, #(:= 3, #0 the empty string, #$(i + 1)$:= #i1, #, := 4 and #) := 5. So given a special constant $c \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$, we define an ecoding

$$\#c := \#(\#i\#, \#\tau_0\#, \ldots \#, \#\tau_{n-1}\#, \#n\#)$$

As before we order strings build from $\mathcal{L}_c$ first by their length and lexicographically with $0 < 1 < 2 < 3 < 4 < 5$. We define potentially infinite ordered lists

$$\Lambda_\tau := [\tau_0, \tau_1, \ldots] \text{ and } \Lambda_c := [c_0, c_1, \ldots]$$

of terms constants and special constants respectively.

Definition B.5.19. Let $\sigma_i \in T^*_{\sigma_0}$ and $c_j \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ a special constant. We say that c_j *is a witness of* σ_i if

1. $i \geq 1$ and σ_i is in special Prenex Normal Form.

2. $\exists x_n$ appears in σ_i.

3. For each $m < n$: If $\exists x_m$ appears in σ_i then $\tau_m \equiv (i, \tau_0, \ldots, \tau_{m-1}, m)$.

We replace $\neg\sigma_0$ by its equivalent formula $\sigma_k \in T^*_{\sigma_0}$, which is in sPNF.
We now construct $T^*_{\sigma_0}(c)$. Suppose we have a sentence $\sigma_i \in T^*_{\sigma_0}$ that is in sPNF and $\forall x_n$ or $\exists x_n$ appear in σ_i, then it is on the form

$$\exists\!\!\!/_0 x_0 \exists\!\!\!/_1 x_1 \ldots \exists\!\!\!/ x_n \ldots \exists\!\!\!/ x_m \sigma_i(x_0, x_1, \ldots, x_n, \ldots, x_m)'$$

where σ_i is quantifier free and in which $x_1, \ldots, x_m$ appear free, hence

$$\sigma_i \equiv \exists\!\!\!/_0 x_0 \exists\!\!\!/_1 x_1 \ldots \exists\!\!\!/ x_n \sigma_{i,n}(x_0, \ldots, x_n)$$

where $x_1, \ldots, x_n$ appear free in $\sigma_{i,n}(x_0, \ldots, x_n)$. Then if $c_j \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ witnesses σ_i,

$$\sigma_i \equiv \exists\!\!\!/_0 x_0 \exists\!\!\!/_1 x_1 \ldots \exists x_n \sigma_{i,n}(x_0, \ldots, x_n).$$

Set

$$\sigma_{i,n}[c_j] :\equiv \sigma_{i,n}(x_0/\tau_0, \ldots, x_{n-1}/\tau_{n-1}, x_n/c_j).$$

Set $T^*_0 := T^*_{\sigma_0}$. Suppose T^*_j is already defined and $c_j \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$ is a special constant Then

$$T^*_{j+1} := \begin{cases} T^*_j & \text{if } c_j \text{ does not witness } \sigma \in T^*_{\sigma_0} \\ \text{Insert } \sigma_{i,n}[c_j] \text{ into } T^*_j & \\ \text{ such that the resulting list is ordered} & \text{otherwise} \\ \text{ with respect to the \#-encodings} & \end{cases}$$

Definition B.5.20. We set $T^*_{\sigma_0}(c) := \bigcup T^*_j$

Definition B.5.21. We define the *height* of term constants, denoted height($\bullet$) by cases in the following way:

1. If τ is a closed term, then height(τ) := 0.

2. If $\tau_0, \ldots, \tau_{n-1}$ are term constants and $F \in \mathcal{L}$ is an n-ary function symbol, then

$$\text{height}(F(\tau_0, \ldots, \tau_{n-1})) := \max(\text{height}(\tau_0), \ldots, \text{height}(\tau_{n-1})).$$

3. If $\tau \equiv (n, \tau_0, \ldots, \tau_{n-1}, i)$ is a special constant, define

$$\text{height}(\tau) := 1 + \max(\text{height}(\tau_0), \ldots, \text{height}(\tau_{n-1})).$$

Lemma B.5.22. *The $\mathcal{L}_c$-theory $T^*_{\sigma_0}(c)$ is consistent.*

Proof. **Step 1: $T^*_{\sigma_0}$ is consistent with respect to $\mathcal{L}_c$:**
Suppose for a contradiction that in $\mathcal{L}_c$

$$T^*_{\sigma_0} \vdash \text{False}.$$

Fix a formal $\mathcal{L}_c$-proof $\sigma_1, \ldots, \sigma_n$ of False from $T^*_{\sigma_0}$ for each special constant c fix a unique variable x_c that does not appear in any φ_i. Note that given an instance of a logical axiom L_i with respect to $\mathcal{L}_c$, if we replace each special constant with its x_c we get another instance of L_i. The same is the case for instances of modus ponens and generalization, since special constants never appear after quantifiers. This means that we obtain a formal proof of $\mathcal{L}$-sentences $\sigma'_1, \ldots, \sigma'_n$ of False from $T^*_{\sigma_0}$ by replacing special constants by variables in the way we just described. But then $\neg\mathrm{Con}(T^*_{\sigma_0})$ with respect to $\mathcal{L}$, which contradicts the fact that with respect to the signature $\mathcal{L}$ the extension $T^*_{\sigma_0}$ is maximally consistent and then in particular consistent.
Step 2: $T^*_{\sigma_0}(c)$ is consistent with respect to $\mathcal{L}_c$:
Suppose for a contradiction that $T^*_{\sigma_0}(c)$ is inconsistent. Using the compactness theorem there is a finite list of $\mathcal{L}_c$-sentences in $T^*_{\sigma_0}(c)$ that are inconsistent. In particular we get

$$\neg\mathrm{Con}(T^*_{\sigma_0} + \{\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_m,n_m}[c_{j_m}]\})$$

for suitable distinct $\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_m,n_m}[c_{j_m}]$. Choose these sentences such that $n_1 + \cdots + n_m + m$ is minimal. WLOG

$$\text{height}(c_{j_m}) \geq \text{height}(c_{j_k})$$

for each k. We thus have that c_{j_m} does not appear in c_{j_k} for $k < m$ for otherwise $c_{j_k} < c_{j_m}$ elaborate at some point. Set

$$\Sigma := \{\sigma_{i_1,n_1}[c_{j_1}], \ldots, \sigma_{i_{m-1},n_{m-1}}[c_{j_{m-1}}]\}.$$

Write

$$c_{j_m} \equiv (i_m, \tau_0, \ldots, \tau_{n-1}, n_m),$$

hence $\sigma_{i_m,n_m}[c_{j_m}] \equiv \sigma_{i_m}(x_0/\tau_0, \ldots, x_{n-1}/\tau_{n-1}, x_n/c_{j_m})$. Note that $\exists x_n$ appears in σ_{i_m} since c_{j_m} witnesses σ_{i_m}, hence

$$\sigma_{i_m,n_m-1}(x_0, \ldots, x_{n-1}) \equiv \exists x_n \sigma_{i_m,n_m}(x_0, \ldots, x_n).$$

Define

$$\widetilde{\sigma}(x_n) := \sigma_{i_m,n_m}(x_0/\tau_0, \ldots, x_{n-1}/\tau_{n-1}, x_n)$$

and see that only x_n appears free in this formula. **Claim:**

$$\neg\text{Con}(T^*_{\sigma_0} + \Sigma + \sigma_{i_m,n_m}[c_{j_m}]) \overset{\text{meta}}{\Longrightarrow} \neg\text{Con}(T^*_{\sigma_0} + \Sigma + \exists x_n \widetilde{\sigma}(x_n)).$$

Suppose $T^*_{\sigma_0} + \Sigma + \sigma_{i_m,n_m}[c_{j_m}]$ is not consistent. Then we get

$$T^*_{\sigma_0} + \Sigma + \sigma_{i_m,n_m}[c_{j_m}] \vdash \text{False}, \tag{12}$$

hence by the deduction theorem

$$T^*_{\sigma_0} + \Sigma \vdash \sigma_{i_m,n_m}[c_{j_m}] \to \text{False}.$$

In this proof replace c_{j_m} by x; a variable that does not occur in σ_{i_m,n_m} or in any of the formulae comprising (12). Note that an instance of a logical axiom or an inference rule is still an instance of the same axiom or inference rule after this replacement. Any sentence in $T^*_{\sigma_0}$ remains unchanged upon such a replacement since these contain no special constants. The same is the case for the sentences in Σ since c_{j_m} is distinct from any other c_{j_k}. This means we get

$$T^*_{\sigma_0} + \Sigma \vdash \widetilde{\sigma}(x) \to \text{False}.$$

With generalization we get

$$T^*_{\sigma_0} + \Sigma \vdash \forall x(\widetilde{\sigma}(x) \to \text{False}).$$

From L_{13} we get

$$T^*_{\sigma_0} + \Sigma \vdash \forall x(\widetilde{\sigma}(x) \to \text{False}) \to (\exists x \widetilde{\sigma}(x) \to \text{False}).$$

By modus ponens

$$T^*_{\sigma_0} + \Sigma \vdash (\exists x \widetilde{\sigma}(x) \to \text{False}).$$

Since x_n does not appear in $\widetilde{\sigma}(x) \to \text{False}$ it is a tautology (which will be used without proof) that

$$T^*_{\sigma_0} + \Sigma \vdash (\exists x_n \widetilde{\sigma}(x_{n_m}) \to \text{False}),$$

which means $\neg\text{Con}(T^*_{\sigma_0} + \Sigma + \exists x_{n_m} \widetilde{\sigma}(x_{n_m}))$. Set $i := i_m$. Let $p \leq n$ be the largest integer such that for every $1 \leq l \leq p$, $\forall x_{n-l}$ appears in σ_i. Then σ_i is on the form

$$\sigma_i \equiv \exists\!\!\!/_0 x_0 \ldots \exists\!\!\!/_{n_m-p-2} x_{n_m-p-2} \exists x_{n_m-p-1} \forall x_{n_m-p} \ldots \forall x_{n-1} \exists x_n \sigma_{i_m,n_m}(x_0, \ldots, x_n)$$

Define

$$\widetilde{\sigma}_p :\equiv \sigma_{i_m,n_m-p-1}(x_0/\tau_0, \ldots, x_{n_m-p-1}/\tau_{n_m-p-1})$$

for $p \leq n_m - 1$. When $p = n_m$, $\exists_{n'} \equiv \forall$ for each $n' < n_m$. In this case we set $\widetilde{\sigma}_p :\equiv \sigma_i \in T^*_{\sigma_0}$. When $p < n_m$, $\exists x_{n_m-p-1}$ appears in σ_i.
In the first case,

$$\widetilde{\sigma}_p \equiv \forall x_0 \ldots \forall x_{n_m-1} \exists x_{n_m} \sigma_{i,n_m}(x_0, \ldots, x_{n_m})$$

then since $\tau_0, \ldots, \tau_{n_m-1}$ are variable free, it follows that $T^*_{\sigma_0} \vdash \exists x_{n_m} \widetilde{\sigma}(x_{n_m})$ by repeated use of L_{10}. The claim together with what we assumed towards contradiction shows that $\neg\mathrm{Con}(T^*_{\sigma_0} + \Sigma + \exists x_{n_m}\widetilde{\sigma})$ and then $\neg\mathrm{Con}(T^*_{\sigma} + \Sigma)$, i.e. since we can deduce $\exists x_{n_m}\widetilde{\sigma}(x_{n_m})$, we on only need to add Σ to $T^*_{\sigma_0}$ to get an inconsistent theory. But this contradicts the minimality of $n_1 + \cdots + n_m + m$.
In the second case, since c_{j_m} witnesses σ_i and $\exists x_{n_m-p-1}$ appears in σ_i, we get

$$\tau_{n_m-p-1} \equiv (i, \tau_0, \ldots, \tau_{n_m-p-2}, n_m - p - 1)$$

which also by definition witnesses σ_i. Note

$$\sigma_{i,n_m-p-1}[\tau_{n_m-p-1}] \equiv \sigma_{i,n_m-p-1}(x_0/\tau_0, \ldots, x_{n_m-p-2}/\tau_{n_m-p-1}, x_{n_m-p-1}/\tau_{n_m-p-1}).$$

From $T^*_{\sigma_0}$ and the above formula, we get

$$\forall x_{n_m-p} \ldots \forall x_{n_m-1} \exists x_{n_m} \sigma_{i,n_m-p-1}[\tau_{n_m-p-1}].$$

Using L_{10} we thus get

$$T^*_{\sigma_0} + \sigma_{i,n_m-p-1}[\tau_{n_m-p-1}] \vdash \exists x_{n_m}\widetilde{\sigma}(x_{n_m}) \overset{\text{meta}}{\iff} T^*_{\sigma_0} \vdash \sigma_{i,n_m-p-1}[\tau_{n_m-p-1}] \to \exists x_{n_m}\widetilde{\sigma}(x_{n_m}).$$

So if we derive False from $T^*_{\sigma_0}+\Sigma+\exists$, then we also derive it from $T^*_{\sigma_0}+\Sigma+\sigma_{i,n_m-p-1}[\tau_{n_m-p-1}]$. But

$$n_0 + \cdots + n_{m-1} + n_m - p - 1 + m < n_0 + \cdots + n_m + m,$$

leading to a contradiction.
We thus conclude that $T^*_{\sigma_0}(c)$ is a consistent $\mathcal{L}_c$-theory. □

B.5.5 Gödel's Completeness Theorem (for Countable Signatures)

Lemma B.5.23. *$T^*_{\sigma_0}(c)$ can be extended to a consistent list $\widetilde{T}$ of $\mathcal{L}_c$-sentences such that the additional sentences contain no quantifiers or free variables and such that for each variable free $\mathcal{L}_c$-sentence σ,*

$$\sigma \in \widetilde{T} \text{ or } \neg\sigma \in \widetilde{T}.$$

Proof. We construct the universal list for variable free $\mathcal{L}_c$-sentences of $T^*_{\sigma_0}(c)$ and go through list, setting $\Theta_0 := T^*_\sigma(c)$ and recursively add such a σ to Θ_{n+1} if $\mathrm{Con}(T^*_{\sigma_0}(c) + \Theta_n + \sigma)$. The same argument as in Lindenbaum's lemma shows that this theory, denoted $\widetilde{T}$, is consistent and has the property that

$$\sigma \in \widetilde{T} \vee \neg\sigma \in \widetilde{T}.$$

□

Remark B.5.24. Note that $\widetilde{T}$ is a maximally consistent theory. Indeed if

We now define a $\mathcal{L}_c$-structure M which we will show in the next is a model of $\widetilde{T}$ and therefor also of T and indeed $T + \neg\sigma_0$.

Lemma B.5.25. *Suppose*

$$\sigma \equiv \forall x_k \forall \ldots \forall x_{n-1} \exists x_n \sigma_{i,n}(x_0/\tau_0, \ldots, x_{k-1}/\tau_{k-1}, x_k, \ldots, x_n)$$

is in $T^*_{\sigma_0}(c)$. Let $\tau_k, \ldots, \tau_{n-1}$ be arbitrary term constants. Then for

$$c :\equiv (i, \tau_0, \ldots, \tau_{n-1}, n),$$

we have that $\sigma_i[c] \in T^*_{\sigma_0}(c) \subset \widetilde{T}$.

Proof. Note that the special constant c will witness σ_i. Indeed,

$$\sigma \equiv \sigma_{i,k}[(i, \tau_0, \ldots, \tau_{k-1}, k)],$$

hence $(i, \tau_0, \ldots, \tau_{k-1}, k)$ witnesses σ_i, and this together with the fact that $\exists x_m$ does not appear in σ_i for $k \leq m \leq n-1$ shows that $(i, \tau_0, \ldots, \tau_{n-1}, n)$ witnesses σ_i, hence

$$\sigma_{i,n}(x_0/\tau_0, \ldots, x_{n-1}/\tau_{n-1}, x_n/c)$$

is in $T^*_{\sigma_0}(c) \subset \widetilde{T}$ by construction. □

Lemma B.5.26. *Suppose*

$$\sigma \equiv \forall x_k \forall \ldots \forall x_n \sigma_{i,n}(x_0/\tau_0, \ldots, x_{k-1}/\tau_{k-1}, x_k, \ldots, x_n)$$

is in $T^*_{\sigma_0}(c)$ where $\sigma_{i,n}$ is quantifier free. Let $\tau_k, \ldots, \tau_n$ be arbitrary term constants. Then

$$\sigma' :\equiv \sigma_{i,n}(x_0/\tau_0, \ldots, x_n/\tau_n)$$

is in $\widetilde{T}$.

Proof. By L_{10} we get $T^*_{\sigma_0}(c) \vdash \sigma'$, hence since $\widetilde{T} \vdash \sigma'$, which by maximal consistency of $\widetilde{T}$ shows that $\sigma' \in \widetilde{T}$. □

Remark B.5.27. Consider a $\sigma_{i,n}[c_j] \in T^*_{\sigma_0}(c)$ with $c_j \equiv (i, \tau_0, \ldots, \tau_{n-1}, n)$. Then the next existential quantifier $\exists\!\!\!/_m \equiv \exists$ with $m < n$, then $\tau_m \equiv (i, \tau_0, \ldots, \tau_{m-1})$ witnesses σ_i, hence $\sigma_{i,m}[c_j] \in T^*_{\sigma_0}(c)$ and recall that

$$\sigma_{i,m} \equiv \exists\!\!\!/_{m+1} x_{m+1} \ldots \exists x_n \exists\!\!\!/_{n+1} x_{n+1} \ldots \exists\!\!\!/_N x_N \sigma_{i,N}(x_0, \ldots, x_N).$$

Then using L_{10} a potential bunch of times as we did in a proof above we can derive

$$\sigma_{i,N}(x_0/\tau_0, \ldots, x_k/\tau_k, x_{k+1}, \ldots, x_N)$$

for any $m \leq k \leq N$ and using that $\widetilde{T}$ is maximally consistent, it follows that this sentence is a an element of said theory. Of no such existential quantifier appear we obviously are in a position where we can just apply L_{10} a potential bunch of times to obtain the same conclusion. The upshot is that for each $\sigma \in T^*_{\sigma_0}(c)$, there is an sPNF sentence $\sigma' \in T^*_{\sigma_0}(c)$,

$$\sigma' \equiv \exists\!\!\!/_0 x_0 \ldots \exists\!\!\!/_n x_n \sigma_{i,n}(x_0, \ldots, x_n)$$

such that

$$\sigma \equiv \exists\!\!\!/_{k+1} x_{k+1} \ldots \exists\!\!\!/_n x_n \sigma_{i,n}(x_0/\tau_0, \ldots, x_k/\tau_k, x_{k+1}, \ldots, x_n)$$

for some k and term constants $\tau_0, \ldots, \tau_k$ with $\tau_k \equiv (i, \tau_0, \ldots, \tau_{k-1}, k)$. This fact WILL BE VERY IMPORTANT SOON AND I AM VERY HAPPY THAT I FINALLY KNOW WHY IT IS TRUE!

B.5.6 Construction of a Model of $\widetilde{T}$ & the Proof of Gödel's Completeness Theorem

First we construct the domain of the $\mathcal{L}_c$-structure. Recall that $\Lambda_\tau = [\tau_0, \tau_1, \ldots]$ is the potentially infinite list of term constants with respect to $\mathcal{L}$. Set

$$A_0 := [].$$

If non of the sentences $\tau_n = \tau_0, \ldots, \tau_n = \tau_{n-1}$ are in $\widetilde{T}$ we set

$$A_{n+1} := A_n + [t_n].$$

If this is not the case, for each $0 \leq m \leq n$ for which $\tau_n = \tau_m$ is in $\widetilde{T}$, append τ_n to the list(s) in A_n which contain t_m. The resulting list, A, will be a potentially infinite list of potentially infinite lists of the form

$$A := [[\tau_{n_0}, \ldots], [\tau_{n_1}, \ldots], \ldots].$$

A simple induction argument shows that each term constant τ shows up in at most one list in A. If so call this list $\widetilde{\tau}$. We define a mapping from $\mathcal{L}_c$ to A, and this mapping will comprise the desired $\mathcal{L}_c$-structure which will be a model of $\widetilde{T}$.

- For constants of $\mathcal{L}$ and special constants of $\mathcal{L}_c$, c

$$c^M := \widetilde{c}$$

- For a function symbol F in $\mathcal{L}_c$, we define

$$F^M(\widetilde{\tau_1}, \ldots, \widetilde{\tau_n}) := F(\widetilde{\tau_1, \ldots, \tau_n}) \quad (\widetilde{\tau_1}, \ldots \widetilde{\tau_n} \in A)\,.$$

- For a relation symbol R in $\mathcal{L}_c$, we define

$$(\widetilde{\tau_1}, \ldots, \widetilde{\tau_n}) \in R^M :\overset{\text{meta}}{\iff} R(\tau_1, \ldots, \tau_n) \in \widetilde{T}$$

Remark B.5.28. Note that this mapping is well-defined Check!

Theorem B.5.29. *The $\mathcal{L}_c$-structure M defined above defines a model for $\widetilde{T}$ and since T is a subset of $\widetilde{T}$, when we restrict $\bullet^M$ to $\mathcal{L}$, it follows that this restriction is a model of $T + \neg\sigma_0$.*

Proof. Note that the converse of $M \vDash \sigma \overset{\text{meta}}{\implies} \sigma \in \widetilde{T}$ since if $M \vDash \sigma$ then $\sigma \in \text{Th}(T) \supset \widetilde{T}$.

Consider first the case where σ is variable free. We prove the statement for such σ by induction in the number of logical operators. In the base case $\sigma \equiv \tau = \tau'$ or, in general, $\sigma \equiv R(\tau_1, \ldots, \tau_n)$ for some n-ary relational symbol R and terms $\tau_1, \ldots, \tau_n, \tau, \tau'$. By definition, given any assignment j, for $I = (M, j)$, $\tau = \tau'$ if and only if $I(\tau) = I(\tau')$ and $R(\tau_1, \ldots, \tau_n) \in \widetilde{T}$ if and only if $(\widetilde{\tau_1}, \ldots, \widetilde{\tau_n}) \in R^M$.

Now fix σ to be a non-atomic sentence and assume that for every sentence σ' containing fewer logical operators than σ that $\sigma \in \widetilde{\sigma'} \overset{\text{meta}}{\iff} M \vDash \sigma'$. It is a fact that one can find a sentence equivalent to σ which only uses the logical operators $\wedge, \neg, \exists$. By the soundness theorem we may WLOG assume that any sentence is of this form. Since σ is variable free, we need only consider two case:

$\sigma \equiv \neg\sigma''$**:** By induction $\sigma'' \in \widetilde{T}$ if and only if $M \vDash \sigma''$. By definition we thus have

$$\neg\sigma'' \notin \widetilde{T} \overset{\text{meta}}{\iff} M \nvDash \neg\sigma'',$$

which is equivalent to

$$\sigma \in \widetilde{T} \overset{\text{meta}}{\iff} M \vDash \sigma$$

$\sigma \equiv \sigma_1 \wedge \sigma_2$**:** By induction $\sigma_i \in \widetilde{T} \overset{\text{meta}}{\iff} M \vDash \sigma_i$ for $i = 1, 2$, hence by elementary considerations in the metatheory it follows that

$$\sigma \equiv \sigma_1 \wedge \sigma_2 \in \widetilde{T} \overset{\text{meta}}{\iff} M \vDash \sigma \equiv \sigma_1 \wedge \sigma_2.$$

We now allow σ to contain variables and prove the statement by induction in the number of variables used in σ. We covered the base case above. So assume that σ is not variable free and suppose that $\sigma' \in \widetilde{T} \overset{\text{meta}}{\iff} M \vDash \sigma'$ for every sentence with less variables than σ. By construction every element in $\widetilde{T}$ not in $T^*_{\sigma_0}(c)$ is a variable free $\mathcal{L}_c$-sentence. Then $\sigma \in T^*_{\sigma_0}(c)$. WLOG σ is in sPNF. There is a $\widetilde{\sigma} \in T^*_{\sigma_0}(c)$ in special prenex normal form,

$$\widetilde{\sigma} \equiv \exists_0 x_0 \exists_1 x_1 \dots \exists_n x_n \sigma_{i,n}(x_0, \dots, x_n)$$

such that for some $k \leq n$ and term constants $\tau_0, \dots, \tau_{k-1}$,

$$\sigma \equiv \exists_k x_k \dots \exists_n x_n \sigma_{i,n}(x_0/\tau_0, \dots, x_{k-1}/\tau_{k-1}, x_k, \dots, x_n)$$

by Remark B.5.27.

Case $\exists_k \equiv \exists$**:** Since σ has variables it must be an element of $T^*_{\sigma_0}(c)$. Note that

$$\sigma' \equiv \exists_{k+1} x_{k+1} \dots \exists_n x_n \sigma_{i,n}(x_0/t_0, \dots, x_k/t_k, x_{k+1}, \dots, x_n) \in T^*_{\sigma_0}(c) \subset \widetilde{T}.$$

By induction $M \vDash \sigma'$ and by L_{11}, $\sigma' \vdash \sigma$, hence by the Soundness Theorem $M \vDash \sigma$.

Case $\exists_k \equiv \forall$**:** We split this case into two cases:

Case 1: There is an existential quantifier appearing after $\exists_k$**:**

In this case for some $k < m \leq n$,

$$\sigma \equiv \forall x_k \dots \forall x_{m-1} \exists x_m \sigma_{i,m}(x_0/\tau_0, \dots, x_{k-1}/\tau_{k-1}, x_k, \dots, x_m).$$

Then setting

$$c_j \equiv (i, \tau_0, \dots, \tau_{m-1}, m),$$

where $\tau_k, \dots, \tau_m$ are arbitrary term constants, by Lemma B.5.25,

$$\sigma'' :\equiv \sigma_{i,m}(x_0/\tau_0, \dots, x_{m-1}/\tau_{m-1}, x_m/c_j) \in \widetilde{T}.$$

By induction it follows that $M \vDash \sigma''$. Take an arbitrary assignment j and consider $I = (M, j)$. Note that then

$$I \vDash \sigma_{i,m}(x_0/\tau_0, \dots, x_{m-1}/\tau_{m-1}, x_m/c_j) \overset{\text{meta}}{\iff} I[x_m \mapsto I(c_j)]\sigma_{i,m}(x_0/\tau_0, \dots, x_{m-1}/\tau_{m-1}, x_m)$$
$$\overset{\text{meta}}{\iff} I \vDash \exists x_m \sigma_{i,m}(x_0/\tau_0, \dots, x_{m-1}/\tau_{m-1}, x_m)$$
$$\overset{\text{meta}}{\iff} I[x_k \mapsto \widetilde{\tau_k}] \dots [x_{m-1} \mapsto \widetilde{\tau_{m-1}}] \vDash \exists x_m \sigma_{i,m}(x_0/\tau_0, \dots, x_{k-1}/\tau_{k-1}, x_k, \dots, x_m)$$

Since $\tau_k, \ldots, \tau_{m-1}$ were taken arbitrarily it follows that

$$I \vDash \forall x_k \ldots \forall x_{m-1} \exists x_m \sigma_{i,m}(x_0/\tau_0, \ldots, x_{k-1}/\tau_{k-1}, x_k, \ldots, x_m) \overset{\text{meta}}{\iff} I \vDash \sigma \overset{\text{meta}}{\iff} M \vDash \sigma$$

Case 2: Every quantifier appearing after $\exists^{\not}_k$ is universal:
In this case

$$\sigma \equiv \forall x_k \ldots \forall x_n \sigma_{i,n}(x_k, \ldots, x_n).$$

Then using Lemma B.5.26,

$$\sigma'' \equiv \sigma_{i,n}(x_0/\tau, \ldots, x_n/\tau_n) \in \widetilde{T}$$

for arbitrary term constants $\tau_k, \ldots, \tau_n$. By induction $M \vDash \sigma''$. Following a similar argument to that of case 1 we obtain

$$M \vDash \sigma.$$

So we conclude that $M \vDash \sigma$ for any $\sigma \in \widetilde{T}$. It thus follows that $M \vDash \widetilde{T}$ and hence that $M \vDash T + \sigma_0$. □

Theorem B.5.30. *(Henkin's Existence Theorem). If $\mathcal{L}$ is a countable signature and T is a consistent then it has a model M. Moreover, if σ_0 is a sentence such that $T \not\vdash \sigma_0$, then there is a model of $T + \neg\sigma_0$ and in particular a model of T.*

Corollary B.5.31. *Every theory that can be completed has a model.*

Definition B.5.32. Let T be an $\mathcal{L}$-theory and σ an $\mathcal{L}$-sentence, then T *is a model of* σ, denoted $T \vDash \sigma$, if for every model M of T we have $M \vDash \sigma$

Corollary B.5.33. *(Gödel's Completeness Theorem) Let $\mathcal{L}$ be a countable signature. Consider an $\mathcal{L}$-theory T and an $\mathcal{L}$-sentence σ. Then*

$$T \vDash \sigma \overset{\text{meta}}{\iff} T \vdash \sigma.$$

Proof. $T \vdash \sigma \overset{\text{meta}}{\implies} T \vDash \sigma$ is the soundness theorem. Suppose $T \vDash \sigma$. If T is inconsistent, then $T \vdash \sigma$. So suppose Con(T). Then if $T \not\vdash \sigma$, by Henkin's Existence Theorem we can find a model of $T + \neg\sigma$, i.e.

$$\text{Exists } M(M \vDash T + \neg\sigma)$$

which is equivalent to

$$\text{Exists } M(M \vDash T \text{ and } M \vDash \neg\sigma) \overset{\text{meta}}{\iff} \text{not(Forall } M(M \vDash T \overset{\text{meta}}{\implies} M \vDash \sigma))$$

□

Corollary B.5.34. *Let T be a theory. If every finite subset of T has a model then T has a model.*

Proof. Indeed if every finite subset of T has a model, then every finite subset of T is consistent, hence by the Compactness Theorem T is consistent, so by Henkin's Existence Theorem, T has a model. □

The above corollary is very useful for doing mathematics. Once we have argued for the formal proofs of basic theorems in ZF(C) and when we have a standard model of ZF(C), we have a way to prove mathematical statements without needing to do formal proofs. For instance instead of doing a formal proof of some sentence σ from the axioms of Group Theory (GT), Ring Theory(RT), Topology (TOP, which is an extension of the axioms of ZF(C)), or (almost) whatever set of axioms we desire, we simply prove $\mathtt{GT} \vDash \sigma$, $\mathtt{RT} \vDash \sigma$ or $\mathtt{TOP} \vDash \sigma$. For example in group theory to prove that the neutral element is unique, we can take an arbitrary model of group theory and prove the result in this model.

B.6 The Axioms and Standard Model of Peano Arithmetic

B.6.1 The Axioms

We introduce to the language of first order predicate logic the following symbols, $\mathcal{L}_{\mathrm{PA}} := \{0, \mathrm{s}, +, \cdot\}$ and these symbols make up the language of Peano Arithmetic (PA). The axioms of Peano arithmetic are as follows.

Axiom(s).

$$
\begin{aligned}
&\mathrm{PA}_0\colon \quad \neg\exists x(\mathrm{s}x = 0)\\
&\mathrm{PA}_1\colon \quad \forall x\forall y(\mathrm{s}x = \mathrm{s}y \to x = y)\\
&\mathrm{PA}_2\colon \quad \forall x(x + 0 = x)\\
&\mathrm{PA}_3\colon \quad \forall x\forall y(x + \mathrm{s}y = \mathrm{s}(x + y))\\
&\mathrm{PA}_4\colon \quad \forall x(x \cdot 0 = 0)\\
&\mathrm{PA}_5\colon \quad \forall x\forall y(x \cdot \mathrm{s}y = (x \cdot y) + x)\\
&\mathrm{PA}_6\colon \quad (\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathrm{s}(x)))) \to \forall x\varphi(x)
\end{aligned}
$$

B.6.2 The Standard Model of PA

For the standard model we let the domain be $\mathbb{N}$, i.e. the naive notion of natural numbers we described earlier consisting of **0** and finite strings of the form **s** . . . **s0**.

B.6.3 Gödels Incompleteness Theorem

C Set Theory

C.1 An axiomatization of set theory

C.1.1 Zermelos axioms

We introduce set theory first via the Zermelo-Frankel axioms of set theory with an added axiom of choice. This will be axioms with respect to the signature $\{\in\}$. For a pair of variables z, X we define $z \in X :=\in (z, X)$, hence $\in$ is a binary relational symbol. This is to be read as "z is an element of X" or "z belongs to X". We now introduce the axioms of Zermelos set theory, starting with the first four.

Axiom(s). *0. The axiom of empty set:* $\exists\emptyset\forall z(\neg(z \in \emptyset))$

1. *The axiom of extensionality:* $\forall X\forall Y(\forall z(z \in X \iff z \in Y) \implies X = Y)$.
2. *The axiom of pairing:* $\forall x\forall y\exists P\forall z\,(z \in P \iff (z = x \vee z = y))$.
3. *The axiom of Union:* $\forall X\exists U\forall z(z \in U \iff \exists w \in X(z \in w))$.

To shorten notation we define $z \notin X :\iff \neg(z \in X)$. with the axioms above in mind we make some definitions. Before that we give a few remarks on the axioms given so far

Remark C.1.1. One should think about the set produced from the axiom of the empty set to be a set that contains no element.
The converse statement in the axiom of extensionality is also true, i.e.

$$\forall X\forall Y(X = Y \implies \forall z(z \in X \iff z \in Y)).$$

If $z \in X$ using substitution rule for formulas, $z \in Y$. Conversely by the same argument, if $z \in Y$ we get that $z \in X$.
The empty set is unique: For if $\emptyset$ and $\emptyset'$ satisfy axiom 0, then $\forall z(z \notin \emptyset \wedge z \notin \emptyset')$, meaning $\forall z(z \in \emptyset \iff z \in \emptyset')$, hence by axiom 1, $\emptyset = \emptyset'$.
The set P obtained from the axiom of pairing is uniquely given by x and y: Indeed, if given x and y, and P and P' are sets satisfying $\forall z(z \in P \iff (z = x \vee z = y))$ and $\forall z(z \in P' \iff (z = x \vee z = y))$, then $z \in P \iff z = x \vee z = y \iff z \in P$, hence by axiom 1 $P = P'$.
Given sets x, y we can therefor define $\{x, y\}$ to be the unique set satisfying the pairing

axiom. We also define $\{x\}$ to be the unique set satisfying the pairing axiom for the pair x, x and such a set is called *the singleton containing* x.

The set of pairs is unordered, i.e. $\forall X \forall Y(\{X, Y\} = \{Y, X\})$. This is an immediate consequence of $\vee$ being commutative and the axiom of extensionality. We will return to the question of defining ordered pairs later on.

On the axiom of union: Loosely, it states that given a set, we may define the union over this set. The axiom of extensionality again implies that this is uniquely determined by the first set in the first universal quantifier presented in formula, i.e. there is a unique set, $\bigcup X$, satisfying $\forall z(z \in \bigcup X \iff \exists w \in X(z \in w))$. We may then form a binary functional symbol $\cup$, *the union of* X *and* Y, defined by

$$\cup(X, Y) := X \cup Y := \bigcup\{X, Y\}$$

Definition C.1.2. We introduce yet another binary relation symbol $\subset$, defined by

$$\subset (X, Y) :\equiv X \subset Y : \iff \forall z(z \in X \implies z \in Y),$$

i.e. if X and Y are sets we get that $X \subset Y$ to be read X *is a subset of* Y if every element of X is also an element of Y.

We introduce a another binary relation $\subsetneq$,

$$X \subsetneq Y : \iff X \subset Y \wedge \neg(X = Y)$$

We then say that X *is a proper subset of* Y

Remark C.1.3. Using the axiom of extensionality we get that if $X \subsetneq Y$ then $\neg(\forall z(z \in X \iff z \in Y))$, hence $\exists z((z \in X \wedge z \notin Y) \vee (z \in Y \wedge z \notin X))$, and since $X \subset Y$, we get $\exists z(z \in Y \wedge z \notin X)$. Note that $\emptyset \subset X$, since proving $\forall z(z \in \emptyset \implies z \in X)$ boils down to the fact that $z \in \emptyset$ definitionally is equal to False.

Definition C.1.4. A set X is called *inductive* if

$$\forall y(y \in X \implies (y \cup \{y\}) \in X).$$

We define

$$\text{ind}(X) : \iff y(y \in X \implies (y \cup \{y\}) \in X)$$

Example C.1.5. Note that $\text{ind}(\emptyset)$ by ex falso.

To get more inductive sets we need the *axiom of infinity*

Axiom(s).

$$\exists I(\emptyset \in I \wedge \text{ind}(I))$$

Remark C.1.6. One notes that for such an I, $\{\emptyset\} = \emptyset \cup \{\emptyset\} \in I$ and by the same argument $\{\emptyset, \{\emptyset\}\} \in I$ so from this we get a way to recursively construct non-empty sets.

The following is an axiom schema called the axiom schema of seperation

Axiom(s). *For each formula $\varphi(z, p_1, \ldots, p_n)$ where all the free variables of φ are among $z, p_1, \ldots, p_n$, we postulate*

$$\forall X \forall p_1 \ldots \forall p_n \exists Y \forall z (z \in Y \iff (z \in X \wedge \varphi(z, p_1, \ldots, p_n)))$$

The above axioms lets us form new sets by imposing some conditions on the elements of the set using a formula, i.e. we can in many instances construct sets of the form $\{z \in X : \varphi(z)\}$, i.e. given a set X and the sufficient data to write a term of the formula φ we can find a set Y such that $z \in Y \iff (z \in X \wedge \varphi(z, _))$. By the usual argument, once the sufficient data is provided to this axiom schema, the set Y which it produces is uniquely given by this data, so we may define

$$\{z \in X : \varphi(z, _)\}$$

as the unique set satisfying the axiom of seperation with respect to φ. Note that $\{z \in X : \varphi(z)\} \subset X$.

Example C.1.7.

1. Consider sets X, Y and consider the formula $\varphi(z, S) \equiv z \in S$. We then define
$$X \cap Y := \{z \in Y : \varphi(z, X)\}.$$
Note that $z \in X \cap Y \iff (z \in X \wedge z \in Y)$, which shows that given arbitrary sets X and Y
$$\{z : z \in X \wedge z \in Y\}$$
can be uniquely defined. We call this set *the intersection of X and Y*, hence we obtain a binary function symbol $\cap$ taking a pair of sets to their intersection.

2. A more general construction than the one above, is to consider $\varphi(z, S) \equiv \forall T \in S(z \in T)$. Than given a set X and applying the axiom schema of seperation to $\bigcup X$ and the formula $\varphi(z, X)$ we get the set
$$\bigcap X := \left\{z \in \bigcup X : \forall Y \in X (z \in Y)\right\}$$
which is a subset of $\bigcup X$. In this way we obtain a unary function symbol $\bigcap$ taking X to the intersection of the elements in X. Note that $X \cap Y = \bigcap \{X, Y\} = \bigcap X \cup Y$.

3. We can also define *set difference* from this axiom schema. Consider sets X and Y and a formula $\varphi(z, Y) \equiv z \notin Y$. Then we can form

$$\{z \in X : z \notin Y\}.$$

The next axiom lets us form the the set of all subsets of any set.

Axiom(s).

$$\forall X \exists PS \forall z (z \in PS \iff z \subset X)$$

By the axiom of extensionality we get that for each X there is a unique PS satisfying $\forall z(z \in PS \iff z \subset X)$. We from now on denote this set 2^X or $\mathcal{P}(X)$. At this point, we have enough axioms to build a lot of mathematics. We could at this point for instant diverge to construct a set of ordinal numbers ω, that assuming the consistency of Z, in a model of Z will let us construct a model of PA. We may also define cartesian products of a pair of sets from which, we could define relations such as orders, equivalence relations and functions and from thereon, we could define functions on $\mathbb{N}$ satisfying the Peano Axioms. Before doing this, however, we will opt to introduce two more axioms.

C.1.2 ZF Set Theory

The axioms of Zermelo set theory with the addition of the following two added axioms is called *Zermelo-Fraenkel set theory*. The first added axiom is in fact an axiom schema called *the axiom schema of replacement*. Before stating the axiom we introduce the notion of class function

Definition C.1.8. Consider a first order formula $\varphi(X, Y, p)$ where X, Y are free variables and p a tuple of variables of φ such that

$$\forall X \exists! Y (\varphi(X, Y, p))$$

We define a unary function symbol F given by

$$F(X) = Y :\iff \varphi(X, Y, p)$$

Such a unary function symbol is called the *class function for* φ.

Axiom(s). *Consider a first order formula $\varphi(x, y, p)$ where x, y and p a tuple of parameters are free variables of φ. We then have*

$$\forall A \forall p (\forall x \in A \exists! y \varphi(x, y, p) \implies \exists B \forall x \in A \exists y \in B \varphi(x, y, p))$$

Remark C.1.9. For a set A and a class function F, we can then find a set B containing the elements of the form $F(x)$ where $x \in A$. We may then consider the set

$$F[A] := \{y \in B : \exists x \in A(y = F(x))\} = \{F(x) : x \in A\}.$$

The axiom was introduced by Fraenkel and Skolem, who independently of each other in 1922 discovered that in Zermelo's axioms it is not possible to prove the existence of the set $\{\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \dots\}$. With this axiom the class function $X \mapsto \mathcal{P}(X)$ gives rise to such a set.

The second axiom is called *the axiom of foundation*

Axiom(s).

$$\forall X \left(\exists z(z \in X) \implies \exists y \in X(y \cap X = \emptyset)\right).$$

This axiom was introduced in 1925 by John von Neumann in order to define ordinal numbers.

Example C.1.10. Suppose for a contradiction that there is a sequence of sets satisfying $x_1 \ni x_2 \ni x_3 \ni \dots$. Consider the set containing these sets $X = \{x_1, x_2, x_3, \dots\}$ (implicitly we are here assuming the existence of $\mathbb{N}$ and use the axiom schema of replacement with the formula $\varphi \equiv T$). Then take an element x_k of X. Note $x_{k+1} \in x_k$ and $x_{k+1} \in X$, hence $x_k \cap X \neq \emptyset$, which in contradiction with the axiom of foundation. We can therefor conclude that there is no set x with $x \in x$, since this would imply the existence of a descending sequence $x \ni x \ni \dots$, which shows that we cannot run into Russels paradox in Zermelo-Fraenkel set theory. Nor can we have a sequence of sets $x_1, \dots, x_n$ satisfying

$$x_1 \in x_2 \in x_3 \in \cdots \in x_n \in x_1$$

for then we would get a sequence $x_1 \ni x_n \ni \cdots \ni x_2 \ni x_1 \ni \dots$. One can also prove that $\forall x(x \notin x)$ simply using the axiom of foundation on the set $\{x\}$. Indeed, there is an element $z \in \{x\}$, so there must be a $y \in \{x\}$ such that $y \cap \{x\} = \emptyset$, which implies that for any $w \in \{x\}$, $w \notin y$. Now note that $y = w = x$, which means $x \notin x$.

C.2 Relations

Definition C.2.1. Consider sets x, y. We define the ordered pair of x, y to be

$$(x, y) := \{\{x\}, \{x, y\}\}.$$

Remark C.2.2. $(x, y) = (x', y') \iff (x = x' \wedge y = y')$. Indeed, if $(x, y) = (x', y')$, then

$$\{x\} = \{x'\} \wedge \{x, y\} = \{x', y'\} \vee \{x\} = \{x', y'\} \wedge \{x, y\} = \{x'\}$$

In the first case we get that $x = x'$ and that $x = x' \wedge y = y' \vee x = y' \wedge y = x'$. In either case it easily follows that $x = x'$ and $y = y'$. In the second case $x = x' \wedge x = y'$ and $x' = x \wedge x' = y$, so trivially $x = x'$ and $y = y'$.

Definition C.2.3. Consider sets X and Y. The axiom schema of seperation, the axiom of powerset and the axiom of union lets us form the set

$$X \times Y := \{(x, y) \in \mathcal{P}\left(\mathcal{P}(X \cup Y)\right) : x \in X \wedge y \in Y\}.$$

Given sets $X_1, \ldots, X_n$ we recursively define $X_1 \times \cdots \times X_n$ to be

$$(X_1 \times \cdots \times X_{n-1}) \times X_n.$$

We denote an element of this set $((x_1, \ldots, x_{n-1}), x_n)$ by $(x_1, \ldots, x_n)$

Remark C.2.4. Using induction on the number of sets we can show that $(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$ if and only if $x_i = x'_i$ for every $i \in \{1, \ldots, n\}$.

Definition C.2.5. An *n-ary relation* on n sets $X_1, \ldots, X_n$ is a subset $R \subset X_1 \times \cdots \times X_n$. For an n-tuple $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$.
A 2-ary relation R on sets X, Y is called a *binary relation*. We define $xRy :\equiv (x, y) \in R$. If R is a relation on $S \times S$ for some set S, we call it a *(binary) relation on S*.

In the following we will investigate different notions of (binary) relations between elements in sets.

C.2.1 Functions

Definition C.2.6. We define a *(set theoretical) function* f from a set X to a set Y, denoted $f : X \to Y$, to be a relation $f \subset X \times Y$ satisfying

$$\forall x \in X \exists! y \in B((x, y) \in f).$$

For an $x \in A$ we take $f(x)$ to be the unique element in Y satisfying $(x, f(x)) \in f$. With this notation if $x = x'$ then $f(x) = f(x')$. For a subset $Z \subset X$, the axiom schema of seperation lets us form the the image of Z under f, which we define as

$$f(Z) := \{f(x) : x \in Z\}.$$

We can also define the *restriction of f to Z* as the set

$$f|_Z := \{(x, f(x)) \in f : x \in Z\}$$

We denote the set of function between X and Y by ${}^XY$, i.e.

$${}^XY := \{f \subset X \times Y : \forall x \in X \exists! y \in Y((x, y) \in f)\}$$

Definition C.2.7. A function $f : X \to Y$ is called *surjective* or *onto* if

$$\forall y \in Y \exists x \in X(f(x) = y).$$

The short hand notation for a function being surjective is $f : X \twoheadrightarrow Y$.

Definition C.2.8. A function $f : X \to Y$ is called *injective* or *one-to-one* if

$$\forall x \forall x'(f(x) = f(x') \implies x = x').$$

The short hand notation for f being injective is $f : X \hookrightarrow Y$.

Definition C.2.9. If a function $f : X \to Y$ is both surjective and injective, then it is called *bijective*. The short hand notation for f being bijective is $f : X \xrightarrow{\sim} Y$.

Remark C.2.10. Note that if f is bijective, then

$$\forall y \in Y \exists! x \in X((x, y) \in f),$$

or more succinctly for each $y \in Y$ we may find a unique $x \in X$ satisfying $y = f(x)$. Indeed given a $y \in Y$, there is an $x \in X$ such that $(x, y) \in f$ (in other words $y = f(x)$) since f is surjective. If x' is another element in X satisfying $(x', y) \in f$. Then $f(x) = y = f(x')$, hence $x = x'$ since f is injective. This lets us form the *the inverse of f*, which we take to be the set

$$f^{-1} := \{(y, x) \in Y \times X : (x, y) \in f\}.$$

This is indeed a function, for if $y \in Y$ then there is a unique $x \in X$ such that $(x, y) \in f$, meaning x is unique such that $(y, x) \in f^{-1}$.

We can also define a way to compose certain compatible functions to form new functions.

Definition C.2.11. Let $f : X \to Y$, $g : Y \to Z$ be arbitrary functions. Then we define a set

$$g \circ f := \{(x, z) \in X \times Z : \exists y \in Y((x, y) \in f \wedge (y, z) \in g)\}.$$

Lemma C.2.12. *Let $f : X \to Y$, $g : Y \to Z$ be arbitrary functions. $g \circ f$ is a function from X to Z and for each $x \in X$, $(g \circ f)(x) = g(f(x))$.*

Proof. Let $x \in X$. Then setting $y := f(x)$, $z := g(f(x))$, $(x, y) \in f$ and $(y, z) \in g$, so $(x, z) \in g \circ f$. If z' is such that $(x, z') \in g \circ f$, then there is a $y' \in Y$ such that $(x, y') \in f$ and $(y', z') \in g$. Since f is a function, $y' = f(x)$ and since g is a function $z' = g(y') = g(f(x)) = z$. □

To get away from explicitly constructing functions as sets we prove the following lemma

Lemma C.2.13. *Let $f, g : X \to Y$ be functions. $f = g$ iff $f(x) = g(x)$ for every $x \in X$.*

Proof. " $\implies$ ": Suppose $f = g$. Let $x \in X$. Then $(x, f(x)) \in X$ by the converse statement of the axiom of extensionality $(x, f(x)) \in g$. Then since g is a function $f(x) = g(x)$.
" $\impliedby$ ": Supppose $f(x) = g(x)$ for every $x \in X$. Let $(x, y) \in f$ be given. Then $(x, y) = (x, f(x)) = (x, g(x)) \in g$, so $f \subset g$. By symmetry $g \subset f$. Then $\forall z(z \in f \iff z \in g)$, so the axiom of extensionality tells us that $f = g$. □

We may then construct a function $f : X \to Y$ only by specifying the image under each element in the *domain* X.

$$f : X \to Y$$
$$x \mapsto f(x)$$

where $f(x)$ is some set. Explictly this postulates that

$$\{(x, y) \in X \times Y : y = f(x)\},$$

so one has to check that $f(x) \in Y$ i.e. check that $f(x)$ is an element of the *codomain* (this to check that it forms a set that is a subset of $X \times Y$), and that it is in particular a function (so one has to check that if $x = x'$ then $f(x) = f(x')$).

Example C.2.14. Here are some simple examples of functions:

1. Let X and Y be sets and fix a $y \in Y$. Consider the function

$$f : X \to Y$$
$$x \mapsto y$$

I.e. the set $f = \{(x, a) \in X \times Y : a = y\}$. Indeed, $f(x) = y \in Y$ for every $x \in X$. If $x = x'$, then $f(x) = y = f(x')$. So f is a function.

2. Let X be a set. Then

$$\begin{aligned} \mathrm{id}_X : X &\to X \\ x &\mapsto x \end{aligned}$$

i.e. the set $\{(x, y) \in X \times X : y = x\}$ defines a function. Indeed, if $(x, y) \in \mathrm{id}_X$, then $(x, y) = (x, x) \in X \times X$. Suppose $x = x'$, then $\mathrm{id}_X(x) = x = x' = \mathrm{id}_X(x')$, so id_X is a function.

Lemma C.2.15. *Let $f : X \to Y$ be a function. Then f is bijective if and only if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If so g is unique.*

Proof. " $\Longrightarrow$:" Set $g := f^{-1}$. Let $x \in X$. Since $(f(x), x) \in f^{-1}$, we get that $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x = \mathrm{id}_X(x)$. So $f^{-1} \circ f = \mathrm{id}_X$. By symmetry $f^{-1} \circ f = \mathrm{id}_Y$.
" $\Longleftarrow$ ": Let $y \in Y$ and set $x = g(y)$. Then

$$f(x) = f(g(y)) = (f \circ g)(y) = y.$$

Suppose $x, x' \in X$ are given such that $f(x) = f(x')$. Then

$$x = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = x'.$$

Then f is both surjective and injective, so f is bijective.
Suppose g is a function satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Let $y \in Y$, then $y = f(x)$ for some x, then $g(y) = g(f(x)) = x = f^{-1}(f(x)) = f^{-1}(y)$, so $g = f^{-1}$. □

Definition C.2.16. For a function $f : X \to Y$ we call a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ the *mutual inverse of f (with respect to composition)*

Remark C.2.17. We have just seen that a function has a mutual inverse if and only if it is bijective and that this inverse is equal to f^{-1}.

C.2.2 Products of Sets

Definition C.2.18. Consider a set I and a set X such that there is a surjective function

$$\begin{aligned} I &\to X \\ i &\mapsto x_i \end{aligned}$$

We call X a *family of sets indexed by I* and it is denoted $\{x_i\} := \{x_i\}_{i \in I}$. We introduce the notation

$$\bigcup_{i \in I} x_i := \bigcup \{x_i\} = \bigcup X$$

and

$$\bigcap_{i\in I} x_i := \bigcap \{x_i\} = \bigcap X.$$

Note that given a class function F and a set I, we get a well-defined function $I \to F[I], i \mapsto F(i)$ using the axiom schema of replacement. So given a class function we can think about $F[I]$ as a family of sets indexed by I.

Definition C.2.19. Let a set I and a family of sets index by I, $\{X_i\}$ be given such that $X_i \neq \emptyset$ for each $i \in I$. We then define *the cartesian/direct product of* $\{X_i\}$ to be the set

$$\prod_{i\in I} X_i := \left\{ f \in {}^{I}\bigcup_{i\in I} X_i : \forall i \in I(f(i) \in X_i) \right\}.$$

With this definition presupposing the construction of the ordinals, we may construct a n-ary relation on an n-fold product of sets.

Definition C.2.20. Let $n \in \omega$ (where ω is the first infinite ordinal). We then define an *n-ary relation* on sets $X_1, \ldots, X_n$ to be a subset $R \subset X_1 \times X_2 \times \cdots \times X_n$ where $X_1 \times X_2 \times \cdots \times X_n := \prod_{i\in n} X_i$.

C.2.3 ZFC: Adding the Axiom of Choice

The axiom of choice is a statement that seems obvious: If we have a set X that does not contain the empty set, then we can pick an element from each $x \in X$. This of course is not a precise statement, since there is no precise notion of "picking" an element that can be derived from the axioms of ZF in any case where we wish to do dish. To be precise, what we mean to say is that we may find a function, $f : X \to \bigcup X$ satisfying $\forall x \in X(f(x) \in x)$ and this statement if one thinks about satisfies our notion of picking an element from each $x \in X$. This is independent of ZF as proven by Paul Cohen in 1964. As it turns out some rather paradoxical things can happen when the axiom of choice (AC) is assumed, but it is in some sense a sufficient and nescessary statement for the study of something as "simple" as for instance finite dimensional vector spaces. In this example we need it to be able to be able to "pick" a basis for a finite dimensional vector space, which is an action the legality of which seems self-evident. Nonetheless it is equivalent to AC. If we take the cartesian product over any set that does contain the empty set (however you wish to interpret this), we run into the problem of proving that this is non-empty being equivalent to AC. So it is certainly nescessary in some cases. However, it allows for a solid ball in $\mathbb{R}^3$ being equidecomposable to two copies of itself. In other words there is way of constructing 2 solid balls in 3 dimensional euclidean space by simply gluing together pieces of 1 such ball in such a way that no stretching is involved.

Axiom(s). *(AC)*

$$\forall X \left(\emptyset \notin X \implies \exists f \left(f \in {}^{X}\bigcup X \wedge \forall x \in X (f(x) \in x) \right) \right).$$

The f in this axiom is called a choice function. We now construct the cartesian product over a set X and show that assuming AC we

Theorem C.2.21. *The following are equivalent*

1. *AC*
2. *For every $X = \{X_i\}_{i \in I}$ where $X_i \neq \emptyset$ for every $i \in I$, $\prod_{i \in I} X_i \neq \emptyset$.*

Proof. Assume AC. Let f be a choice function on X. By definition of the cartesian product, $f \circ (i \mapsto X_i) \in \prod_{i \in I} X_i$. Assume 2. Let X that does not have $\emptyset$ as a member. We may view X as a family of sets indexed by X using id_X. Then using the assumption there is a function $f : X \to \bigcup_{x \in X} x = \bigcup X$ such that $f(x) \in x$ for every $x \in X$, meaning f is a choice function. □

C.2.4 Orderings

Definition C.2.22. Let X be a non-empty set. A *Partial order* on X is a relation $\leq$ on X satisfying,

1. reflexivity,
$$x \sim x \text{ for every } x \in X$$
2. antisymmetry,
$$x \leq y \text{ and } y \leq x \implies x = y \text{ for every } x, y \in X$$
3. transitivity,
$$x \leq y \text{ and } y \leq z \implies x \leq z \text{ for every } x, y, z \in X.$$

Remark C.2.23. Given a partial order $\leq$, we have that $\geq$ is a partial order as well.

Example C.2.24. Let X be a set and $\mathcal{X} \subset 2^X$. Then $\subset := \{(A, B) \in \mathcal{X} \times \mathcal{X} : \forall x (x \in A \implies x \in B)\}$ defines a partial order on $\mathcal{X}$.
Another example is that of $\leq$ on $\mathbb{N}$,$\mathbb{Z}$ or $\mathbb{Q}$.

Definition C.2.25. Let X be a set with a partial order and $\{x_i\}_{i\in\mathbb{N}} \subset X$ be a sequence. We say that $\{x_i\}_{i\in\mathbb{N}}$ is *descending with respect to* $\leq$ if $x_i \geq x_{i+1}$ for every $i \geq 0$ and *ascending with respect to* $\leq$ if $x_i \leq x_{i+1}$ for every $i \geq 0$. A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to *stabilize* if there is a non-negative integer n such that $x_n = x_{n+d}$ for every $d \geq 0$.

Definition C.2.26. A partial order $\leq$ on a non-empty set X is called *total order* if for every $x, y \in X$, $x \leq y$ or $y \leq x$.

Definition C.2.27. Let X be a set with a partial order $\leq$. A subset Y of X is called a *chain* if $\leq$ defines a total order on Y.

Remark C.2.28. Any ascending/descending sequence $\{x_i\}_{i\in\mathbb{N}}$ is a chain and is denoted

$$x_1 \leq x_2 \leq \dots \text{ respectively } x_1 \geq x_2 \geq \dots,$$

these are called *ascending/descending chains*

We give the following axiom which one check is equivalent to the axiom of choice

Axiom(s). *(Zorn's Lemma) Let $X \neq \emptyset$ be a set with a partial order $\leq$ such that for every chain $C \subset X$ there exists an $x \in C$ such that $c \leq x$ for every $c \in C$, (i.e. there is an upper bound x for C in C). Then there is a maximal element in $m \in X$, i.e. for every $y \in X$ if $m \leq y$, then $m = y$.*

Theorem C.2.29. *In ZF Zorn's Lemma is equivalent to AC.*

Proof. Do at some point □

Example C.2.30. In certain situations we do not need to assume Zorn's Lemma.

1. Suppose X is a non-empty finite set with n elements and a partial order $\leq$. Then X has a maximal element. Indeed, this is easily proven by induction in n. If X has one element this is trivially maximal. Consider for $n \geq 1$ $X = \{x_1, \dots, x_{n+1}\}$. Then by induction $\{x_1, \dots, x_n\}$ has a maximal element x_i. Then $\max_{\leq}(x_i, x_{n+1})$ is a maximal element of X.

2. A topology τ on some set X has X as a maximal element

We give a reformulation of every chain having a maximal/minimal element

Lemma C.2.31. *Let $X \neq \emptyset$ be a set with a partial ordering $\leq$. Every ascending/descending sequence in X stabilizes if and only if every chain C in X has a upper/lower bound in C.*

Proof. We only check the ascending case since a descending sequence is just an ascending sequence with respect to $>$ and a minimal element is just a maximal element with respect to $>$.

" $\implies$ ":We prove the contrapositive. Suppose $C \subset X$ is a chain that does not have an upper bound in C. Let $c_1 \in C$. Then there exists $c_2 \in C$ such that $c_1 < c_2$. Continuing this process recursively we get a sequence $\{c_i\}_{i\in\mathbb{N}}$ such that $c_i < c_{i+1}$ for every $i \geq 0$, hence this is a sequence in X that does not stabilize.

" $\impliedby$ ": Let $\{x_i\}_{i\in\mathbb{N}}$ be an ascending sequence in X. Then $\{x_i\}_{i\in\mathbb{N}}$ is a chain. Then there exists a $x_n \in \{x_i\}_{i\in\mathbb{N}}$ such that $x_j \leq x_n$ for every $j \geq 1$. Now since $x_n \leq x_{n+d}$ for every $d \geq 0$ it follows that $x_n = x_{n+d}$ for every $d \geq 0$, hence $\{x_i\}_{i\in\mathbb{N}}$ stabilizes. □

Definition C.2.32. A total order $\leq$ on a set X is called a *well-ordering* if every non-empty subset $Y \subset X$ has an element $y_0 \in Y$ such that for every $y \in Y$, $y_0 \leq y$. If X is a set that can be well ordered by a total order $\leq$, we write $\text{wo}_{\leq}(X)$.

Definition C.2.33. *The well-ordering principle* refers to the statement

$$\forall X(X \neq \emptyset \implies \exists \leq \subset X \times X \ \text{wo}_{\leq}(X))$$

We will later see that the well-ordering principle is in fact equivalent to the axiom of choice.

C.2.5 Equivalence Relations

Definition C.2.34. Let X be a non-empty set. We define an *equivalence relation* to be a relation $\sim$ on X satisfying

1. reflexivity,
$$x \sim x \text{ for every } x \in X$$
2. symmetry,
$$x \sim y \implies y \sim x \text{ for every } x, y \in X$$
3. transitivity
$$x \sim y \ \wedge y \sim z \implies x \sim z \text{ for every } x, y, z \in X.$$

Here we define $x \sim y$ to mean $(x, y) \in \sim$. For an $x \in X$ we define the *equivalence class under $\sim$ represented by x* to be the set

$$[x]_{\sim} := \{y \in X : y \sim x\}.$$

We denote the set of equivalence classes under $\sim$ by $X/\sim$.

Lemma C.2.35. *Let X be a non-empty set and $\sim$ an equivalence relation on X. Let $x, y \in X$. Then*

$$[x]_\sim = [y]_\sim \iff x \sim y$$

Proof. "$\Longleftarrow$": Let $z \in [x]_\sim$, then $z \sim x$ and $z \sim y$, since also $z \in [y]_\sim$. Then $x \sim z$ (by symmetry) and $z \sim y$, implying $x \sim y$ by transitivity.
" $\Longleftarrow$ ": If $x \sim y$, then $x \in [y]_\sim$. By symmetry $y \sim x$, hence $y \in [x]_\sim$. □

Lemma C.2.36. *Let X be a non-empty set and $\sim$ an equivalence relation on X. The function*

$$\pi : X \to X/\sim$$
$$x \mapsto [x]_\sim$$

is a well-defined surjective function.

Proof. Suppose $x = y$, then $x \sim y$, hence by Lemma C.2.35 $p(x) = [x]_\sim = [y]_\sim = p(y)$. Let $[x]_\sim \in X/\sim$. Then $\pi(x) = [x]_\sim$, hence π is surjective. □

Definition C.2.37. Let X be a set. P a predicate. We say that $P(x)$ *is true for all but finitely many* $x \in X$, if there exists a $Y \subset X$, such that $P(x)$ is true for all $x \in X \setminus y$

C.3 ordinal theory

C.3.1 The Set ω

Definition C.3.1. Let I_0 be an inductive non-empty set. Consider the formula $\varphi(X) \equiv \emptyset \in X \wedge \text{ind}(X)$. Using power sets and seperation we get the set

$$\omega := \bigcap \{X \in \mathcal{P}(I_0) : \emptyset \in X \wedge \text{ind}(X)\}$$

Lemma C.3.2. *ω is the smallest with the property of being inductive and containing $\emptyset$, i.e. it is a subset of every other set with these properties. Immediately it is the unique smallest set with the property of being inductive and containing $\emptyset$.*

Proof. The emptyset is an element of each $X \in \mathcal{P}(I_0)$ satisfying $\emptyset \in X$ and $\text{ind}(X)$, so $\emptyset \in \omega$. If $x \in \omega$, then for every $X \in \mathcal{P}(I_0)$ with $\emptyset \in X$ and $\text{ind}(X)$, $x \cup \{x\} \in X$, hence $x \cup \{x\} \in \omega$, implying that ω is inductive. Let I be any inductive set with $\emptyset \in I$. Consider $X_I := \omega \cap I \subset \omega \subset I_0$. Note that clearly $X_I \in \mathcal{P}(I_0)$, $\emptyset \in X_I$ and $\text{ind}(X_I)$. Then $\omega \subset X_I$. This means $\omega = \omega \cap I \subset I$. So ω is contained in every inductive set containing the emptyset. □

We thus get a kind of induction principle on ω:

$$\forall A(\text{ind}(A) \wedge A \subset \omega \implies A = \omega).$$

So if we are given a formula φ, to prove $\forall x \in \omega \varphi(x)$, it is sufficient to prove $\varphi(\emptyset) \wedge \forall x(\varphi(x) \implies \varphi(x \cup \{x\}))$, since then $A := \{x \in \omega : \varphi(x)\}$, satisfies $\text{ind}(A) \wedge A \subset \omega$.

C.3.2 ordinal numbers

Definition C.3.3. Let $z \in X$. Then z is $\in$-*minimal in* X if $\forall y(y \in z \implies y \notin X)$. We denote this $\min_{\in}(z, X)$.

Definition C.3.4. A set X is *ordered by* $\in$ if

$$\forall y_1, y_2 \in X(y_1 \in y_2 \vee y_2 \in y_1 \vee y_1 = y_2).$$

We denote this formula by $\text{ord}_{\in}(X)$.

Definition C.3.5. A set X is *well-ordered by* $\in$ if

$$\text{ord}_{\in}(X) \wedge \forall Y \in \mathcal{P}(X)(Y \neq \emptyset \implies \exists z \in Y \min_{\in}(z, Y)).$$

We denote this formula by $\text{wo}_{\in}(X)$.

Definition C.3.6. A set X is *transitive* if

$$\forall y(y \in X \implies y \subset X).$$

This formula is denoted by $\text{trans}(X)$.

Remark C.3.7. If $z \in y \in x$ and $\text{trans}(x)$, then $z \in y \subset x$, hence $z \in x$.

Definition C.3.8. An *ordinal number* is a set α such that

$$\text{trans}(\alpha) \wedge \text{wo}_{\in}(\alpha).$$

We denote this formula by $\text{ordinal}(\alpha)$.

Remark C.3.9. Informally, we define Ω to be the collection of every ordinal number. By $\alpha \in \Omega$ we mean $\text{ordinal}(\alpha)$.

Theorem C.3.10. *1. If $\alpha \in \Omega$, then $\alpha = \emptyset$ or $\emptyset$ xor $\emptyset \in \alpha$.*

2. Without assuming the axiom of foundation, if $\alpha \in \Omega$, then $\alpha \notin \alpha$.

3. *If* $\alpha, \beta \in \Omega$*, then* $\alpha \in \beta$ xor $\beta \in \alpha$ xor $\alpha = \beta$.

4. *If* $\alpha \in \beta \in \Omega$*, then* $\alpha \in \Omega$.

5. *If* $\alpha \in \Omega$*, then* $\alpha \cup \{\alpha\} \in \Omega$.

6. Ω *is transitive and well-ordered by* $\in$*, i.e the collection is transitive and ordered by* $\in$*, and every non-empty collection in* Ω *has an* $\in$*-minimal element.*

Proof. 1.α is well-ordered by $\in$. If $\alpha = \emptyset$, we are done. So if $\alpha \neq \emptyset$, then since $\alpha \in \mathcal{P}(\alpha)$ there is a $z \in \alpha$ such that $\min_{\in}(\alpha, z)$. Consider an element of α, $x \neq \emptyset$. Pick $y \in x$. By transitivity, $x \subset \alpha$, so $y \in \alpha$. Then $y \in x \wedge y \in \alpha$, hence $\neg \min_{\in}(x, \alpha)$. Then $\emptyset = z \in \alpha$.
2. Suppose for a contradiction that $\alpha \in \alpha$. Then $\{\alpha\} \in \mathcal{P}(\alpha)$. There is then a $z \in \{\alpha\}$ that is $\in$-minimal for $\{\alpha\}$. We must have $z = \alpha$. So $\min_{\in}(\alpha, \{\alpha\})$. But then for $\alpha \in \alpha$, $\alpha \notin \{\alpha\}$ leading to a contradiction.
3. By 2. the three cases a mutually exclusive. Since then we cannot have $\alpha \in \beta \wedge \beta \in \alpha$, since then by transitivity $\alpha \in \alpha$. Similary if $\alpha \in \beta \wedge \alpha = \beta$ or $\beta \in \alpha \wedge \alpha = \beta$, we would get $\alpha \in \alpha$. Suppose $\alpha \neq \beta$. Then WLOG $x \in \alpha \wedge x \notin \beta$, so $\alpha \setminus \beta \in \mathcal{P}(\alpha) \setminus \emptyset$.
We claim that $\alpha \cap \beta$ **is the** $\in$**-minimal element of** $\alpha \setminus \beta$**:** Let γ be an $\in$-minimal element of $\alpha \setminus \beta$. Using trans(α) and $\gamma \in \alpha$, we get that

$$\forall u(u \in \gamma \implies u \in \alpha) \quad (*)$$

And using $\min_{\in}(\gamma, \alpha \setminus \beta)$, $\forall u(u \in \gamma \implies u \notin \alpha \setminus \beta)$ or equivalently $u(u \in \gamma \implies (u \notin \alpha \vee u \in \beta))$. Then using $(*)$, we get that $\forall u(u \in \gamma \implies u \in \beta)$. So $\gamma \subset \alpha \cap \beta$. Conversely, suppose for a contradiction that there is a $w \in \alpha \cap \beta \setminus \gamma$. Then since $\text{ord}_{\in}(\alpha)$, $w \notin \gamma$ and $\gamma \neq w$ (since $\gamma \notin \beta \ni w$), we must have that $\gamma \in w \in \beta$, which implies that $\gamma \in \beta$ by transitivity of β. But this contradicts, $\gamma \in \alpha \setminus \beta$. So $\gamma = \alpha \cap \beta$ as we claimed.
Now if $\beta \setminus \alpha$ was non-empty, then $\alpha \cap \beta$ would also be its $\in$-minimal element. But then $\alpha \cap \beta \notin \beta \wedge \alpha \cap \beta \in \beta$, leading to a contradiction. Then $\beta \setminus \alpha = \emptyset$, hence $\beta \subset \alpha$, meaning $\beta = \alpha \cap \beta \in \alpha$.
4. Let $\alpha \in \beta \in \Omega$. Let $y_1, y_2 \in \alpha$. By the transitivity of β, $y_1, y_2 \in \beta$ and since $\text{ord}_{\in}(\beta)$, it follows readily that $y_1 \in y_2 \vee y_1 = y_2 \vee y_1 \ni y_2$, hence $\text{ord}_{\in}(\alpha)$. Let $x \in \mathcal{P}(\alpha)$ be non-empty. By transitivity of β, $x \in \mathcal{P}(\beta)$, and since $(wo)_{\in}(\beta)$, there is a $y \in x$ satisfying $\min_{\in}(y, x)$, so $\text{wo}_{\in}(\alpha)$. Let $\gamma \in \alpha$, and let $\delta \in \gamma$. By trans(β), $\delta \in \beta$, and using $\text{ord}_{\in}(\beta)$, $\delta \in \alpha \vee \delta = \alpha \vee \alpha \in \delta$. Suppose $\delta = \alpha \vee \alpha \in \delta$. Then for $X := \{\alpha, \gamma, \delta\} \in \mathcal{P}(\beta)$, we have each element of X is an element of some element member of X, hence $\forall x \in X \neg(\min_{\in}(x, X))$. So we conclude $\delta \in \alpha$, hence trans(α). Hence $\alpha \in \Omega$.
5. If $\beta \in (\alpha \cup \{\alpha\})$, then $\beta \in \alpha \vee \beta = \alpha$. In either case $\beta \subset \alpha$, since in the first case we

can use trans(α) and in the second case we trivially have $\forall x(x \in \beta \implies x \in \alpha)$. We thus have $\beta \subset \alpha \subset \alpha \cup \{\alpha\}$. Note that if $\beta, \gamma \in \alpha \cup \{\alpha\}$ then $\beta, \gamma \in \alpha \vee \beta \in \alpha = \gamma \vee \gamma \in \alpha = \beta$. In the first case clearly $\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta$ since $\text{ord}_{\in}(\alpha)$. In the second and third case we also get this since $\beta \in \gamma \vee \gamma \in \beta$. So $\alpha \cup \{\alpha\}$ is ordered by $\in$. Let $x \in \mathcal{P}(\alpha \cup \{\alpha\})$ be a non-empty set. If $x = \{\alpha\}$, then we have already seen that α is $\in$-minimal of x. In the case where $x \cap \alpha \neq \emptyset$, this is a non-empty subset of α so it has a $\in$-minimal element, z say, which implies $z \cap x = \emptyset$, for otherwise there would be an element of z that is also in x, y say, which would contradict $\min_{\in}(z, x \cap \alpha)$, for then using that α is transitive we would have $y \in \alpha$, meaning $y \in \alpha \cap x$.
6. Ω is transitive by 4. and ordered by $\in$ by 3. Let C be a non-empty collection of ordinals. If C is singleton, then this element is $\in$-minimal for C. If C is not a singleton, then given any two distinct elements of C, $\delta, \delta' \in C$, we have that $\delta \in \delta'$ or $\delta' \in \delta$, since $\text{ord}_{\in}(\Omega)$. We may thus pick a $\delta_0 \in C$ such that $\delta_0 \cap C \neq \emptyset$. Set $x := \delta_0 \cap C$. Let $\alpha \in x$. Pick an $\in$-minimal element in $x \cap (\alpha \cup \{\alpha\})$, γ say. Since $\gamma \in \alpha \cup \{\alpha\}$ and $\alpha \cup \{\alpha\} \in \Omega$, $\gamma \subset \alpha \cup \{\alpha\}$. Let $\gamma' \in \gamma$. Then $\gamma' \in \alpha \cup \{\alpha\}$. But $\gamma' \notin x \cap (\alpha \cup \{\alpha\})$. Then $\gamma' \notin x$, which means γ is an $\in$-minimal element of x and hence therefor also in C (recall that by definition of x, $\gamma \in C$). □

Remark C.3.11. The above theorem implies that Ω cannot be a set, since if it were Ω would an ordinal number by 6. but then $\Omega \in \Omega$ leading to a contradiction by 2. We still wish to think about objects such as Ω as some kind of collection of elements. We will call a collection of sets satisfying some condition which cannot be described as a sets a *proper class*. Another example is the proper class of sets. I.e. the sets defined by tautology. If this were a set, then trivially it would have itself as an element, contradicting the axiom of foundation. The formal notion of a *class* will left undiscussed for now. It is indeed possible to define this notion rigourosly by extending the ZFC axioms to the Von Neumann–Bernays–Gödel axioms, which we will do later.

Definition C.3.12. For a pair of ordinal number α and β we define

$$\alpha < \beta :\iff \alpha \in \beta$$
$$\alpha \leq \beta :\iff \alpha = \beta \vee \alpha < \beta.$$

Remark C.3.13. This in some sense defines a total order on the proper class of ordinal numbers and on the nose a total order on ever set of ordinal numbers.

Definition C.3.14. For a set X, define $X + 1 := X \cup \{X\}$.

Corollary C.3.15. *1. If $A \subset \Omega$ is a set of ordinals (a set whose elements are ordinals), then $\bigcup A \in \Omega$.*

2. *If* $\alpha, \beta \in \Omega$ *and* $\alpha \in \beta$*, then* $\alpha + 1 \subset \beta$*. So* $\alpha + 1$ *is the smallest ordinal containing* α*.*

3. *For every ordinal* α*, either* $\alpha = \bigcup \alpha$ *or there is an ordinal* β *such that* $\alpha = \beta + 1$*.*

Proof. 1. Let $\beta \in \bigcup A$. Then there is some $\gamma \in A$ such that $\beta \in \gamma$, then $\beta \in \Omega$ by 4. of the last theorem. This means that $\bigcup A$ is a set of ordinal numbers. Then we can apply 6. of the last theorem to get that $\bigcup\ A$ is well-ordered by $\in$. Let $\beta \in \bigcup A$. Again $\beta \in \gamma$ for some $\gamma \in A$. Since γ is transitive, we get that $\beta \subset \gamma$ and since $\gamma \subset \bigcup A$, so $\beta \subset \bigcup A$.
2. $\{\alpha\} \subset \beta$ and using trans(β), $\alpha \subset \beta$, so $\alpha + 1 \subset \beta$.
3. In any case since α is transitive, $\bigcup \alpha \subset \alpha$. Suppose $\bigcup \alpha \neq \alpha$. Then there is an $\in$-minimal element β of $\alpha \setminus \bigcup \alpha$. In particular $\beta \in \alpha$, so by 2. $\beta + 1 \subset \alpha$. If $\alpha \in \beta + 1$, then $\alpha \in \alpha$ contradicting 2. of the prior theorem. If $\beta + 1 \in \alpha$, then $\beta \in \bigcup \alpha$, which contradicts the choice of β. Then we must have $\alpha = \beta + 1$ since $\beta + 1$ is also an ordinal number. □

Definition C.3.16. An ordinal α is called a *succesor ordinal* if there is an ordinal β satisfying $\alpha = \beta + 1$. If on the other hand $\alpha = \bigcup \alpha$, it is called a *limit ordinal*.

C.3.3 ω and ordinal numbers

Lemma C.3.17.

$$\forall x \in \omega(\text{trans}(x)).$$

Proof. Consider the set

$$T := \{x \in \omega : \forall y(y \in x \implies y \subset x)\}.$$

Clearly $\emptyset \in T$. Suppose $x \in T$. Let $y \in x \cup \{x\}$. If $y \in x$, then $y \subset x \subset x \cup \{x\}$. If $y \in \{x\}$, then $y = x \subset x \cup \{x\}$, hence □

Lemma C.3.18.

$$\forall x \in \omega(\text{ord}_\in(x)).$$

Proof. Consider the set

$$O := \{x \in \omega : \text{ord}_\in(x)\}.$$

Trivially, $\emptyset \in O$. Let $x \in O$. Suppose ord$_\in(x)$. Let $y_1, y_2 \in x \cup \{x\}$. If $y_1, y_2 \in x$, then $y_1 \in y_2 \vee y_1 = y_2 \vee y_2 \in y_1$. If $y_1, y_2 \in \{x\}$, then $y_1 = y_2$. If $y_1 \in x$ and $y_2 \in \{x\}$, then $y_2 = x$, hence $y_1 \in y_2$. The case $y_1 \in \{x\}$ and $y_2 \in x$ likewise gives that $y_2 \in y_1$. It follows that $y_1 \in y_2 \vee y_1 = y_2 \vee y_2 \in y_1$, hence ord$_\in(x \cup \{x\})$. It follows that $O = \omega$, hence $\forall x \in \omega(\text{ord}_\in(x))$. □

Lemma C.3.19.

$$\forall x \in \omega \forall y \in \mathcal{P}(x)(y \neq \emptyset \implies \exists z(\min_{\in}(z, y))).$$

With the previous series of lemmas in mind, it follows that each element of ω is an ordinal. Every non-empty element of ω is a succesor ordinal.

Proof. Consider the set

$$A := \{x \in \omega : \forall y \in \mathcal{P}(x)(y \neq \emptyset \implies \exists z(\min_{\in}(z, y)))\}$$

Trivially $\emptyset \in A$. Let $x \in A$ and $y \in \mathcal{P}(x \cup \{x\})$ with $y \neq \emptyset$. If $\{x\} \cap y = \emptyset$, then $y \in \mathcal{P}(x)$, hence there is a z with $\min_{\in}(z, y)$. If $x \in y$, then $x \subset y$, hence $y = x \cup \{x\}$. Let $z \in x$ be such that $\min_{\in}(z, x)$. Then for $\alpha \in z$, $\alpha \notin x$. Moreover $\alpha \notin \{x\}$, for if it were, then $\alpha = x$, meaning $z \in \alpha$, and by transitivity this would mean $z \subset x = \alpha$, but then there would be a $\beta \in z$ with $\beta \in x$ contradicting minimality. So we conclude that $\min_{\in}(z, x \cup \{x\})$. It follows that $A = \omega$.
With previous lemmas in mind, it follows that $\forall x \in \omega(\text{wo}_{\in}(x) \wedge \text{trans}(x))$, hence $\forall x \in \omega(x \in \Omega)$. Note also that if $x \in \omega \setminus \emptyset$, then $x = y \cup \{y\}$ for some $y \in \omega$. This is readily seen with an induction argument: Indeed, $\{\emptyset\} = \emptyset \cup \{\emptyset\}$, and if $x = y \cup \{y\}$ for some $y \in \omega$, then $x \cup \{x\} = y \cup \{y\} \cup \{y \cup \{y\}\} = y' \cup \{y'\}$, where $y' = y \cup \{y\} \in \omega$. So every non-empty element of ω is a succesor ordinal. The empty set is a limit ordinal. □

Lemma C.3.20. *We have that*

$$\omega = \bigcup \omega$$

Proof. Let $x \in \bigcup \omega$, then $x \in y$ for some $y \in \omega$. Then $x \in \omega$, hence $x' := x \cup \{x\} \in \omega$ and $x'' := x' \cup \{x'\} \in \omega$. Then $x \cup \{x\} \in x''$, hence $x \cup x \in \bigcup \omega$. □

Proposition C.3.21. *ω is a limit ordinal.*

Proof. This follows from ω being a family of ordinals, $\omega = \bigcup \omega$ and Corollary C.3.15. 1. □

C.3.4 The Well-ordering Principle & the Axiom of Choice

Definition C.3.22. Let M be a non-emptyset. Set $\mathcal{P}^*(M) := \mathcal{P}(M) \setminus \{\emptyset\}$. Assuming the axiom of choice, pick a choice function

$$f : \mathcal{P}^*(M) \to \bigcup \mathcal{P}^*(M) = M$$

Let an ordinal $\alpha \in \Omega$ be given. An injective function

$$w_\alpha : \alpha \hookrightarrow M$$

is called an *f-set* if for all $\gamma \in \alpha$

$$w_\alpha(\gamma) = f(M \setminus \{w_\alpha(\delta) : \delta \in \gamma\}).$$

The data of a set w being an f-set for an ordinal α is distilled into the formula

$$f\text{−set}(w, \alpha) \equiv \alpha \in \Omega \wedge w \text{ is an } f\text{-set},$$

for future reference.

Lemma C.3.23. *Assume* AC *and let a set M and a choice function f as in the above definition be given. Consider an ordinal $\alpha \in \Omega$. There is at most one f-set for α.*

Proof. Suppose for a contradiction that there are two distinct f-sets w_α and w'_α. Consider the non-empty set $\{\beta \in \alpha : w_\alpha(\beta) \neq w'_\alpha(\beta)\} \in \mathcal{P}(\alpha)$. Since α is an ordinal, this set has an $\in$-minimal element γ. Now if $\delta \in \gamma$, then $w_\alpha(\delta) = w'_\alpha(\delta)$. But then

$$w_\alpha(\gamma) = f(M \setminus \{w_\alpha(\delta) : \delta \in \gamma\}) = f(M \setminus \{w'_\alpha(\delta) : \delta \in \gamma\}) = w'_\alpha(\gamma),$$

leading to a contradiction. □

Remark C.3.24. Consider f-sets w_α and w_β with $\beta \in \alpha$. Then for every $\delta \in \beta$,

$$w_\alpha|_\beta(\delta) = w_\alpha(\delta) = f(M \setminus \{w_\alpha(\delta) : \delta \in \beta\}).$$

Hence $w_\alpha|_\beta$ is an f-set and is therefor equal to w_β by the above lemma.

Theorem C.3.25. *The well-ordering principle is equivalent to the axiom of choice.*

Proof. " $\Longrightarrow$ ": Let X be a family of non-empty sets. Let $\leq$ be a well-ordering on $\bigcup X$. For each $x \in X$, let $m_x \in x$ denote the element that is minimal with respect to $\leq$. Then

$$\begin{aligned} f : X &\to \bigcup X \\ x &\mapsto m_x \end{aligned}$$

defines a choice function.
" $\Longleftarrow$ ": We claim that we may form the set

$$S := \{w_\alpha : f\text{−set}(w_\alpha, \alpha) \wedge \alpha \in \Omega\}$$

Let N be the elements n of M for which there is an f-set w, for some ordinal such that for some ordinal δ, $w(\delta) = n$. We may form this set by the axiom of seperation. Consider the formula

$$(\exists\alpha \in \Omega \exists w(f\text{−set}(w,\alpha) \implies \exists\delta \in \alpha(w(\delta) = n)) \implies y = w) \vee \neg(...) \implies y = \emptyset.$$

This gives rise to the class function

$$G(n) := \begin{cases} w & \text{if } w \text{ is an } f\text{-set that maps to } n \\ \emptyset & \text{otherwise} \end{cases}$$

Then $S = G(N)$ by the axiom schema of replacement I don't like this. Consider now the formula

$$((\exists\alpha\ f\text{−set}(w,\alpha)) \implies y = \alpha) \vee (\neg(\exists\alpha\ f\text{−set}(w,\alpha)) \implies y = \emptyset).$$

This induces the class function F defined by

$$F(w) = \begin{cases} \alpha & \text{if } w \text{ is an } f - \text{set with domain } \alpha \in \Omega \\ \emptyset & \text{otherwise} \end{cases}$$

By the axiom schema of replacement $F(S) = \{F(w) : w \in S\}$ is a family of ordinals, hence $\alpha_0 := \bigcup F(S)$ is an ordinal. For $w_\alpha, w_\beta \in S$, write $w_\alpha \preceq w_\beta$ if $\alpha \leq \beta$. This is easily verified to be a well-ordering on S, since essentially we get that the well-ordering is inherited by the well-ordering on families of ordinals due to $\alpha \mapsto w_\alpha$ being order preserving. Set $w_{\alpha_0} := \bigcup S = \{x : \exists w \in S(x \in w)\} = \{(\delta, w_\alpha(\delta) : \delta \in \alpha \wedge w_\alpha \in S)\}$. The claim is now that this is an f-set, $\alpha_0 \to M$. Let $(\delta, w_\alpha(\delta)), (\delta', w_{\alpha'}(\delta')) \in w_{\alpha_0}$ with $\delta = \delta'$. WLOG $\alpha \leq \alpha'$ and then $w_{\alpha'}(\delta') = w_\alpha(\delta') = w_\alpha(\delta)$, so w_{α_0} is well-defined. Suppose $\delta, \delta' \in \alpha_0$ is such that $w_{\alpha_0}(\delta) = w_{\alpha_0}(\delta')$. As before for some ordinal α, $w_\alpha(\delta) = w_{\alpha_0}(\delta) = w_{\alpha_0}(\delta') = w_\alpha(\delta')$, hence $\delta = \delta'$. Moreover, for $\delta \in \alpha_0$, we see that for a suitable $\alpha \in F(w_\alpha)$ for some $w_\alpha \in S$,

$$w_{\alpha_0}(\delta) = w_\alpha(\delta) = f(M \setminus \{w_\alpha(\gamma) : \gamma \in \delta\}) = f(M \setminus \{w_{\alpha_0}(\gamma) : \gamma \in \delta\}).$$

It follows that w_{α_0} is the maximal f-set, i.e. if w is an f-set, then $w \leq w_{\alpha_0}$. Consider now

$$M' := \{m \in M : \exists\gamma(w_{\alpha_0}(\gamma) = m)\}.$$

Then $M = M'$, since otherwise

$$w_{\alpha_0} \cup \{(\alpha_0, f(M \setminus M'))\}$$

is an f-set strictly greater than w_{α_0}. It follows that $w_{\alpha_0} : \alpha_0 \to M$ is a bijection. We then get a well-ordering on M, where for $x, y \in M$, we write $x \leq y$ if $w_{\alpha_0}^{-1}(x) \leq w_{\alpha_0}^{-1}(y)$. $\square$

C.3.5 Transfinite Recursion and Transfinite Induction

We provide without proof for now? recursion and induction principles that generalize to the proper class of ordinals.

Theorem C.3.26. *(Transfinite Recursion) Let F be a class function defined for all sets. Then there is a unique class function G, defined on Ω such that for each $\alpha \in \Omega$,*

$$G(\alpha) = F\left(G|_\alpha\right)$$

where

$$G|_\alpha := \{(\beta, G(\beta)) : \beta \in \alpha\}$$

Theorem C.3.27. *(Transfinite Induction) Suppose $\varphi(\alpha)$ is an $\{\in\}$-formula such that*

1. $\varphi(0)$.
2. $\forall\alpha \in \Omega(\varphi(\alpha) \to \varphi(\alpha + 1))$.
3. $(\forall\beta < \alpha(\varphi(\beta))) \to \varphi(\alpha)$ *if $\alpha \in \Omega$ is a limit ordinal.*

Then $\varphi(\alpha)$ for every $\alpha \in \Omega$.

C.4 Models of ZF(C)

C.4.1 The standard model V of ZF(C)

In this section we construct a model of ZF under the assumption that ZF is consistent. Then there is a model $(M, \in^{\mathbb{M}})$ of ZF. We define a set for each ordinal $\alpha \in \Omega$,

$$V_\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \bigcup_{\beta\in\alpha} & \text{if } \alpha \text{ is a limit ordinal} \\ \mathcal{P}(V_\alpha) & \text{if } \alpha \text{ is a succesor ordinal} \end{cases}$$

We claim that any element in M corresponds to some element of one of these sets. To see this consider the class function that is the empty set on a sets that are not set functions and where for x that is a function

$$F(x) := \begin{cases} \emptyset & \text{if } x = \emptyset \\ \mathcal{P}(x(\beta)) & \text{if dom } x \text{ is the succesor of an ordinal } \beta \\ \bigcup_{\delta\in\text{dom } x}\delta & \text{if dom } x \text{ is a nonempty limit ordinal} \\ \emptyset & \text{otherwise} \end{cases}$$

By transfinite recursion there is a class function G on Ω such that

$$G(\alpha) = F\left(G|_{\alpha}\right).$$

Note that

$$G|_{\alpha}$$

is a function $\alpha \to G[\alpha]$, where $G[\alpha]$ is a set by the axiom schema of replacement. We see that

$$G(\alpha) = V_{\alpha}$$

for each $\alpha \in \Omega$, thus we have the following lemma on the above construction

Lemma C.4.1. *V_{α} is a set for each $\alpha \in \Omega$.*

Moreover we have the following

Lemma C.4.2. *1. V_{α} is inductive for each α.*

2. The class $\mathbf{V} := \bigcup_{\alpha\in\Omega}$ is inductive.

3. if $\alpha \in \beta \in \Omega$, then $V_{\alpha} \subset V_{\beta}$.

4. For $\alpha \in \Omega$, if $\alpha \subset V_{\alpha}$ and $\alpha \in V_{\alpha+1}$

Proof. do! □

C.4.2 Non-standard models of ZF(C)

C.4.3 Gödel's Model of ZF and of ZFC: The Relative Consistency of ZF and AC

C.5 A Model of PA in V

We aim to construct a $\mathcal{L}_{\mathrm{PA}}$-structure within V which we denote $\mathbb{N}_{\omega}$ whose domain is ω and show that this is a model of PA, where $\mathcal{L}_{\mathrm{PA}} = \{0, \mathrm{s}, +, \cdot\}$. We then define

$$0^{\mathbb{N}_{\omega}} := \emptyset \in \omega$$

$$\mathrm{s}^{\mathbb{N}_{\omega}} : \omega \to \omega$$

$$n \mapsto n + 1 := n \cup \{n\}$$

On ω we define a binary operation, called addition, recursively by $n+0 := n$ and $n+\mathrm{s}(m) = \mathrm{s}(n+m)$, hence we define

$$+^{\mathbb{N}_\omega} : \omega \times \omega \to \omega$$
$$(n, m) \mapsto n + m$$

We also define a binary operation, called multiplication, recursively by $n \cdot 0 = 0$ and $n \cdot \mathrm{s}(m) = n \cdot m + n$, hence we define

$$\cdot^{\mathbb{N}_\omega} : \omega \times \omega \to \omega$$
$$(n, m) \mapsto nm := n \cdot m$$

Theorem C.5.1.

$$\mathbb{N}_\omega \vDash \mathrm{PA}$$

Proof. Fix an assignment j. We prove that $\mathbb{N}_\omega \vDash \mathrm{PA}_i$ for each $i \in \{0, 1, \ldots, 6\}$. For $i = 0$, the prove will use the inductive definition on the complexity of verifying validity of a formula. This is a little cumbersome, so for the other axioms we will use a more natural language approach.

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_0$": We need to show that $\mathbb{N}_\omega \vDash \neg\exists x(\mathrm{s}(x) = 0)$. Note that this means, we need to prove that $\mathbb{N}_\omega \vDash \forall x(\mathrm{s}(x) \neq 0)$. Let $n \in \omega$ be given. We then need to prove that $\mathbb{N}_\omega[x \mapsto n] \vDash \mathrm{s}(x) \neq 0$, i.e. that $\mathbb{N}_\omega[x \mapsto n] \vDash \mathrm{s}(x) = 0$ is not true. We thus have to verify that

$$\mathbb{N}_\omega[x \mapsto n](\mathrm{s}(x)) = \mathrm{s}^{\mathbb{N}_\omega}(\mathbb{N}_\omega[x \mapsto n](x)) = \mathrm{s}^{\mathbb{N}_\omega}(n) = n \cup \{n\}$$

is not equal to $0^{\mathbb{N}_\omega} = \emptyset$, but this is indeed obvious since $n \in n \cup \{n\}$.

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_1$": Let n and m in ω be given. Suppose $n \neq m$. Then $n \in m$ or $m \in n$. In the first case, by Corollary C.3.15 2. $n + 1 \subset m$, hence $n + 1 \in m$ or $n + 1 = m$. In either case since $m \subsetneq m + 1$, $n + 1 \subsetneq m + 1$ and in particular $n + 1 \neq m + 1$.I feel like I have explained some facts about ordinals multiple times different places – collect in lemma!

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_2$": The result follows from the base case of the definition of addition on ω.

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_3$": The result follows from the recursion step of the definition of addition on ω. "$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_4$": Again it follows from the base case recursive definition of multiplication on ω.

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_5$": Again it follows from the recursion step in the definition of multiplication on ω.

"$(\mathbb{N}_\omega, j) \vDash \mathrm{PA}_6$": Let a $\mathcal{L}_{\mathrm{PA}}$-formula φ with $x \in \mathrm{free}(\varphi)$ be given. We need to show that

under the given interpretation, if

$$\psi :\equiv \varphi(\emptyset) \wedge \forall n \in \omega(\varphi(n) \to \varphi(n+1)),$$

then $\forall n \in \omega(\varphi(n))$. Set

$$E := \{n \in \omega : \neg\varphi(n)\}.$$

If we can show that

$$E = \emptyset$$

then we are done. Suppose $E \neq \emptyset$. Then E has an $\in$-minimal element m. Since we assumed ψ, $m \neq \emptyset$ and m cannot be the succesor of some m' for then $\neg\varphi(m')$ (since we with ψ assumed $\varphi(n) \to \varphi(n+1)$ which is equivalent to $\neg\varphi(n+1) \to \neg\varphi(n)$) and hence $m' \in E$ which would contradict the minimality of m. We must therefor have that E is empty. □

Theorem C.5.2. $\mathbb{N} \simeq \mathbb{N}_\omega$

Proof. DO! □

Corollary C.5.3. *ω is an ordered semi-ring, which we will from now on denote* $\mathbb{N}$.

Proof. If one finishes the The Natural Number Game one can show that you can derive the axioms of an ordered semi-ring in PA. It thus follows from the Soundness Theorem that any model of PA will be an ordered semi-ring. □

C.6 NGB Set Theory: A Formal Treatment of Classes

C.7 MK Set Theory: Quantification over Classes

D Category Theory

D.0.1 Initial Definitions

Definition D.0.1. A category $\mathcal{C}$ is a pair $(\mathrm{Ob}(\mathcal{C}), \mathrm{Hom}(\mathcal{C}))$ where

1. $\mathrm{Ob}(\mathcal{C})$ denotes a class of *objects*.
2. $\mathrm{Hom}(\mathcal{C})$ denotes a class of *morphisms*.
3. A morphism f in $\mathrm{Hom}(\mathcal{C})$ is a binary relation between between members A, B of $\mathrm{Ob}(\mathcal{C})$. We denote it by $f : A \to B$.

4. For objects A, B in $\mathrm{Ob}(\mathcal{C})$ we denote the class of morphisms from A to B by $\mathrm{Hom}(A, B)$.

5. There is a binary operation $\circ$ on the class of *composable* morphisms called *composition* such that for composable morphisms $f : B \to C$ and $g : A \to B$ we have a morphism

$$fg := f \circ g : A \to C,$$

such that

$$(f \circ g) \circ h = f \circ (g \circ h)$$

where $f : C \to D$, $g : B \to C$ and $h : A \to B$ for objects A, B, C, D in $\mathrm{Ob}(\mathcal{C})$. Furthermore for each object X in $\mathrm{Ob}(\mathcal{C})$ there is a morphism $\mathbb{1}_X : X \to X$ called the *identity morphism* such that

$$\mathbb{1}_B f = f = f \mathbb{1}_A,$$

for a morphism $f : A \to B$.

Remark D.0.2. Here the notion of "class" depends on foundations. For instance, it could mean a set in the sense of ZFC or it could mean a proper class in the sense of NGB I guess what I really mean is that objects and morphisms live in a model of ZFC/NGB but I am not sure.

Definition D.0.3. In a category, a morphism $f : X \to X$ is called an *endomorphism*.

Definition D.0.4. Let $\mathcal{C}$ be a category. An *isomorphism* $f : A \to B$ is a morphism in $\mathrm{Hom}(\mathcal{C})$ such that there is another morphism $f^{-1} : B \to A$ satisfying,

$$ff^{-1} = \mathbb{1}_B \text{ and } f^{-1}f = \mathbb{1}_A.$$

An endomorphism that is an isomorphism is called an *automorphism*.

Definition D.0.5. A *subcategory* $\mathcal{D}$ of a category $\mathcal{C}$ is a subclass of $\mathrm{Ob}(\mathcal{C})$ together with a subclass of $\mathrm{Hom}(\mathcal{C})$ that constitutes a category.

Remark D.0.6. equivalently a subcategory of $\mathcal{C}$ is a subclass $\mathrm{Ob}(\mathcal{D})$ of $\mathrm{Ob}(\mathcal{C})$ and a subclass $\mathrm{Hom}(\mathcal{D})$ of $\mathrm{Hom}(\mathcal{C})$ such that each domain A and codomain B for a morphism in $\mathrm{Hom}(\mathcal{D})$, A, B are elements of $\mathrm{Ob}(\mathcal{D})$ and such that $\mathrm{Hom}(\mathcal{D})$ is closed under composition.

Definition D.0.7. Let $\mathcal{C}$ be a category and $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, A)$. f is called a *retraction* of g and g a section of f if $fg = \mathbb{1}_B$

Lemma D.0.8. *Let $\mathcal{C}$ be a category, $f \in \mathrm{Hom}(A, B)$. Suppose $g, h \in \mathrm{Hom}(B, A)$ are respectively a retraction and a section of f. Then $g = h$ and f is an isomorphism. It follows that a morphism can have at most one inverse*

Proof. Indeed

$$g = g\mathbb{1}_A = gfh = \mathbb{1}_B h = h.$$

Hence f is an isomorphism with $f^{-1} = g = h$. Let f_1 and f_2 be inverses of an isomorphism f. Note that both f_1 and f_2 is both a section and a retraction. Therefor, by the first statement, $f_1 = f_2$. □

Definition D.0.9. Let $\mathcal{C}$ be a category. We define *the opposite category of* $\mathcal{C}$ denoted $\mathcal{C}^{\mathrm{op}}$ to be the category with $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$ and where a morphism $f : A \to B$ in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}})$ is a morphism $f : B \to A$ in $\mathrm{Hom}(\mathcal{C})$

Remark D.0.10. The above indeed does define a category. We define $\circ^{\mathrm{op}}$ by

$$f \circ^{\mathrm{op}} g = g \circ f : C \to A$$

where $f : B \to C$ and $g : A \to B$ are morphisms in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}})$. Then for morphisms $f : C \to D$, $g : B \to C$ and $h : A \to B$

$$(f \circ^{\mathrm{op}} g) \circ^{\mathrm{op}} h = h(gf) = (hg)f = f \circ^{\mathrm{op}} (g \circ^{\mathrm{op}} h).$$

Furthermore, we define the identity morphisms in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}})$ to be the identity morphisms in $\mathrm{Hom}(\mathcal{C})$, hence

$$f \circ^{\mathrm{op}} \mathbb{1}_A = \mathbb{1}_A f = f \text{ and } \mathbb{1}_B \circ^{\mathrm{op}} f = f\mathbb{1}_B = f.$$

Definition D.0.11. A category $\mathcal{C}$ is *locally small* if $\mathrm{Hom}(A, B)$ is a set for every object A, B in $\mathrm{Ob}(\mathcal{C})$. It is *small* if $\mathrm{Ob}(\mathcal{C})$ is a set.

Definition D.0.12. In a category $\mathcal{C}$ a morphism $f \in \mathrm{Hom}(A, B)$ is a *monomorphism* if for every pair of morphisms $g_1, g_2 \in \mathrm{Hom}(C, A)$,

$$fg_1 = fg_2 \implies g_1 = g_2.$$

It is called an *epimorphism* if for every pair of morphisms $h_1, h_2 \in \mathrm{Hom}(B, D)$,

$$h_1 f = h_2 f \implies h_1 = h_2$$

Remark D.0.13. Note that an epimorphism is just a monomorphism in $\mathcal{C}^{\mathrm{op}}$ and vice versa

Lemma D.0.14. *Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism. Then if f is an isomorphism, then it is a monomorphism and an epimorphism.*

Proof. Suppose f is an isomorphism. Let $g_1, g_2 : C \to A$ with $fg_1 = fg_2$. Then

$$g_1 = f^{-1}fg_1 = f^{-1}fg_2 = g_2.$$

Using the dual result, we get that for every $h_1, h_2 : B, D$, if $h_1 f = h_2 f$ then $h_1 = h_2$. We thus conclude that f is both an epimorphism and a monomorphism.

□

Definition D.0.15. A category for which the converse of the above lemma holds is called *balanced*.

Lemma D.0.16. *In the category of* Set*, we have for a function $f : A \to B$*

1. *f is epi if and only if it is surjective.*
2. *f is mono if and only if is injective*

In particular Set *is balanced.*

Proof. 1. Suppose f epimorphism. Note that in general for a subset $C \subset B$, if for each set X and each pair of maps $\lambda_1, \lambda_2 : B \to X$, $\lambda_1|_C = \lambda|_C$ implies $\lambda_1 = \lambda_2$, then $C = B$. Our assumption tells us that for each pair of functions $h_1, h_2 : B \to X$, $h_1|_{\text{im } f} = h_2|_{\text{im } f}$ implies $h_1 = h_2$, hence im $f = B$, meaning f is surjective.
Suppose conversely that f is surjective. Let $h_1, h_2 \in \text{Hom}(X, B)$ and suppose $h_1 f = h_2 f$. Let $b \in B$. Then there is an $a \in A$ such that $b = f(a)$, hence

$$h_1(b) = h_1(f(a)) = h_1 f(a) = h_2 f(a) = h_2(f(a)) = h_2(b).$$

Since b was chosen arbitrarily it follows that $h_1 = h_2$.
2. Suppse f is mono. Suppose $a_1, a_2 \in A$ are given such that $f(a_1) = f(a_2)$. Consider the functions, $g_i : \{*\} \to A, * \mapsto a_i$. Then

$$fg_1 = fg_2,$$

hence $g_1 = g_2$, which implies $a_1 = g_1(*) = g_2(*) = a_2$.
Suppose f is injective. Consider $g_1, g_2 : C \to A$ with $fg_1 = fg_2$. Let an arbitrary $x \in C$ be given. Then

$$f(g_1(x)) = f(g_2(x)) \implies g_1(x) = g_2(x).$$

Since x was chosen arbitrarily this means $g_1 = g_2$. □

Remark D.0.17. Note that surjectivity implies epi, and dually injectivity, in any subcategory of Set.

Example D.0.18. There are non balanced categories. Consider the canonincal embedding

$$f_{\mathbb{Q}} : \mathbb{Z} \hookrightarrow \mathbb{Q}$$
$$n \mapsto n$$

a morphism in the category of rings. Since it is injective it is a monomorphism. Suppose $h_1, h_2 : \mathbb{Q} \to R$ are ring homomorphism such that

$$h_1 f_{\mathbb{Q}} = h_2 f_{\mathbb{Q}}$$

these are both ring homomorphisms $\mathbb{Z} \to \mathbb{Q}$. We shall later see that there is only one such homomorphism, namely

$$f_R : \mathbb{Z} \to R$$
$$n \mapsto \sum_1^n 1$$

The vocabulary that captures this fact is that $\mathbb{Z}$ is the initial object in Ring. We conclude that $h_1 = h_2$ and hence that $f_{\mathbb{Q}}$ is epi. However, since $f_{\mathbb{Q}}$ is not surjective it cannot be an isomorphism.

Lemma D.0.19. *Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism*

1. *If f is a section of some $h : B \to A$, then it is a mono.*
2. *If f is a retraction of some $h : B \to A$, then it is epi.*

Proof. 1. Consider $g_1, g_2 : C \to A$ and suppose $fg_1 = fg_2$. Since f is a section of h, then $hf = \mathbb{1}_A$. Then

$$g_1 = (hf)g_1 = h(fg_1) = h(fg_2) = (hf)g_2 = g_2.$$

2. Since a retraction is just a section in the opposite category, this result is dual to 1. □

Remark D.0.20. If a morphism is a section of some morphism, then it is called a *split monomorphism*. If a morphism is a retraction of some morphism it is called a *split epimorphism*.

Lemma D.0.21. *For every category $\mathcal{C}$, a morphism $f : A \to B$ is an isomorphism if and only if it is mono and split epi. Dually it is an isomorphism if and only if it is mono and split epi.*

Proof. One direction follows from Lemma D.0.1. Suppose f is mono and split epi. Since it is split epi, there is a $g : B \to A$ such that $fg = \mathbb{1}_B$. Note that

$$f(gf) = (fg)f = \mathbb{1}_B f = f = f\mathbb{1}_A,$$

hence $gf = \mathbb{1}_A$ by the assumption that f is mono. □

Lemma D.0.22. *Consider a category $\mathcal{C}$. Then*

1. *The composition of two monos is a mono.*
2. *For $f : A \to B$ and $g : B \to C$ if gf is a mono, then so is f.*

Dually

1. *The composition of two epis is an epi.*
2. *For For $f : A \to B$ and $g : B \to C$ if gf is an epi, then so is g.*

Proof. 1. Consider monos $f : B \to C$ and $g : A \to B$. Suppose $h_1, h_2 : X \to A$ are given such that $(fg)h_1 = (fg)h_2$. Then

$$f(gh_1) = f(gh_2) \implies gh_1 = gh_2 \implies h_1 = h_2,$$

hence fg is a mono.
2. Consider $h_1, h_2 : X \to A$ such that $fh_1 = fh_2$. Then $(gf)h_1 = (gf)h_2$, hence by assumption, $h_1 = h_2$. We thus conclude that f is mono. □

Remark D.0.23. From the above result it follows that $\mathrm{Ob}(\mathcal{C})$ with morphisms being monos/epis in $\mathrm{Hom}(\mathcal{C})$, defines a subcategory of $\mathcal{C}$.

Definition D.0.24. Let $\mathcal{C}$ be a locally small category and fix an object X in $\mathcal{C}$. Given a morphism $f : A \to B$, define

$$f_* : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$$
$$g \mapsto fg$$

we also define the dual concept

$$f^* : \mathrm{Hom}(A, Y) \to \mathrm{Hom}(B, X)$$
$$h \mapsto hf$$

Remark D.0.25. Note that $f_*^{\mathrm{op}} = f^*$, i.e. for $f^{\mathrm{op}} : B \to A$, the morphism f interpreted as a morphism of $\mathcal{C}^{\mathrm{op}}$ we have

$$f_*^{\mathrm{op}} : \mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(X, B) = \mathrm{Hom}(B, X) \to \mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(X, A) = \mathrm{Hom}(A, X)$$
$$h : B \to X \mapsto f \circ^{\mathrm{op}} h = hf$$

hence $f_*^{\mathrm{op}} = f^*$

Lemma D.0.26. *For a locally small category $\mathcal{C}$ and morphism $f : A \to B$, the following are equivalent*

1. *f is an isomorphism*
2. *For every object X in $\mathcal{C}$, f_* is a bijection.*
3. *For every object X in $\mathcal{C}$, f^* is a bijection*

Proof. "1. $\implies$ 2.": Let $g : B \to A$ be the inverse of f. Then given $h \in \mathrm{Hom}(X, A)$,

$$(g_*f_*)h = g_*(f_*h) = g_*(fh) = gfh = h \implies g_*f_* = \mathbb{1}_{\mathrm{Hom}(X,A)}$$

and similarly one proves that $f_*g_* = \mathbb{1}_{\mathrm{Hom}(X,B)}$.
"2. $\implies$ 1.": Consider $f_* : \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, B)$. This by assumption has an inverse $\lambda : \mathrm{Hom}(B, B) \to \mathrm{Hom}(B, A)$. Set $g := \lambda(\mathbb{1}_B)$. Then for each

$$fg = f\lambda(\mathbb{1}_B) = f_*(\lambda(\mathbb{1}_B)) = (f_*\lambda)(\mathbb{1}_B) = \mathbb{1}_{\mathrm{Hom}(B,B)}(\mathbb{1}_B) = \mathbb{1}_B.$$

Now consider the function

$$\mu : \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, B)$$
$$h \mapsto fh$$

which is just another instance of an "f_*", but we use different notation to not confuse it with $f_* : \mathrm{Hom}(B, B) \to \mathrm{Hom}(B, A)$. Then μ has an inverse μ^{-1}. Note then thats

$$\mu(gf) = f(gf) = (fg)f = \mathbb{1}_B f = f\mathbb{1}_A = \mu(\mathbb{1}_A)$$

hence

$$gf = (\mu^{-1}\mu)(gf) = \mu^{-1}(\mu(gf)) = \mu^{-1}(\mu(\mathbb{1}_A)) = (\mu^{-1}\mu)(\mathbb{1}_A) = \mathbb{1}_A.$$

"1. $\iff$ 3.": This is dual to "1. $\iff$ 2.".

□

Lemma D.0.27. *Consider a locally small category $\mathcal{C}$ and a morphism $f : A \to B$.*

1. *f is a split epimorphism if and only if for each X, $f_* : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is surjective.*

2. *Dually, f is a split monomorphism if and only if for each X, $f^* : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is injective.*

Proof. 1. Suppose f is a split epi. Then f has a right inverse g, i.e. $fg = \mathbb{1}_B$. Let $h : X \to B$ be given. Then $f_*(gh) = f(gh) = (fg)h = h$, hence f_* is surjective.
Suppose that $f_* : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is surjective for each object X. Then $f_* : \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, B)$ is surjective. hence for some $g : B \to A$,

$$\mathbb{1}_B = f_*(g) = fg$$

hence f is a retraction of g and therefor a split epimorphism. 2. This is dual to 1. □

D.0.2 Examples of Categories

Example D.0.28. Consider the proper class of sets and the class of functions. This defines a category Set where composition is given by function composition and the identity morphism is the identity function.

Example D.0.29. The following are some examples of subcategories of Set.

1. There is the category of finite sets Fin, whose objects are finite sets and whose morphisms are maps.

2. Top has the proper class of topological spaces as objects and continuous functions as morphisms.

3. Htpy is the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. That is given continous maps $f, g : X \to Y$, these maps are *homotopic* if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. This defines an equivalence relation, $\sim$ say. We define $[g]_\sim[f]_\sim := [gf]_\sim$. Suppose $f, f' : X \to Y$ are homotopic via a map $I : X \times [0, 1] \to Y$ and $g, g' : Y \to Z$ are homotopic via $J : Y \times [0, 1] \to Z$. We define

$$H : X \times [0, 1] \to Z$$
$$(x, t) \mapsto J(I(x, t), t)$$

Then $H(x,0) = J(I(x,0),0) = J(f(x),0) = gf(x)$ and $H(x,1) = J(I(x,1),1) = J(f'(x),1) = g'f'(x)$, so this is homotopy, hence composition is well-defined. One readily verifies associativity. The identity morphism is the homotopy class of the identity map.

4. Given a (preferably algebraically closed) field K, a multiprojective variety over K is a solution to a set of polynomial equations (a socalled algebraic set) in a set of the form $\mathbb{A}^n(K) \times \mathbb{P}^{n_1}(K) \times \cdots \times \mathbb{P}^{n_m}(K)$ that cannot be written as the union of two proper subsets that are sets of solutions to systems of polynomial equations. The set of varieties in such a set forms a subbasis for a topology called the Zariski topology. A subset U of a multiprojective variety V is called a variety if it is open in Zariski subspace topology on V. The collection of all such varieties is the objects in a category Var_K. To discuss how the morphisms work in this category one has to go into a long explanation about some constructions in algebraic geometry. The jist is the following: Given a set A, $\mathrm{Hom}^{\mathrm{Set}}(A,K)$ defines a ring. In our context we are interested in the subring $\Gamma(A)$ of this ring consisting of maps that are essentially evaluation of fractions of polynomials. The morphisms of Var_K are then continuous maps of varieties $\varphi : X \to Y$ such given any open subset $U \subset Y$ and $f \in \Gamma(U)$, we have that $f \circ \varphi \in \Gamma\left(\varphi^{-1}(U)\right)$. One checks that this defines a subcategory of Top.

5. As a subcategory of Var_K we have AffVar_K, the category of affine varieties, whose objects are varieties isomorphic to irreducible algebraic sets in $\mathbb{A}^n$ for some n.

6. As a subcategory of Var_K we have $\mathrm{ProjVar}_K$, the category of projective varieties, whose objects are varieties isomorphic to irreducible algebraic sets in $\P^n$ for some n.

7. As another subcategory of Top, we have $\mathrm{Man}_{\mathbb{R}}$, the category of smooth manifolds over $\mathbb{R}$ with morphisms being smooth maps. One can also consider $\mathrm{Man}_{\mathbb{C}}$ and as a subcategory RS of riemann surfaces.

8. There is the category Meas of measure spaces and measurable functions.

9. Mon has the proper class of monoids as objects and monoid homomorphisms as morphisms.

10. Group has groups as objects and group homomorphisms as morphisms.

11. Ab has abelian groups as objects and group homomorphisms as morphisms.

12. Ring has (unital) rings as objects and ring homomorphism as morphisms.

13. Rng has (not necessarily unital rings) as objects and maps that are monoid homomorphisms with respect to both addition and multiplication.

14. Given a ring, R ${}_R\mathrm{Mod}$ is the category with objects being left R-modules with left R-module homomorphisms.

15. There is a dual notion of a category of right R-modules Mod_R.

16. Given rings R, S, there is a category ${}_R\mathrm{Mod}_S$ of (R, S)-bimodules and (R, S)-bimodule homomorphisms.

17. Given a field K we have the category of vector spaces, Vect_K. As a subcategory there is the category of finite dimensional vector spaces $\mathrm{Vect}_K^{\mathrm{fd}}$

18. Given a commutative ring R, we have the category of algebras over R, denoted Alg_R.

19. Indeed, asking that a set is endowed with some structure and finding a class of maps that preserves this structure and is closed under composition readily leads to a category.

Example D.0.30. Given any subcategory $\mathcal{C}$ of Set. We can form $\mathcal{C}_*$ consisting of a pair (X, x) where $X \in \mathrm{Ob}(\mathcal{C})$ and $x \in X$ where a morphism $f : (X, x) \to (Y, y)$ is a map $f : X \to Y$ such that $f(x) = y$. This is the category of *pointed sets in* $\mathcal{C}$. For example, we could consider the category of pointed sets Set_* or the category of pointed topological spaces Top_*.

Example D.0.31. Given a (commutative say) ring R we can form the category of *chain complexes* Ch_R whose objects are families of R-modules $\{M_i\}_{i\in\mathbb{Z}}$ together with a family of R-module maps $\{d_i : M_i \to M_{i-1}\}_{i\in\mathbb{Z}}$ such that $d_{i-1}d_i = 0$ which are called *boundary maps*. A morphism of chain complexes $f_\bullet : M_\bullet \to M'_\bullet$ is a collection of maps $\{f_i : M_i \to M'_i\}$ such that the square

$$\begin{array}{ccc} M_i & \xrightarrow{d_i} & M_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ M'_i & \xrightarrow{d'_i} & M'_{i-1} \end{array}$$

commutes. Composition of $f_\bullet : M_\bullet \to M'_\bullet$ and $g_\bullet : M'_\bullet \to M''_\bullet$ is given by $\{g_i f_i : M_i \to M''_i\}$. Indeed, the square

$$\begin{array}{ccc} M_i & \xrightarrow{d_i} & M_{i-1} \\ \downarrow{\scriptstyle g_i f_i} & & \downarrow{\scriptstyle g_{i-1} f_{i-1}} \\ M''_i & \xrightarrow{d''_i} & M''_{i-1} \end{array}$$

commutes, since

$$\begin{array}{ccc} M_i & \xrightarrow{d_i} & M_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ M'_i & \xrightarrow{d'_i} & M'_{i-1} \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_{i-1}} \\ M''_i & \xrightarrow{d''_i} & M''_{i-1} \end{array}$$

commutes. The identity morphism is $\{\mathrm{id}_{M_i}\}_{i \in \mathbb{Z}}$.

Example D.0.32. Given a preordered set $(X, \leq)$, we can construct a category whose objects are X and where a morphisms are $\leq$. Composition of morphisms $a \leq b$ and $b \leq c$ is $a \leq c$. Indeed if $a \leq b$ and $b \leq c$, then $a \leq c$, so this is well-defined. One readily verifies that this is associative using transitivity a few times. The identity morphism is $a \leq a$ with exists by reflexivity.
This could for instance be an ordinal $\alpha \in \Omega$.

Definition D.0.33. A small category $\mathcal{C}$ is called a *groupoid* if every f in $\mathrm{Hom}(\mathcal{C})$ is an isomorphism

Example D.0.34. Let $(M, \cdot, e)$ be a monoid. We define a category $\mathrm{B}M$ whose class of objects is $\{M\}$ and whose morphisms are $n, m \in M$ and where composition of morphisms $n, m \in M$ is given by

$$nm := n \cdot m.$$

Since M is a monoid for $m_1, m_2, m_3 \in M$, we have

$$m_1(m_2 m_3) = (m_1 m_2) m_3.$$

The identity morphism is e. When $(G, ;e)$ is a group we get that every morphism is an isomorphism, since for each $g \in G$ there is a $g^{-1} \in G$ such that

$$g g^{-1} = g^{-1} g = e$$

Note that a groupoid with one object X is a group in the following sense: The set $\mathrm{Hom}(X, X)$ forms a group.

Example D.0.35. Consider a topological space X. We construct the *fundemental groupoid of* X to be the category with objects X and morphisms being homotopy classes of paths $f_{x,y} : [0,1] \to X$ such that for $f_1, f_2 \in [f]_{x,y}$, $f_1(0) = x = f_2(0)$ and $f_1(1) = y = f_2(1)$. Here composition of paths $[f]_{x,y}$ and $[g]_{y,z}$ is the homotopy class of $g \bullet f$, where

$$(g \bullet f)(t) := \begin{cases} f(2t) & \text{if } t \in [0, 1/2] \\ g(2t-1) & \text{if} t \in (1/2, 1] \end{cases}$$

one proves that this is independent of representatives by showing that one can construct a homotopy of any pair of composed paths. The identity morphism for an object $x \in X$ is $[0,1] \to X, t \mapsto x$. It is clear that this is a groupoid, since given a path $f : [0,1] \to X$ we construct $f^{-1} : [0,1] \to X, t \mapsto f(1-t)$.

Definition D.0.36. The *maximal groupoid* of a category $\mathcal{C}$ is the subcategory of $\mathcal{C}$ whose objects are $\mathrm{Ob}(\mathcal{C})$ and whose morphisms are the isomorphisms of $\mathrm{Hom}(\mathcal{C})$

Remark D.0.37. The maximal groupoid is a subcategory. Indeed, The domain and codomain of an isomorphism are trivially in $\mathrm{Ob}(\mathcal{C})$. Suppose $f : A \to B$ and $g : B \to C$ are isomorphisms. Then

$$f^{-1}g^{-1}gf = f^{-1}\mathbb{1}_B f = f^{-1}f = \mathbb{1}_A$$

and

$$gff^{-1}g^{-1} = g\mathbb{1}_B g^{-1} = gg^{-1} = \mathbb{1}_C.$$

Hence $gf : A \to C$ is an isomorphism with inverse $f^{-1}g^{-1}$.

Definition D.0.38. A category $\mathcal{C}$ is *discrete*, if for each morphism $f : X \to Y$, $Y = X$ and $f = \mathrm{id}_X$.

Example D.0.39. Take any non-empty set X as objects and the relation

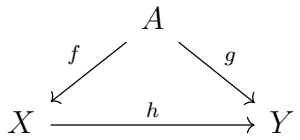
$$=:= \{(x, y) \in X : x = y\}$$

to be morphisms. A composition of a morphism $x = y$ and a morphism $y = z$ is the morphism $x = z$ which exists by transitivity. The identity morphism is $x = x$ for each $x \in X$. For a triple of composable morphisms we have

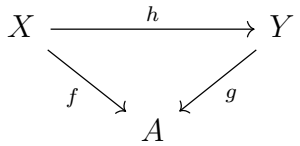
$$\begin{aligned}(x = y)((y = z)(z = w)) = (x = y)(y = w) = (x = w) = (x = z)(z = w) \\ = ((x = y)(y = z))(z = w).\end{aligned}$$

Note that given a morphism $x = y$, then $(x, x) = (x, y)$,hence $x = x$ and $x = y$ are the same morphism. It follows that $(X, =)$ is a discrete category.
In general, given a set X and an equivalence relation $\sim$, the pair $(X, \sim)$ defines a groupoid.

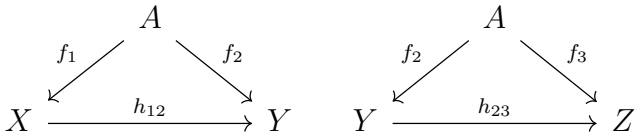
Definition D.0.40. Let $\mathcal{C}$ be a category and A an object in $\mathcal{C}$. *The slice category of $\mathcal{C}$ under A* denoted $A/\mathcal{C}$ is the category whose objects are morphisms in $\text{Hom}(\mathcal{C})$ with domain A and where a morphism from $f : A \to X$ and $g : A \to Y$ is a map $h : X \to Y$ such that



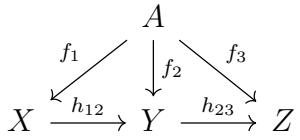
commutes. *The slice category of $\mathcal{C}$ over A* denoted $\mathcal{C}/A$ has morphisms with codomain A as objects and a morphism from $f : X \to A$ to $g : Y \to A$ is a morphism $h : X \to Y$ satisfying



Remark D.0.41. Both these constructions are indeed categories: Consider morphisms h_{12} between $f_1 : A \to X$ & $f_2 : A \to Y$ and h_{23} between $f_2 : A \to Y$ & $f_3 : A \to Z$. Then we have commutative diagrams



to obtain the commutative diagram



hence $h_{23}h_{12}$ is a morphism between f_1 and f_3. For an object $f : A \to X$ in $A/\mathcal{C}$ define the identity morphism to be $\mathbb{1}_X$. We thus get that associativity of composition and the identity morphisms being neutral with respect to composition is inherited from this being true in $\mathcal{C}$. Reversing arrows we get that $\mathcal{C}/A$ is also a category.

D.0.3 Functors

Definition D.0.42. Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. A *Covariant functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a mapping $\mathcal{F}$, denoted $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, which assigns to each object A in $\text{Ob}(\mathcal{C}_1)$ to an object $\mathcal{F}(A)$ in $\text{Ob}(\mathcal{C}_2)$ and to each morphism in $\text{Hom}(\mathcal{C}_1)$, $f : A \to B$ a morphism in $\text{Hom}(\mathcal{C}_2)$, $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ such that

1. for every object X in $\text{Ob}(\mathcal{C}_1)$, $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$.

2. for every pair of morphisms $f : B \to C$ and $g : A \to B$ in $\mathrm{Hom}(\mathcal{C}_1)$, $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

Definition D.0.43. Consider categories $\mathcal{C}_1$ and $\mathcal{C}_2$. A *contravariant functor* $\mathcal{F}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ is a covariant functor between $\mathcal{C}_1^{\mathrm{op}}$ and $\mathcal{C}_2$.

Lemma D.0.44. *Consider two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ with a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$. If $f : A \to B$ is an isomorphism in* $\mathrm{Hom}(\mathcal{C}_1)$*, then $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(A)$ is an isomorphism in* $\mathrm{Hom}(\mathcal{C}_2)$.

Proof. Indeed,

$$\mathcal{F}(f)\mathcal{F}\left(f^{-1}\right) = \mathcal{F}\left(ff^{-1}\right) = \mathcal{F}(\mathbb{1}_B) = \mathbb{1}_{\mathcal{F}(B)} \text{ and } \mathcal{F}\left(f^{-1}\right)\mathcal{F}(f) = \mathcal{F}\left(f^{-1}f\right) = \mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}.$$

□

Example D.0.45. Suppose that there, for a category $\mathcal{C}$, is a well-defined assignment $\mathcal{F}$ of objects in $\mathcal{C}$ to integers and of morphisms $A \to B$ to $\mathcal{F}(A) \leq \mathcal{F}(B)$. This will define a functor from $\mathcal{C}$ to $(\mathbb{Z}, \leq)$ called *an integer invariant of objects in $\mathcal{C}$*. Indeed, $\mathcal{F}(A) \leq \mathcal{F}(A)$, hence $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$, so $\mathbb{1}_{\mathcal{F}(A)} = \mathcal{F}(A)$. Given morphisms $g : A \to B$, $f : B \to C$ in $\mathrm{Hom}(\mathcal{C})$,

$$\mathcal{F}(A) \leq \mathcal{F}(B) \text{ and } \mathcal{F}(B) \leq \mathcal{F}(C),$$

implying $\mathcal{F}(A) \leq \mathcal{F}(C)$, hence $\mathcal{F}(A \xrightarrow{fg} C) = \mathcal{F}(A \xrightarrow{f} B)\mathcal{F}(B \xrightarrow{g} C)$. Integer invariants are useful in a large variety of contexts. Some examples are:

1. Dimension (of vector spaces) defines an integer invariant of objects in $\mathrm{Vect}^{\mathrm{fd}}$ with injective maps.

2. Transcendence degree over a field K of field extensions over K is an integer the category of algebraic extensions over K with embeddings. A closely related notion, when K is algebraically closed, is dimension of varieties in Var_K.

3. For a commutative ring R, $\mid$ defines a preorder on $R[x_1, \ldots, x_n] \setminus 0$. An integer invariant on the category associated with this poset is the degree.

4. Genus in the category of category of compact smooth manifolds.

5. For a set X, the finite subsets of X is a poset with respect to inclusion

6. For the the posets $(\mathbb{R}, \leq)$ or $(\mathbb{Q}, \leq)$, the floor- and ceiling functions define an integer invariant.

One can generalize the above to any poset X to a notion of *invariants over X of objects in $\mathcal{C}$*, which is useful if we for intance want to consider invariants over $\mathbb{Z} \cup \{\infty\}$, relevant to for instance dimension of vector spaces in Vect, degree for the poset $(R[x_1, \ldots, x_n], |)$ with the convention that deg $0 = \infty$ or cardinality for a family of subsets of some X. For a measure space (X, Ω, μ), inclusion defines a poset on Ω and μ defines an invariant over $[0, \infty]$ of Ω-measurable sets.

Example D.0.46. The assignment

$$\mathcal{P} : \text{Set} \to \text{Set}$$
$$X \mapsto \mathcal{P}(X) := \{A : A \subset X\}$$
$$(f : X \to Y) \mapsto (A \mapsto f(A))$$

where for each $A \in \mathcal{P}(X)$,

$$f(A) := \{f(x) \in Y : x \in A\}$$

defines a functor. Indeed, for $f : X \to Y$, $g : Y \to Z$,

$$gf(A) = \{gf(x) \in Z : x \in A\} = \{g(f(x)) \in Z : x \in A\} = g(\{f(x) \in Y : x \in A\}) = g(f(A)).$$

Moreover,

$$\text{id}_X(A) = A = \text{id}_{\mathcal{P}(X)}(A).$$

Example D.0.47. Given any function $f : X \to Y$, when considering both X and Y with the structure of discrete category, we get a functor

$$X \to Y$$
$$x \mapsto f(x)$$
$$\mathbb{1}_x \mapsto \mathbb{1}_{f(x)}$$

Example D.0.48. Consider a pair of monoids M, N and a functor $\mathcal{F} : \text{B}M \to [B]N$ this is just a function $M \to N, x \mapsto \mathcal{F}(x)$. By functoriality $\mathcal{F}(e_M) = e_N$ and $\mathcal{F}(xy) = \mathcal{F}(x)\mathcal{F}(y)$, so in particular $\mathcal{F}$ corresponds to a monoid homomorphism. Conversely, any monoid homomorphism gives rise to functor of monoids. Similar considerations can be made to a ring R.

Definition D.0.49. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is called an *endofunctor*.

Definition D.0.50. For each subcategory $\mathcal{C}$ of Set there is a *forgetful functor*

$$\mathcal{C} \to \text{Set}$$
$$X \mapsto X$$
$$f : X \to Y \mapsto f : X \to Y$$

In general, given a pair of subcategories of Set, $\mathcal{C}$ and $\mathcal{D}$ say, such that $\mathcal{D}$ is a subcategory of $\mathcal{C}$, we can construct a forgetful functor from $\mathcal{D}$ to $\mathcal{C}$ in the same fashion.

Example D.0.51. Consider for instance the forgetful functor taking rings $(R, +, \cdot)$ to the underlying set R, or intermediately, the underlying additive group $(R, +)$ in Group and ring maps to maps or intermediately to group maps.

Example D.0.52. Another example is the forgetful functor

$$\text{Mon} \to \text{Set}_*$$
$$(M, \cdot, e) \mapsto (M, e)$$
$$f : (M, \cdot_M, e_M) \to (N, \cdot_N, e_N) \mapsto (f : (M, e_M) \to (N, e_N)), x \mapsto f(x)$$

which is well-defined since monoid homomorphisms fix the neutral element.

Example D.0.53. Given an algebraically closed field K, there is a contravariant functor

$$\Gamma : \text{Var}_K \to \text{FinGenAlg}_K$$
$$X \mapsto \Gamma(X)$$
$$\varphi : X \to Y \mapsto \widetilde{\varphi} : \Gamma(Y) \to \Gamma(X)$$

where FinGenAlg_K is the category of finitely generated algebras over K.

Definition D.0.54. Consider a locally small category $\mathcal{C}$. We then for each object X in $\mathcal{C}$ get a covariant functor

$$\text{Hom}(X, _) : \mathcal{C} \to \text{Set}$$
$$Y \mapsto \text{Hom}(X, Y)$$
$$f : A \to B \mapsto f_* : \text{Hom}(X, A) \to \text{Hom}(X, B)$$

and a contravariant functor

$$\text{Hom}(_, X) : \mathcal{C} \to \text{Set}$$
$$Y \mapsto \text{Hom}(Y, X)$$
$$f : A \to B \mapsto f^* : \text{Hom}(B, X) \to \text{Hom}(A, X)$$

We say that these are the functors *represented by* X.

Remark D.0.55. Let $f : B \to C$ and $g : A \to B$ be given. Then for each $h \in \text{Hom}(X, A)$

$$(fg)_*(h) = (fg)h = f(gh) = f_*(gh) = f_*(g_*(h)) = f_*g_*(h) \implies (fg)_* = f_*g_*.$$

One easily checks that $(\mathbb{1}_A) = \mathbb{1}_{\text{Hom}(X,A)}$. This verifies that $\text{Hom}(X, _)$ is a covariant functor. Note that $\text{Hom}(_, X)$ is just the covariant functor

$$\text{Hom}(X, _) : \mathcal{C}^{\text{op}} \to \text{Set}$$

and is therefor a contravariant functor from $\mathcal{C} \to \text{Set}$.

Definition D.0.56. For categories $\mathcal{C}$ and $\mathcal{C}'$ we define the *the product category of $\mathcal{C}$ and $\mathcal{C}'$* to be category $\mathcal{C} \times \mathcal{C}'$ whose objects are ordered pairs (X, X') where X is an object of $\mathcal{C}$ and X' is an object of $\mathcal{C}'$ and whose morphisms are ordered pairs of morphisms, denoted $(f, f') : (X, X') \to (Y, Y')$.
Given a pair of morphisms in the product, $(f, f') : (X, X') \to (Y, Y')$ and $(g, g') : (Y, Y') \to (Z, Z')$, composition is defined as

$$(g, g')(f, f') := (gf, g'f') : (X, X') \to (Z, Z').$$

Given an object (A, A'), we define $\mathbb{1}_{(A,A')} = (\mathbb{1}_A, \mathbb{1}_{A'})$ How do we know such ordered pairs exist in general?

Remark D.0.57. Consider morphisms $(f, f') : (Z, Z') \to (W, W')$, $(g, g') : (Y, Y') \to (Z, Z')$ and $(h, h') : (X, X') \to (Y, Y')$. Then

$$\begin{aligned}(f, f')((g, g')(h, h')) = (f, f')(gh, g'h') = (f(gh), f'(g'h')) = ((fg)h, (f'g')h') \\ = (fg, f'g')(h, h') = ((f, f')(g, g'))(h, h').\end{aligned}$$

Moreover,

$$(h, h')\mathbb{1}_{(X,X')} = (h\mathbb{1}_X, h'\mathbb{1}_{X'}) = (h, h') = (\mathbb{1}_Y h, \mathbb{1}_{Y'}h') = \mathbb{1}_{(Y,Y')}(h, h').$$

One can thus conclude that $\mathcal{C} \times \mathcal{C}'$

Definition D.0.58. Given a locally small category $\mathcal{C}$ we define the *two-sides represented functor*

$$\begin{gathered}\text{Hom}(_, _) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set} \\ (X, Y) \mapsto \text{Hom}(X, Y) \\ (f, h) : (A, B) \to (C, D) \mapsto (f^*, h_*) : \text{Hom}(A, B) \to \text{Hom}(C, D)\end{gathered}$$

where $f : C \to A$ as a morphism in $\mathcal{C}$ and $(f^*, h_*)(g) := hgf$ for $g \in \text{Hom}(A, B)$.

Lemma D.0.59. *Consider categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and covariant functors, $\mathcal{F} : \mathcal{C} \to \mathcal{D}, \mathcal{G} : \mathcal{D} \to \mathcal{E}$. Then*

$$\mathcal{G}\mathcal{F} : \mathcal{C} \to \mathcal{E}$$
$$X \mapsto \mathcal{G}(\mathcal{F}(X))$$
$$f : Y \to Z \mapsto \mathcal{G}(\mathcal{F}(f)) : \mathcal{G}(\mathcal{F}(Y)) \to \mathcal{G}(\mathcal{F}(Z))$$

is a functor called the composition of $\mathcal{G}$ with $\mathcal{F}$.

Proof. Indeed, for morphisms $f : X \to Y$ and $g : Y \to Z$,

$$\begin{aligned}(\mathcal{G}\mathcal{F})(gf) &= \mathcal{G}(\mathcal{F}(gf)) = \mathcal{G}(\mathcal{F}(g)\mathcal{F}(f)) = \mathcal{G}(\mathcal{F}(g))\mathcal{G}(\mathcal{F}(f)) \\ &= (\mathcal{G}\mathcal{F})(g)(\mathcal{G}\mathcal{F})(f).\end{aligned}$$

Moreover,

$$(\mathcal{G} \circ \mathcal{F})(\mathbb{1}_X) = \mathcal{G}(\mathcal{F}(\mathbb{1}_X)) = \mathcal{G}(\mathbb{1}_{\mathcal{F}(X)}) = \mathbb{1}_{\mathcal{G}(\mathcal{F}(X))} = \mathbb{1}_{(\mathcal{G}\mathcal{F})(X)}.$$

□

Lemma D.0.60. *Composition of functors is associative.*

Proof. Consider functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}, \mathcal{G} : \mathcal{B} \to \mathcal{C}, \mathcal{H} : \mathcal{A} \to \mathcal{B}$. Given an object A in $\mathcal{A}$,

$$\begin{aligned}&(\mathcal{F} \circ (\mathcal{G} \circ \mathcal{H}))(A) = \mathcal{F}((\mathcal{G} \circ \mathcal{H})(A)) = \mathcal{F}(\mathcal{G}(\mathcal{H}(A))) = (\mathcal{F} \circ \mathcal{G})(\mathcal{H}(A)) \\ &((\mathcal{F} \circ \mathcal{G}) \circ \mathcal{H})(A).\end{aligned}$$

Once also easily checks that for each morphism $f : X \to Y$

$$(\mathcal{F} \circ (\mathcal{G} \circ \mathcal{H}))(f) = ((\mathcal{F} \circ \mathcal{G}) \circ \mathcal{H})(f),$$

hence

$$\mathcal{F} \circ (\mathcal{G} \circ \mathcal{H}) = (\mathcal{F} \circ \mathcal{G}) \circ \mathcal{H}.$$

□

Remark D.0.61. In addition to functor composition being associative, there is an identity functor for each category

$$\mathcal{I}\mathcal{D}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$
$$X \mapsto X$$
$$f : Y \to Z \mapsto f : Y \to Z$$

for which

$$\mathcal{F}\mathcal{I}\mathcal{D}_{\mathcal{C}} = \mathcal{F} = \mathcal{I}\mathcal{D}_{\mathcal{D}}\mathcal{F}.$$

Example D.0.62. Note that it is not true that if a functor maps a morphism to an isomorphism, then said morphism is an isomorphism. Indeed, consider for example any functor $\mathcal{F}$ from a category $\mathcal{C}$ to the category with one object and one morphism (which will need to be the identity morphism), $\mathbb{1}$. This will take any object to the one object and any morphism to identity morphism.

Definition D.0.63. Let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{E} \to \mathcal{C}$ be functors. We define the *comma category* $\mathcal{F} \downarrow \mathcal{G}$ whose objects are triples $(D \in \mathcal{D}, E \in \mathcal{E}, f : \mathcal{F}(D) \to \mathcal{G}(E))$. A morphism $(D, E, f) \to (D', E', f')$ is a pair of morphisms $(h : D \to D', k : E \to E')$ so that

$$\begin{array}{ccc} \mathcal{F}(D) & \xrightarrow{f} & \mathcal{G}(E) \\ \downarrow{\scriptstyle \mathcal{F}(h)} & & \downarrow{\scriptstyle \mathcal{G}(k)} \\ \mathcal{F}(D') & \xrightarrow{f'} & \mathcal{G}(E') \end{array}$$

commutes

Remark D.0.64. Consider morphisms

$$(h : D \to D', k : E \to E') : (D, E, f) \to (D', E', f')$$

and

$$(h' : D' \to D'', k' : E' \to E'') : (D', E', f') \to (D'', E'', f'').$$

We define composition by

$$(h', k')(h, k) := (h'h, k'k)$$

To see that this is well defined note that

$$\begin{array}{ccc} \mathcal{F}(D) & \xrightarrow{f} & \mathcal{G}(E) \\ \downarrow{\scriptstyle \mathcal{F}(h'h)} & & \downarrow{\scriptstyle \mathcal{G}(k'k)} \\ \mathcal{F}(D'') & \xrightarrow{f''} & \mathcal{G}(E'') \end{array}$$

commutes since $\mathcal{F}(h'h) = \mathcal{F}(h')\mathcal{F}(h)$, $\mathcal{G}(k'k) = \mathcal{G}(k')\mathcal{G}(k)$ and the diagram

$$\begin{array}{ccc} \mathcal{F}(D) & \xrightarrow{f} & \mathcal{G}(E) \\ \downarrow{\scriptstyle \mathcal{F}(h)} & & \downarrow{\scriptstyle \mathcal{G}(k)} \\ \mathcal{F}(D') & \xrightarrow{f'} & \mathcal{G}(E') \\ \downarrow{\scriptstyle \mathcal{F}(h')} & & \downarrow{\scriptstyle \mathcal{G}(k')} \\ \mathcal{F}(D'') & \xrightarrow{f''} & \mathcal{G}(E'') \end{array}$$

commutes. Clearly composition is associative. Define the identity morphism for (D, E, f) to be $(\mathbb{1}_D, \mathbb{1}_E)$. Note that this is indeed a morphism in $\mathcal{F} \downarrow \mathcal{G}$, since

$$\begin{array}{ccc} \mathcal{F}(D) & \xrightarrow{f} & \mathcal{G}(E) \\ {\scriptstyle \mathcal{F}(\mathbb{1}_D)=\mathbb{1}_{\mathcal{F}(D)}}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(\mathbb{1}_E)=\mathbb{1}_{\mathcal{G}(E)}} \\ \mathcal{F}(D) & \xrightarrow{f} & \mathcal{G}(E) \end{array}$$

clearly commutes. It is also clear that this is the identity morphism in $\mathcal{F} \downarrow \mathcal{G}$.

Definition D.0.65. Given functors $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{E} \to \mathcal{C}$ we define projection functors

$$\begin{gathered} \text{dom} : \mathcal{F} \downarrow \mathcal{G} \to \mathcal{D} \\ (D, E, f) \mapsto D \\ (h, k) : (D_1, E_1, f_1) \to (D_2, E_2, f_2) \mapsto h : D_1 \to D_2 \end{gathered}$$

and

$$\begin{gathered} \text{cod} : \mathcal{F} \downarrow \mathcal{G} \to \mathcal{E} \\ (D, E, f) \mapsto E \\ (h, k) : (D_1, E_1, f_1) \to (D_2, E_2, f_2) \mapsto k : E_1 \to E_2 \end{gathered}$$

Remark D.0.66. Note that $C/\mathcal{C}$ is just a comma category; $\mathcal{F}_C \downarrow \mathcal{ID}_{\mathcal{C}}$, where $\mathcal{F}_C : \mathbb{1} \to \mathcal{C}$ is the constant functor from the category with one object and one morphism and $\mathcal{ID}_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$. Indeed an object in this category is a triple $(* \in \mathbb{1}, X \in \mathcal{C}, f : C \to X)$, note that there is a unique such triple when given any morphism $f : C \to X$, so we may represent such a triple more succinctly by just providing a morphism. A morphism in $\mathcal{F}_C \downarrow \mathcal{ID}_{\mathcal{C}}$ from $f : C \to X$ to $g : C \to Y$ is a pair $(h : X \to Y, \text{id}_C : C \to C)$ such that

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle \text{id}_C}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{g} & Y \end{array}$$

commutes. Note that the second data point in this pair is redundant and that this is equivalent to

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ X & & \xrightarrow{h} & & Y \end{array}$$

commuting. So the data specifying the two categories is the same. Note that projection functor simply map a morphism $f : C \to X$ to its domain and codomain respectively. We can also construct $\mathcal{C}/C$ as $\mathcal{ID}_{\mathcal{C}} \downarrow \mathcal{F}_C$.

D.0.4 Functors on Groups

Fix a group G and a category $\mathcal{C}$. A functors $\mathrm{B}G \to \mathcal{C}$, maps G to an object X in $\mathcal{C}$ and a group element $g \in G$ to an endomorphism $g_\star : X \to X$ with

$$(gh)_\star = g_\star h_\star$$

and $e_\star = \mathbb{1}_X$. Such a functor is called an *action*. When $\mathcal{C} = \mathrm{Set}$, it is a G-set. When $\mathcal{C} = \mathrm{Vect}_K$ for some field K it is called a *G-representation*. Each endomorphism induced by G is an isomorphism, again, by functoriality.

D.0.5 Categories of Categories

When our foundations are suitably developed we may consider a category of categories when we have the last lemma of the prior section in mind. Naively one cannot consider such a category, since the question of whether such a category is an object in the category of categories leads to an instance of Russel's paradox. Instead one needs to consider some more refined concepts such as the following,

Proposition D.0.67. *Consider the class of small categories with morphisms being functors. This defines a locally small category denoted* Cat.

Proof. Note that the class of subclasses of the proper class of all sets is itself a class (in e.g. Morse-Kelley set theory). Note that a functor between categories, whose objects and class of morphisms are sets, is a function. I.e. such a functor is a function

$$\mathrm{Ob}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}') \times \mathrm{Hom}(\mathcal{C}').$$

Then the class of all such functors is a subclass of the class of all functions $?$. By Lemma D.0.60 and Remark D.0.61, Cat is a category and it is locally small since $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is a set for each pair of small categories $\mathcal{C}, \mathcal{D}$. □

Remark D.0.68. The collection of all locally small categories with functors as morphisms, which will be denoted CAT, will similarly define a category, which won't be locally small.

From the above we get a notion of isomorphism of categories with this we have for instance the following result

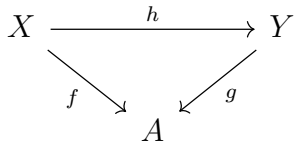
Proposition D.0.69. *Given a category $\mathcal{C}$ and an object A, then*

$$\mathcal{C}/A \simeq (A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

Proof. Define a functor

$$\mathcal{F} : \mathcal{C}/A \to (A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

which maps an object $f : X \to A$ to itself. Note that the objects of $(A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is the same as the objects of $(A/\mathcal{C}^{\mathrm{op}})$, which are morphisms of $\mathcal{C}^{\mathrm{op}}$ with domain A. Such morphisms are exactly morphisms of $\mathcal{C}$ with codomain A. It follows that the assignment of objects is well-defined. Moreover $\mathcal{F}$ maps a morphism



to itself. Indeed, a morphism from $\lambda \in \mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(A, X)$ to $\mu \in \mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(B, X)$ in $(A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is a morphism in $A/\mathcal{C}^{\mathrm{op}}$ from μ to λ, $\nu \in \mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(Y, X)$ such that $\lambda = \nu \circ^{\mathrm{op}} \mu$. In $\mathcal{C}$, this corresponds to $\nu : X \to Y$ being a morphism satisfying $\lambda = \mu \circ \nu$. This is exactly a morphism in $\mathcal{C}/A$. It follows that $\mathcal{F}$ is well-defined on morphisms as well. The above arguments also clearly shows that the functor

$$\mathcal{G} : (A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}/A$$

taking objects and morphisms to themselves gives a well-defined functor whose left and right inverse is $\mathcal{F}$. It follows that $\mathcal{C}/A \simeq (A/\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ at least when this category can be shown to live in some category. □

Example D.0.70. Another example of an isomorphism of categories is the functor

$$(_)^{\mathrm{op}} : \mathrm{CAT} \to \mathrm{CAT}$$
$$\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$$
$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \mapsto \mathcal{F}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$$

where for a given functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$

$$\mathcal{F}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$$
$$X \mapsto \mathcal{F}(X)$$
$$f : X \to Y \mapsto f^{\mathrm{op}} : Y \to X$$

Note that clearly $(_)^{\mathrm{op}}(_)^{\mathrm{op}} = \mathcal{ID}_{\mathrm{CAT}}$.

Example D.0.71. Consider the category Set^{∂} whose objects are sets and whose morphisms are partial functions. For a given set X, we define

$$X_+ := X \cup \{X\}$$

and for a given partial map $f : X \dashrightarrow Y$ we define

$$f_+ : X_+ \to Y_+$$

where

$$f_+(x) := \begin{cases} f(x) & \text{if } x \in \mathrm{dom}\ f \\ Y & \text{otherwise} \end{cases}$$

We thus get a functor

$$\begin{gathered} (_)_+ : \mathrm{Set}^{\partial} \to \mathrm{Set}_* \\ X \mapsto X_+ \\ f : X \dashrightarrow Y \mapsto f : X_+ \to Y_+ \end{gathered}$$

Indeed, given partial maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$, note that we have a partial map $gf : X \dashrightarrow Z$ with

$$\mathrm{dom}\ gf = \mathrm{dom}\ f \cap f^{-1}(\mathrm{dom}\ g).$$

If $x \in \mathrm{dom}\ gf$, then $f(x) \in \mathrm{dom}\ g$, hence

$$(gf)_+(x) = gf(x) = g(f(x)) = g_+(f(x)) = g_+(f_+(x)) = g_+f_+(x).$$

Suppose $x \notin \mathrm{dom}\ gf \subset X_+$. Then $x \notin \mathrm{dom}\ f$ or $x \notin f^{-1}(\mathrm{dom}\ g)$. In the first case $f(x) = Y$, hence

$$(gf)_+(x) = Z = g_+(Y) = g_+(f_+(x)) = g_+f_+(x).$$

Suppose $x \in \mathrm{dom}\ f$. Then $f(x) \notin \mathrm{dom}\ g$, hence $g_+(f_+(x)) = Z$, hence

$$(gf)_+(x) = Z = g_+(f_+(x)) = g_+f_+(x).$$

We conclude that $(gf)_+ = g_+f_+$. We define a left inverse to $(_)_+$, by

$$\begin{gathered} \mathcal{U} : \mathrm{Set}_* \to \mathrm{Set}^{\partial} \\ (X, x) \mapsto X \setminus \{x\} \\ f : (X, x) \to (Y, y) \mapsto \mathcal{U}(f) : X \setminus \{x\} \dashrightarrow Y \setminus \{y\} \end{gathered}$$

where

$$\operatorname{dom} \mathcal{U}(f) := X \setminus f^{-1}(y)$$

and on the domain of $\mathcal{U}(f)$, $\mathcal{U}(f)(a) = f(a)$. On objects X,

$$\mathcal{U}(X_+) = \mathcal{U}(X \cup \{X\}, X) = (X \cup \{X\}) \setminus \{X\} = X.$$

and on morphisms $f : X \dashrightarrow Y$,

$\operatorname{dom} \mathcal{U}(f_+) = (X \cup \{X\}) \setminus (f_+)^{-1}(Y) = (X \cup \{X\}) \setminus (X \setminus \operatorname{dom} f \cup \{X\}) = \operatorname{dom} f \cap X = \operatorname{dom} f$.

Moreover for $x \in \operatorname{dom} f$

$$\mathcal{U}(f_+)(x) = f_+(x) = f(x),$$

hence

$$\mathcal{U}(f_+) = f.$$

On the other hand

$$(\mathcal{U}(X, x))_+ = (X \setminus \{x\})_+ = (X \setminus \{x\}) \cup \{X \setminus \{x\}\}.$$

Thus $\mathcal{U}$ and $(_)_+$ are not mutual inverses. However,

$$(X \setminus \{x\}) \cup \{X \setminus \{x\}\}$$
$$X \ni y \mapsto y$$
$$X \setminus \{x\} \mapsto x$$

Defines a bijection. So every set $X \in \mathrm{Set}^{\partial}$ is isomorphic to $\mathcal{U}(Y, y)$ for some $(Y, y) \in \mathrm{Set}_+$. Moreover, given a map of pointed sets $f : (X \cup \{X\}, X) \to (Y \cup \{Y\}, Y)$, $f|_{X \setminus f^{-1}(Y)} : X \setminus f^{-1}(Y) \to Y$ gives rise to partial map $g : X \dashrightarrow Y$ where for $x \in \operatorname{dom} g$,

$$g_+(x) = f(x)$$

and for $x \notin \operatorname{dom} g$,

$$g_+(x) = Y = f(x).$$

It thus follows that $g_+ = f$. Additionally, suppose for partial maps $f, g : X \dashrightarrow Y$ that

$$f_+ = g_+.$$

If $x \in \operatorname{dom} f$, then $Y \neq f(x) = f_+(x) = g_+(x)$, hence $g_+(x) = g(x)$, meaning $x \in \operatorname{dom} g$. It follows by symmetry that $\operatorname{dom} f = \operatorname{dom} g$ and thus readily that $f = f_+|_{\operatorname{dom} f} = g_+|_{\operatorname{dom} g} = g$. Later we will see that this shows that $(_)_+$ and $\mathcal{U}$ defines and equivalence of categories between Set^{∂} and Set_+.

D.0.6 Natural Transformations

Definition D.0.72. A *natural transformation* of functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$,

$$\alpha : \mathcal{F} \Rightarrow \mathcal{G}$$

is a collection of morphisms

$$\{\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X) : X \in \text{Ob}(\mathcal{C})\}$$

called *components* such that for any morphism $f : X \to X'$,

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\alpha_X} & \mathcal{G}(X) \\ \downarrow{\scriptstyle \mathcal{F}(f)} & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(X') & \xrightarrow{\alpha_{X'}} & \mathcal{G}(X') \end{array}$$

commutes

Remark D.0.73. A natural transformation is also denoted

$$\begin{array}{ccc} & \overset{\mathcal{F}}{\curvearrowright} & \\ \mathcal{C} & \Downarrow \alpha & \mathcal{D} \\ & \underset{\mathcal{G}}{\curvearrowright} & \end{array}$$

Definition D.0.74. A *natural isomorphism* is a natural transformation for which every component is an isomorphism. If two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ are naturally isomorphic via a natural transformation α, we write $\alpha : \mathcal{F} \cong \mathcal{G}$

Lemma D.0.75. *Suppose $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural isomorphism with components $\{\alpha_X\}$. Then $\{\alpha_X^{-1}\}$ defines the components of a natural isomorphism $\alpha^{-1} : \mathcal{G} \Rightarrow \mathcal{F}$.*

Proof. Let $f : X \to X'$ be a morphism. We need to check that the diagram

$$\begin{array}{ccc} \mathcal{G}(X) & \xrightarrow{\alpha_X^{-1}} & \mathcal{F}(X) \\ \downarrow{\scriptstyle \mathcal{G}(f)} & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathcal{G}(X') & \xrightarrow{\alpha_{X'}^{-1}} & \mathcal{F}(X') \end{array}$$

commutes. Indeed, using the fact that $\alpha_{X'}\mathcal{F}(f) = \mathcal{G}(f)\alpha_X$,

$$\mathcal{F}(f)\alpha_X^{-1} = \alpha_{X'}^{-1}\alpha_{X'}\mathcal{F}(f)\alpha_X^{-1} = \alpha_{X'}^{-1}\mathcal{G}(f)\alpha_X\alpha_X^{-1} = \alpha_{X'}^{-1}\mathcal{G}(f).$$

□

Lemma D.0.76. *Let $\mathcal{C}$ be a locally small category. Consider a morphism $f \in \mathrm{Hom}(X,Y)$. Then the collection of maps*

$$\{f_* : \mathrm{Hom}(Z,X) \to \mathrm{Hom}(Z,Y), \mathrm{Hom}(Z,X) \ni g \mapsto fg \in \mathrm{Hom}(Z,Y) : Z \in \mathrm{Ob}(\mathcal{C})\}$$

defines a natural transformation $f_* : \mathrm{Hom}(_,X) \Rightarrow \mathrm{Hom}(_,Y)$. By duality $f^* : \mathrm{Hom}(Y,_) \Rightarrow \mathrm{Hom}(X,_)$ defines a natural transformation. Moreover if $f \neq f'$ for parallel morphisms f, f', then $f_* \neq f'_*$

Proof. Let a morphism $h : A \to B$ be given. Then

$$\begin{array}{ccc} \mathrm{Hom}(A,X) & \xrightarrow{{}_A f_*} & \mathrm{Hom}(A,Y) \\ \downarrow{\scriptstyle {}_Y h^*} & & \downarrow{\scriptstyle {}_X h^*} \\ \mathrm{Hom}(B,X) & \xrightarrow{{}_B f_*} & \mathrm{Hom}(B,Y) \end{array}$$

commutes by associativity of composition. Suppose $f \neq f'$. Then ${}_X f_*(\mathbb{1}_X) \neq {}_X f'_*(\mathbb{1}_X)$, hence ${}_X f_* \neq {}_X f'_*$, which implies $f_* \neq f'_*$. □

Example D.0.77. Let V a vector space over some field K. Let for any vector space W over K, W^* denote $\mathrm{Hom}(W,K)$, *the dual vector space of* W, i.e. $(_)^* = \mathrm{Hom}(_,K)$. Consider the linear maps

$$\begin{aligned} {}_V\mathrm{ev} : V &\to V^{**} \\ v &\mapsto \mathrm{ev}_v \end{aligned}$$

where ev_v is the linear map

$$\begin{aligned} V^* &\to K \\ (l : V \to K) &\mapsto l(v) \end{aligned}$$

The collection $\{{}_W\mathrm{ev} : W \to W^{**} : W \in \mathrm{Ob}(\mathrm{Vect}_K)\}$, defines a natural transformation of $\mathcal{ID}_{\mathrm{Vect}_K}$ with $(_)^{**}$. The square

$$\begin{array}{ccc} V & \xrightarrow{{}_V\mathrm{ev}} & V^{**} \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l^{**}} \\ W & \xrightarrow{{}_W\mathrm{ev}} & W^{**} \end{array}$$

commutes. Indeed,

$$\begin{aligned} ((_W\mathrm{ev}l)(v))(\phi) = \mathrm{ev}_{l(v)}(\phi) = \phi(l(v)) = \phi l(v) = \mathrm{ev}_v(\phi l) = (\mathrm{ev}_v l^*)(\phi) = (l^{**}(\mathrm{ev}_v))(\phi) \\ = ((l^{**}{}_V\mathrm{ev})(v))(\phi) \end{aligned}$$

D.0.7 Equivalence of Categories

Definition D.0.78. Let $\mathbb{2}$ denote the category with two objects $0, 1$ and a single non-identity arrow $0 \to 1$.

We consider two functors

$$\iota_i : \mathbb{1} \to \mathbb{2}$$
$$* \mapsto i$$
$$\mathrm{id}_* \mapsto \mathrm{id}_i$$

for $i = 0, 1$.

Lemma D.0.79. *Fix functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$. There is a bijection of natural transformations $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ and* ***witnessing functors*** *$\mathcal{H} : \mathcal{C} \times \mathbb{2} \to \mathcal{D}$ such that*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\mathcal{I}_0} & \mathcal{C} \times \mathbb{2} & \xleftarrow{\mathcal{I}_1} & \mathcal{C} \\ & \searrow^{\mathcal{F}} & \downarrow{\scriptstyle \mathcal{H}} & \swarrow_{\mathcal{G}} & \\ & & \mathcal{D} & & \end{array}$$

commutes. Here $\mathcal{I}_i : X \mapsto (X, i), f : Y \to Z \mapsto (f, \mathrm{id}_i)$

Proof. Let a functor $\mathcal{H}$ be given as described. For each object $X \in \mathcal{C}$, define

$$\alpha^{\mathcal{H}}_X = \mathcal{H}(\mathrm{id}_X, 0 \to 1)$$

This is a morphism with domain

$$\mathcal{H}(X, 0) = \mathcal{H}\mathcal{I}_0(X) = \mathcal{F}(X)$$

and codomain

$$\mathcal{H}(X, 1) = \mathcal{H}\mathcal{I}_1(X) = \mathcal{G}(X).$$

We see that the collection

$$\left\{\alpha^{\mathcal{H}}_X : \mathcal{F}(X) \to \mathcal{G}(X) \mid X \in \mathcal{C}\right\}$$

defines the components of a natural transformation, as

$$\begin{aligned} \mathcal{G}(f)\alpha_X &= \mathcal{G}(f)\mathcal{H}(\mathrm{id}_X, 0 \to 1) = \mathcal{H}\mathcal{I}_\infty(f)\mathcal{H}(\mathrm{id}_X, 0 \to 1) \\ &= \mathcal{H}(f, \mathrm{id}_1)\mathcal{H}(\mathrm{id}_X, 0 \to 1) \\ &= \mathcal{H}(f, 0 \to 1) = \mathcal{H}(\mathrm{id}_{X'}, 0 \to 1)\mathcal{H}(f, \mathrm{id}_0) \\ &= \alpha_{X'}\mathcal{H}\mathcal{I}_0(f) = \alpha_{X'}\mathcal{F}(f). \end{aligned}$$

We denote this natural transformation by $\alpha^{\mathcal{H}}$. Given a natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$, define a functor

$$\begin{gathered}\mathcal{H}_\alpha : \mathcal{C} \times \mathbb{2} \to \mathcal{D} \\ (X, 0) \mapsto \mathcal{F}(X) \\ (X, 1) \mapsto \mathcal{G}(X) \\ (f, \mathrm{id}_0) \mapsto \mathcal{F}(f) \\ (f, \mathrm{id}_1) \mapsto \mathcal{G}(f) \\ (f : X \to Y, 0 \to 1) \mapsto \alpha_X\end{gathered}$$

One readily verifies that $\mathcal{F} = \mathcal{H}\mathcal{I}_0$ and that $\mathcal{G} = \mathcal{H}\mathcal{I}_1$. Note that by definition

$$\alpha_X^{\mathcal{H}_\alpha} = \mathcal{H}(\mathrm{id}_X, 0 \to 1) = \alpha_X.$$

Suppose we are given two witnessing functors $\mathcal{H}, \mathcal{H}'$ with $\alpha^{\mathcal{H}} = \alpha^{\mathcal{H}'}$. For each object $X \in \mathcal{C}$

$$\mathcal{H}(X, 0) = \mathcal{H}\mathcal{I}_0(X) = \mathcal{F}(X) = \mathcal{H}'\mathcal{I}_0(X) = \mathcal{H}'(X, 0).$$

By a similar computation $\mathcal{H}(X, 1) = \mathcal{H}'(X, 1)$. Moreover, similar computations show that $\mathcal{H}(f, \mathrm{id}_i) = \mathcal{H}'(f, \mathrm{id}_i)$ for $i = 0, 1$. It remains to check that $\mathcal{H}(f, 0 \to 1) = \mathcal{H}'(f, 0 \to)$ which readily follows from the fact that $\alpha_X^{\mathcal{H}} = \alpha_X^{\mathcal{H}'}$ for each X. □

Definition D.0.80. Consider categories $\mathcal{C}$ and $\mathcal{D}$. An *equivalence of categories* is a pair of functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha : \mathcal{I}\mathcal{D}_{\mathcal{C}} \cong \mathcal{G}\mathcal{F}$ and $\beta : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{F}\mathcal{G}$. We say two categories are *equivalent* if there is an equivalence of categories between them and in this case we may write $\mathcal{C} \cong \mathcal{D}$.

Remark D.0.81. Equivalence of categories is an equivalence relation.

- Two copies of $\mathcal{I}\mathcal{D}_{\mathcal{C}}$ together with two copies of components $\{\mathrm{id}_X : X \in \mathcal{C}\}$ defines an equivalence of categories of $\mathcal{C}$ with itself.
- Suppose $\mathcal{C} \cong \mathcal{D}$. Then there are functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : \mathcal{I}\mathcal{D}_{\mathcal{C}} \cong \mathcal{G}\mathcal{F}$ and $\beta : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{F}\mathcal{G}$. This data by definition also expresses that $\mathcal{D} \cong \mathcal{C}$, when we implicitly use the commutativity of conjuction.
- Suppose $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{D} \cong \mathcal{E}$. There are then functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : \mathcal{I}\mathcal{D}_{\mathcal{C}} \cong \mathcal{G}\mathcal{F}$ and $\beta : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{F}\mathcal{G}$. Moreover there are functors $\mathcal{H} : \mathcal{D} \to \mathcal{E}$, $\mathcal{I} : \mathcal{E} \to \mathcal{D}$ with natural isomorphisms $\gamma : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{I}\mathcal{H}$ and $\kappa : \mathcal{I}\mathcal{D}_{\mathcal{E}} \cong \mathcal{H}\mathcal{I}$. Set

$$\mathfrak{F} := \mathcal{H}\mathcal{F} \text{ and } \mathfrak{G} := \mathcal{G}\mathcal{I}.$$

Consider

$$\mu := \left\{ \mu_X := \mathcal{G}(\gamma_{\mathcal{F}(X)})\alpha_X : X \to \mathfrak{G}\mathfrak{F}(X) \mid X \in \mathcal{C} \right\}$$

This is a natural isomorphism. Indeed,

$$\begin{aligned}
\mathfrak{G}\mathfrak{F}(f)\mu_X &= \mathcal{G}\mathcal{I}\mathcal{H}\mathcal{F}(f)\mathcal{G}(\gamma_{\mathcal{F}(X)})\alpha_X = \mathcal{G}(\mathcal{I}\mathcal{H}(\mathcal{F}(f))\gamma_{\mathcal{F}(X)})\alpha_X \\
&= \mathcal{G}(\gamma_{\mathcal{F}(X')}\mathcal{F}(f))\alpha_X = \mathcal{G}(\gamma_{\mathcal{F}(X')})\mathcal{G}\mathcal{F}(f)\alpha_X \\
&= \mathcal{G}(\gamma_{\mathcal{F}(X')})\alpha_{X'} f = \mu_{X'} f.
\end{aligned}$$

so μ is a natural transformation of $\mathcal{I}\mathcal{D}_{\mathcal{C}}$ to $\mathfrak{G}\mathfrak{F}$. Moreover, every component of μ is a composition of isomorphism, hence every component of μ is an isomorphism. We construct a natural isomorphims $\nu : \mathcal{I}\mathcal{D}_{\mathcal{E}} \cong \mathfrak{F}\mathfrak{G}$, by considering the collection of morphisms

$$\nu := \left\{ \nu_Y := \mathcal{H}(\beta_{\mathcal{I}(Y)})\kappa_Y : Y \to \mathcal{H}\mathcal{F}\mathcal{G}\mathcal{I}(Y) \mid Y \in \mathcal{E} \right\}.$$

We thus conclude that $\mathcal{C} \cong \mathcal{E}$.

Definition D.0.82. Let $\mathcal{C}, \mathcal{D}$ be locally small categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ a functor. We say that $\mathcal{F}$ is *full* if $\mathcal{F}(\bullet) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y)), f \mapsto \mathcal{F}(f)$ is surjective for each pair of objects X, Y in $\mathcal{C}$. It is *faithful* if $\mathcal{F}(\bullet)$ is injective. If $\mathcal{F}(\bullet)$ is both full and faithful, we say that $\mathcal{F}(\bullet)$ is *fully faithful*.

Definition D.0.83. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is *essentially surjective on objects* if for every $D \in \mathrm{Ob}(\mathcal{D})$ there is a $C \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{F}(C) \simeq D$.

Definition D.0.84. A faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an *embedding* if it is injective on objects, i.e. if $\mathcal{F}(C) = \mathcal{F}(C')$ implies $C = C'$ for every pair of objects $C, C' \in \mathcal{C}$. If $\mathcal{F}$ is fully faithful and injective on objects, then it is a *full embedding*.

Remark D.0.85. Given a full embedding $\mathcal{F} : \mathcal{C} \hookrightarrow \mathcal{D}$, we get a subcategory of $\mathcal{D}$ called *the full subcategory of $\mathcal{D}$ spanned by $\mathcal{F}$*, denoted $\mathcal{F}(\mathcal{C})$ whose objects are $\mathcal{F}(X)$ where $X \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are morphism $f : \mathcal{F}(X) \to \mathcal{F}(Y)$. Incidentally, $f = \mathcal{F}(g)$ for some $g : X \to Y$. So we may aswell characterise morphism as being of the form $\mathcal{F}(g) : \mathcal{F}(X) \to \mathcal{F}(Y)$ where $g : X \to Y$ is a morphism in $\mathcal{C}$. By functoriality, this subclass of morphisms is closed under composition, hence the full subcategory of $\mathcal{D}$ spanned by $\mathcal{F}$ is a subcategory of $\mathcal{D}$.

Lemma D.0.86. *Given a morphism $f : A \to B$ and isomorphisms $\alpha : A \simeq A'$, $\beta : B \simeq B'$ there is a unique morphism $f : A' \to B'$ so that the diagrams*

$$\begin{array}{ccc} A & \xleftarrow{\alpha} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\beta]{} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\beta]{} & B' \end{array} \quad \begin{array}{ccc} A & \xleftarrow{\alpha} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[\beta]{} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[\beta]{} & B' \end{array}$$

commute.

Proof. **Diagram 1:** Pick $f' := \beta f \alpha$. Suppose $f'' : A' \to B'$ is another morphism making the diagram commute. Then $f'' = \beta f \alpha = f'$.
Diagram 2: Note that $f' = \beta f \alpha^{-1}$ is the unique morphism making

$$\begin{array}{ccc} A & \xleftarrow{\alpha^{-1}} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

commute, hence it is the unique morphism such that $\alpha f' = \alpha\alpha^{-1}\beta f = \beta f$.
Diagram 3: Apply diagram 1 to $\alpha : A' \to A$, f and $\beta^{-1} : B' \to B$.
Diagram 4: Apply diagram 1 to α^{-1}, f and β^{-1}. □

Lemma D.0.87. *Consider a fully faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. For each $X, Y \in \mathrm{Ob}(\mathcal{C})$*

$$X \simeq Y \iff \mathcal{F}(X) \simeq \mathcal{F}(Y).$$

Proof. Consider an isomorphism $h : \mathcal{F}(X) \to \mathcal{F}(Y)$. By fullness, there is an $f : X \to Y$ such that $\mathcal{F}(f) = h$. Moreover there is a $g : Y \to X$ such that $\mathcal{F}(g) = h^{-1}$. Then

$$\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f) = h^{-1}h = \mathbb{1}_{\mathcal{F}(X)} = \mathcal{F}(\mathbb{1}_X) \implies gf = \mathbb{1}_X.$$

By symmetry, $fg = \mathbb{1}_Y$. □

Theorem D.0.88. *A functor that is one functor comprising an equivalence of categories is fully faithful and essentially surjective on objects. For locally small categories, assuming the axiom of choice any fully faithful functor that is essentially surjective on objects is one functor in an equivalence of categories.*

Proof. Consider functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ with natural isomorphism $\alpha : \mathcal{I}\mathcal{D}_{\mathcal{C}} \cong \mathcal{G}\mathcal{F}$ and $\beta : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{F}\mathcal{G}$.
Suppose $f, g \in \mathrm{Hom}(X, Y)$ are given such that $\mathcal{F}(f) = \mathcal{F}(g)$. Note that

$$\alpha_Y f = \mathcal{F}(f)\alpha_X = \mathcal{F}(g)\alpha_X = \alpha_Y g \implies f = g$$

since α_Y is an isomorphism and thus a monomorphism. We thus conclude that $\mathcal{F}$ is faithful, and by a symmetric argument so is $\mathcal{G}$.
Consider any morphism $h \in \text{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$. Then we have commutative diagrams

$$\begin{array}{ccc} \mathcal{GF}(X) & \xleftarrow[\alpha_X]{} & X \\ \downarrow{\scriptstyle \mathcal{G}(h)} & & \downarrow{\scriptstyle f} \\ \mathcal{GF}(Y) & \xleftarrow[\alpha_Y]{} & Y \end{array}$$

for some unique $f : X \to Y$. We thus by naturality get a commutative diagram

$$\begin{array}{ccc} \mathcal{GF}(X) & \xleftarrow[\alpha_X]{} & X \\ \downarrow{\scriptstyle \mathcal{GF}(f)} & & \downarrow{\scriptstyle f} \\ \mathcal{GF}(Y) & \xleftarrow[\alpha_Y]{} & Y \end{array}$$

Using the result of Lemma D.0.86 diagram 2 to these two diagrams, by uniqueness, $\mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(h)$. By the faithfulness of $\mathcal{G}$,

$$\mathcal{F}(f) = h.$$

We thus conclude that $\mathcal{F}$ (and $\mathcal{G}$) are fully faithful functors.
Consider an object $D \in \mathcal{D}$. Set $C = \mathcal{G}(D)$. Then

$$\begin{array}{ccc} D & \xrightarrow{\beta_D} & \mathcal{F}(C) = \mathcal{FG}(D) \\ \downarrow{\scriptstyle \text{id}_D} & & \downarrow{\scriptstyle \text{id}_{\mathcal{F}(C)}} \\ D & \xrightarrow{\beta_D} & \mathcal{F}(C) = \mathcal{FG}(D) \end{array}$$

hence $D \simeq \mathcal{F}(C)$.
Consider a fully faithul functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ that is essentially surjective on objects. We construct the functor

$$\mathcal{G} : \mathcal{D} \to \mathcal{C}$$
$$D \mapsto \mathcal{G}(D)$$
$$f : D_1 \to D_2 \mapsto \mathcal{G}(f) : \mathcal{G}(D_1) \to \mathcal{G}(D_2)$$

to make $\mathcal{F}$ and $\mathcal{G}$ equivalence of categories. For objects, we do this by for each $D \in \text{Ob}(\mathcal{D})$ considering

$$\mathcal{C}_D := \{C \in \text{Ob}(\mathcal{C}) : \mathcal{F}(C) \simeq D\}$$

to obtain a family of sets indexed by objects in $\mathcal{D}$

$$\mathcal{X} := \{\mathcal{C}_D : D \in \text{Ob}(\mathcal{D})\}.$$

Since $\mathcal{F}$ is essentially surjective on objects, this is a family of non-empty set, hence by the axiom of choice there is a well-defined assigment

$$\begin{aligned} \mathcal{G} : \mathrm{Ob}(\mathcal{D}) \to \mathcal{X} \to \bigcup \mathcal{X} \subset \mathrm{Ob}(\mathcal{C}) \\ D \mapsto \mathcal{C}_D \mapsto \mathcal{G}(D). \end{aligned}$$

Hence $\mathcal{F}\mathcal{G}(D) \simeq D$. Denote this isomorphism by α_D. Fix a morphism $f : D \to D'$

$$\begin{array}{ccc} \mathcal{F}\mathcal{G}(D) & \xrightarrow{\alpha_D} & D \\ {\scriptstyle \exists! h}\,\vdots\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{F}\mathcal{G}(D') & \xrightarrow[\alpha_{D'}]{} & D' \end{array}$$

Since $\mathcal{F}$ is full there is a morphism $\mathcal{G}(f) : \mathcal{G}(D) \to \mathcal{G}(D')$, such that $\mathcal{F}(\mathcal{G}(f)) = h$. This assignment is well-defined, since if $k : \mathcal{G}(D) \to \mathcal{G}(D')$ is another morphism with $\mathcal{F}(k) = h = \mathcal{F}(\mathcal{G}(f))$, then by faithfulness of $\mathcal{F}$, $k = \mathcal{G}(f)$.
We now check that $\mathcal{G}$ is a functor. Consider morphisms $f : X \to Y$ and $g : Y \to Z$. Note that

$$\mathcal{F}(\mathcal{G}(g)\mathcal{G}(f))\alpha_X = \mathcal{F}(\mathcal{G}(g))\mathcal{F}(\mathcal{G}(f))\alpha_X = \mathcal{F}(\mathcal{G}(g))\alpha_Y f = \alpha_Z g f,$$

hence $\mathcal{G}(g)\mathcal{G}(f) = \mathcal{G}(gf)$. We also have that

$$\mathcal{F}(\mathrm{id}_{\mathcal{G}(X)})\alpha_X = \mathrm{id}_{\mathcal{F}\mathcal{G}(X)}\alpha_X = \alpha_X \mathrm{id}_X = \mathcal{F}(\mathcal{G}(\mathrm{id}_X))\alpha_X \implies \mathrm{id}_{\mathcal{G}(X)} = \mathcal{G}(\mathrm{id}_X).$$

Note that for any morphism $f : D \to D'$ by definition

$$\begin{array}{ccc} D & \xrightarrow{\alpha_D^{-1}} & \mathcal{F}\mathcal{G}(D) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathcal{F}\mathcal{G}(f)} \\ D' & \xrightarrow{(\alpha_{D'})^{-1}} & \mathcal{F}\mathcal{G}(D') \end{array}$$

commutes, so we have a natural isomorphism $\alpha : \mathcal{I}\mathcal{D}_{\mathcal{D}} \simeq \mathcal{F}\mathcal{G}$.
Fix an object $C \in \mathcal{C}$. Since $\mathcal{F}$ is fully faithful, there is an isomorphism $\beta_C : C \to \mathcal{G}\mathcal{F}(C)$ such that $\mathcal{F}(\beta_C) = \alpha^{-1}_{\mathcal{F}(C)} : \mathcal{F}(C) \to \mathcal{F}\mathcal{G}\mathcal{F}(C)$. Fix a morphism $f : C \to C'$. Then

$$\begin{aligned} \mathcal{F}(\mathcal{G}\mathcal{F}(f)\beta_C) &= \mathcal{F}(\mathcal{G}\mathcal{F}(f))\mathcal{F}(\beta_C) = \mathcal{F}\mathcal{G}(\mathcal{F}(f))\alpha^{-1}_{\mathcal{F}(C)} = \alpha^{-1}_{\mathcal{F}(C')}\mathcal{F}(f) = \mathcal{F}(\beta_{C'})\mathcal{F}(f) \\ &= \mathcal{F}(\beta_{C'} f). \end{aligned}$$

By the faithfulness of $\mathcal{F}$ we thus have that

$$\mathcal{G}\mathcal{F}(f)\beta_C = \beta_{C'} f,$$

hence $\beta : \mathcal{I}\mathcal{D}_{\mathcal{D}} \cong \mathcal{G}\mathcal{F}$. We thus conclude that $\mathcal{C} \cong \mathcal{D}$. □

Example D.0.89. We aim to show a an equivalence of categories implicitly known from the linear algebra of finite dimensional vector space, which is for a fixed field K that the study of $n \times n$ matrices over K is the "same" as the study of finite dimensional vector spaces. To make this concrete, we consider two locally small categories:

- Mat_K, whose objects are $\mathbb{Z}_{\geq 1}$ and whose morphisms are $\{\mathrm{Mat}_{n\times m}(K) : n, m \in \mathbb{Z}_{\geq 1}\}$. composition is given by matrix multiplication. Note we know (or will know) that for an $n \times m$ and an $m \times l$ matrix the resulting matrix from matrix multiplication is an $n \times l$ matrix. We also know that matrix multiplication is associative and the identity matrix is the neutral element with respect to matrix multiplication and thus plays the role of identity morphism in this context.

- $\mathrm{Vect}_K^{\mathrm{fd}}$, whose objects is the proper class of finite dimensional vector spaces and whose morphism are linear maps. This is a subcategory of Set.

We aim to show that $\mathrm{Mat}_K \cong \mathrm{Vect}_K^{\mathrm{fd}}$. To do this, we first introduce the category $\mathrm{Vect}_K^{\mathrm{basis}}$. An object in this category is a pair $V, \mathcal{V}$, where V is a finite dimensional vector space and $\mathcal{V}$ is a basis of V. The morphisms are linear maps. Let $\mathcal{E}_n$ denote the standard basis of K^n. Consider the assignment

$$K^{(_)} : \mathrm{Mat}_K \to \mathrm{Vect}_K^{\mathrm{basis}}$$
$$n \mapsto (K^n, \mathcal{E}_n)$$
$$\mathrm{Mat}_{n\times m}(K) \ni A \mapsto (L_A : K^m \to K^n, v \mapsto Av)$$

one readily verifies that constitutes a functor. With some general linear algebra knowledge and it also easy to verify that it is fully faithful and essentially surjective on objects. Indeed, if V is an n-dimensional vector space with basis $\mathcal{V} = \{v_1, \ldots, v_n\}$, $(V, \mathcal{V}) \simeq K^n$, by the map

$$K^n \ni (a_1, \ldots, a_n) \mapsto \sum_1^n a_i v_i,$$

showing that $K^{(_)}$ is essentially surjective on objects.
If $L_A = L_B$ for some $n \times m$ matrices A, B, we know that $A_{ij} = (L_A e_j)_i = (L_B e_j)_i = B_{ij}$, for each i and j, hence $A = B$. This means $K^{(_)}$ is faithful.
Consider a linear map $L : (K^m, \mathcal{E}_m) \to (K^n, \mathcal{E}_n)$. Set $A := (L(e_j)_i)$. For an arbitrary $v \in K^m$ we have

$$L_A(v) = (L(e_j)_i)v = (\sum_1^m L(e_j)v_j) = \sum_1^m L(v_j e_j) = L(v).$$

We thus get that $L = L_A$, which means $K^{(_)}$ is full.
Consider the forgetful functor $\text{Vect}_K^{\text{basis}} \to \text{Vect}_K^{\text{fd}}$ which forgets the basis on objects. Clearly this is also a full faithful functor that is essentially surjective on objects. By Theorem D.0.88 we then get that $\text{Mat}_K \simeq \text{Vect}_K^{\text{fd}}$.

Definition D.0.90. A category is *connected* if for each pair of objects X, Y there is a finite sequence of objects

$$X_0 := X, X_1, \ldots, X_{n-1}, X_n := Y$$

and for each $i \in \{0, \ldots, n-1\}$ there is a morphism $f : X_i \to X_{i+1}$ or $f : X_{i+1} \to X_i$.

Proposition D.0.91. *A connected groupoid $\mathcal{G}$ is equivalent to the category induced by the automorphism group $G_X := \text{Aut}(X)$ for any object $X \in \text{Ob}(\mathcal{G})$.*

Proof. We construct a functor,

$$\mathcal{F} : \text{B}G_X \to \mathcal{G}$$
$$G_X \mapsto X$$
$$(f : G_X \to G_X) \mapsto (f : X \to X)$$

This is full since any morphism $f : X \to X$ in $\mathcal{G}$ is an isomorphism, hence f is a morphism in G_X and by definition $\mathcal{F}(f : G_X \to G_X) = f : X \to X$. It is also trivially faithful. Note that given an object Y in $\mathcal{G}$, there is a sequence of objects $X_0 = X, \ldots, X_n = Y$ and a morphism $f_i : X_i \to X_{i+1}$ or $f : X_{i+1} \to X_i$ for $i = 0, \ldots, n-1$. Each f_i is an isomorphism, hence $\mathcal{F}(G_X) = X \simeq Y$. This implies that $\mathcal{F}$ is essentially surjective on objects. We conclude that $\mathcal{G} \cong \text{B}G_X$ for each $X \in \text{Ob}(\mathcal{G})$. □

The upshot of the above result is that, as categories, a connected groupoid is just an automorphism group. In particular, a one object groupoid is, again, as categories, the same as the automorphism group of its one object. This makes precise the fact that, in category theory, groups are just one object groupoids.

Corollary D.0.92. *The category of one object groupoids, $\mathcal{GRP}_1$, is equivalent to the category of groups* Grp.

Proof. We construct the functor

$$\text{Aut} : \mathcal{GRP}_1 \to \text{Grp}$$
$$(\mathcal{G}, X) \mapsto \text{Aut}(X)$$
$$\mathcal{F} : (\mathcal{G}, X) \to (\mathcal{G}', X') \mapsto \text{Aut}(X) \ni f \mapsto \mathcal{F}(f) \in \text{Aut}(X')$$

This is a full. Indeed consider $\sigma : \text{Aut}(X) \to \text{Aut}(X')$. We construct a functor

$$\mathcal{F}_\sigma : (\mathcal{G}, X) \to (\mathcal{G}, X')$$
$$X \mapsto X'$$
$$g : X \to X \mapsto \sigma(g) : X' \to X'$$

clearly $\text{Aut}(\mathcal{F}_\sigma) = \sigma$. Suppose $\text{Aut}(\mathcal{F}_0) = \text{Aut}(\mathcal{F}_1)$. Take a morphism $g : X \to X$. Then $\mathcal{F}_0(g) = \text{Aut}(\mathcal{F}_0)(g) = \text{Aut}(\mathcal{F}_1(g)) = \mathcal{F}_1(g)$, hence $\mathcal{F}_0 = \mathcal{F}_1$. We conclude that Aut is faithful. Take an arbitrary group $G \in \text{Grp}$. Then $G \simeq \text{Aut}(\text{B}G, G)$ by the prior proposition. The result thus follows □

Another application is to a basic result in algebraic topology

Example D.0.93. Consider a path-connected topological space X. In this example we see that given any pair of points x, x' in X,

$$\pi(X, x) \simeq \pi(X, x').$$

Note upon considering x as an object of $\Pi(X)$, the fundemental groupoid of X, we see that $\pi(X, x) = \text{Aut}(x)$. Since X is path-connected there is a morphism between any pair objects in $\Pi(X)$, hence the fundemental groupoid of X is connected. We then get that $\text{B}\pi(X, x) \cong \Pi(X) \cong \text{B}\pi(X, x')$ by Proposition D.0.91. A fully faithful functor between categories with only one object is automatically a bijection on objects. It follows that $\pi(X, x) \simeq \pi(X, x')$.

Definition D.0.94. Consider a locally small category $\mathcal{C}$. We define the *isomorphism class* of an object X in $\mathcal{C}$ to be the class of all objects isomorphic to X.

Definition D.0.95. A category $\mathcal{C}$ is *skeletal*, if each isomorphism class of $\mathcal{C}$ contains a single object.

Lemma D.0.96. *An equivalence of skeletal categories is an isomorphism of categories.*

Proof. Suppose $\mathcal{C}$ and $\mathcal{D}$ are equivalent skeletal categories via functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$. We aim to show that $\mathcal{F}\mathcal{G}$ and $\mathcal{G}\mathcal{F}$ are not just naturally isomorphic to but equal to the identity functors. Indeed for each object X in $\mathcal{C}$, $\mathcal{G}\mathcal{F}(X) \simeq X$, since $\mathcal{G}\mathcal{F} \cong \mathcal{I}\mathcal{D}_\mathcal{C}$. Since $\mathcal{C}$ is skeletal, it follows that $\mathcal{G}\mathcal{F}(X) = X$. A similar argument for $\mathcal{F}\mathcal{G}$ shows that $\mathcal{F}$ and $\mathcal{G}$ are mutual inverses on functors. Using that $\mathcal{F}$ and $\mathcal{G}$ fully faithful one easily sees that they are also mutual inverses on morphisms and the result follows. □

Proposition D.0.97. *Consider a category $\mathcal{C}$. There is a skeletal category* Sk$\mathcal{C}$ *equivalent to $\mathcal{C}$. It is unique up to isomorphism.*

Proof. The objects of Sk$\mathcal{C}$ are given by choosing one object from each isomorphism class (here implicitly we use some version of the axiom of choice, so some assumption on smallness is probably appropriate as to not discuss foundations). The morphisms are those with domain and codomain in the aforementioned objects. The inclusion of Sk$\mathcal{C}$ is trivially fully faithful and essentially surjective on objects. We thus conclude that Sk$\mathcal{C} \cong \mathcal{C}$. Another skeletal category equivalent to $\mathcal{C}$ is by transitivity equivalent to Sk$\mathcal{C}$. This equivalence is upgraded to an isomorphism of categories via the prior lemma. □

We have seen that equivalence of categories does not necessarily preserve smallness. For instance Mat_K is small while $\mathrm{Vect}_K^{\mathrm{fd}}$ is locally small but not small since the collection of finite dimensional vector spaces is a proper class. Indeed for any non-empty set X, we can form $K[X]$, the free vector space generated by X. If $\mathrm{Vect}_K^{\mathrm{fd}}$ was a set, then $K[\{X\}]$ would be an element of $\mathrm{Vect}_K^{\mathrm{fd}}$ for every set X. Then $S := \{K[\{X\}] : X \text{ is a set}\}$ is a subset of $\mathrm{Vect}_K^{\mathrm{fd}}$. But then $\bigcup S$ is a set, which every set is an element of. But no such set can exist as an immediate consequence axiom of foundation (cf. Example C.1.10). In this sense smallness is an ugly notion. We can introduce a weaker notion of smallness, which is preserved by equivalence of categories.

Definition D.0.98. A category is *essentially small* if it is equivalent to a small category.

Another property not preserved by equivalence is discreteness. Take for instance the example of a poset $(X, \leq)$ with non-trivial arrows, eg. $X = \mathbb{N}$. This also has an associated discrete category (cf. Example D.0.39). There is a fully faithful functor, surjective on objects,

$$X \to (X, \leq)$$
$$x \mapsto x$$
$$x \mapsto x \leq x$$

hence $X \cong (X, \leq)$. Again we may consider the class of *essentially discrete categories*, i.e. categories that are equivalent to a discrete category.

Proposition D.0.99. *If $\mathcal{C}$ is a locally small category that is equivalent to some category $\mathcal{D}$, then $\mathcal{D}$ is locally small.*

Proof. There is a fully faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ that is essentially surjective on objects. Let X and Y be objects in $\mathcal{D}$. Then $\alpha : X \simeq \mathcal{F}(A)$ and $\beta : Y \simeq \mathcal{F}(B)$ for some objects A, B in $\mathcal{C}$. We consider the first order formula

$$\varphi :\equiv y = \mathcal{F}(f).$$

Note that for the set $\mathrm{Hom}(A, B)$, for every $f \in \mathrm{Hom}(A, B)$ we have that $\mathcal{F}(f)$ is the unique set satifying $\varphi(f, \mathcal{F}(f))$. Via the axiom schema of replacemen, this gives rise to a class function, $f \mapsto \mathcal{F}(f)$, whose image $\mathcal{F}(\mathrm{Hom}(A, B)) := \{\mathcal{F}(f) : f \in \mathrm{Hom}(A, B)\}$ is a set. Let $g : \mathcal{F}(A) \to \mathcal{F}(B)$. since $\mathcal{F}$ is full there is a morphism $f : A \to B$ such that $\mathcal{F}(f) = g$. Then $g \in \mathcal{F}(\mathrm{Hom}(A, B))$, so $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ is the set $\mathcal{F}(\mathrm{Hom}(A, B))$. We also have a class function, $k \mapsto \beta k \alpha^{-1}$. This takes the set $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ to the set $\beta\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))\alpha^{-1}$. Consider $g : X \to Y$. Then $h := \beta^{-1} g \alpha$ is a morphism in $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$. Then $g \in \beta\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))\alpha^{-1}$. We thus conclude that $\mathrm{Hom}(X, Y)$ is the set $\beta\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))\alpha^{-1}$. It follows that $\mathcal{D}$ is a locally small category. □

D.0.8 Diagrams

Definition D.0.100. A *(commutative) diagram* in a category $\mathcal{C}$ is a functor from a small category, called the *indexing category*, to $\mathcal{C}$.

Example D.0.101. An example arises the category $\mathbb{2} \times \mathbb{2}$. This category has objects $(0, 0), (0, 1), (1, 0), (1, 1)$ and arrows $(\mathrm{id}_i, \mathrm{id}_j)$ where $i, j = 0, 1$ and $(\mathrm{id}_i, 0 \to 1)$, $(0 \to 1, \mathrm{id}_i)$ and $(0 \to 1, 0 \to 1)$. We thus have a category with four objects, and five non-trivial arrows: one arrow from $(0, 0)$ to each of the other objects, one arrow from $(0, 1)$ and $(1, 0)$ to $(1, 1)$. We may illustrate this as a directed graph.

$$\begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & \searrow & \downarrow \\ (1,0) & \longrightarrow & (1,1) \end{array}$$

A diagram $D : \mathbb{2} \times \mathbb{2} \to \mathcal{C}$ maps the four objects in $\mathbb{2} \times \mathbb{2}$ to four objects in $\mathcal{C}$, A, B, C, D say. Moreover the five non-trivial arrows of $\mathbb{2} \times \mathbb{2}$ to morphisms $f_{0001} : A \to B$, $f_{0010} : A \to C$, $f_{0011} : A \to D$, $f_{0111} : B \to D$ and $f_{1011} : C \to D$. Functoriality assures that the diagram (in the informal sense),

$$\begin{array}{ccc} A & \xrightarrow{f_{0001}} & B \\ {\scriptstyle f_{0010}}\downarrow & \searrow^{f_{0011}} & \downarrow{\scriptstyle f_{0111}} \\ C & \xrightarrow[f_{1011}]{} & D \end{array}$$

commutes, i.e. $f_{0111} f_{0001} = f_{0011} = f_{1011} f_{0010}$

In general, the abstract definition of diagram lets us study the consequences of a list of morphism relations in terms of directed graph underlying these. This proof technique is called *diagram chasing*.

Lemma D.0.102. *Functors preserve commutative diagram.*

Proof. Indeed given a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and a diagram $D : \mathcal{I} \to \mathcal{C}$. Then $\mathcal{F}D : \mathcal{I} \to \mathcal{D}$ is a diagram in $\mathcal{D}$. □

Lemma D.0.103. *Consider a composable sequence of morphisms $f_1, \ldots, f_n$ in $\mathcal{C}$. If*

$$f_j f_{j-1} \cdots f_{i+1} f_i = g_m \cdots g_1$$

for some composable morphisms $g_1, \ldots, g_m$ in $\mathcal{C}$, then

$$f_n \cdots f_1 = f_n \cdots f_{j+1} g_m \cdots g_1 f_{i-1} \cdots f_1$$

Proof. Indeed, this is just a result of pre-composing with $f_{i-1} \cdots f_1$ and post-composing with $f_n \cdots f_{j+1}$ on both sides of the equality. □

Definition D.0.104. An object $I \in \text{Ob}(\mathcal{C})$ is *initial* if for each $X \in \text{Ob}(\mathcal{C})$ there is a unique morphism $I \to X$.
An object $T \in \text{Ob}(\mathcal{C})$ is *terminal* if for each $X \in \text{Ob}(\mathcal{C})$ there is a unique morphism $X \to T$, i.e. T is an initial object in $\mathcal{C}^{\text{op}}$.

Example D.0.105.

1. 0 is both initial and terminal in $\text{Group}, \text{Ab}, \text{Mod}_R, \text{Vect}_K$. It is also terminal in Ring (the category of initial rings) but not initial since there is no ring homomorphism from the zero ring.

2. $\mathbb{Z}$ is an initial object in Ring. Indeed

$$f_R : \mathbb{Z} \to R$$
$$n \mapsto \sum_1^n 1_R$$

defines a ring homomorphism and for any ring homomorphism $f : \mathbb{Z} \to R$ and integer $n \in \mathbb{Z}$,

$$f(n) = f\left(\sum_1^n 1\right) = \sum_1^n f(1) = \sum_1^n 1_R = f_R(n).$$

3. The rational numbers are initial in Field_0, the category of fields of characteristic 0. Indeed, there is a functor from the category of integral domains to the category of fields given by

$$R \mapsto \text{Frac}(R)$$
$$f : R \to S \mapsto \text{Frac}(f) : \text{Frac}(R) \ni \frac{x}{y} \mapsto \frac{f(x)}{f(y)} \in \text{Frac}(S)$$

We claim that $\text{Frac}(f_R)$ is the unique ring map from $\text{Frac}(\mathbb{Z}) = \mathbb{Q}$ to K for any characteristic 0 field K. Note that given a homomorphism $f : \mathbb{Q} \to K$, $f|_{\mathbb{Z}} : \mathbb{Z} \to K$ is equal to f_K, hence

$$f(\frac{a}{b}) = f(a)f(b)^{-1} = f_K(a)f_K(b)^{-1} = \frac{f_K(a)}{f_K(b)} = \text{Frac}(f_K)(\frac{a}{b}),$$

for each $\frac{a}{b} \in \mathbb{Q}$, hence $\text{Frac}(f_K) = f$.

4. If a preordered set has a minimum this will be the initial object. For instance $0 \in \mathbb{N}$ is the initial object.

5. For a monoid M, the category BM only has an initial or terminal object when M is trivial.

6. The empty set, $\emptyset$, is the initial object in Set. Indeed, the only function with domain $\emptyset$ is $\emptyset$.

7. In Set_*, $(\{*\}, *)$ is both initial and terminal.

8. In Cat, the category $(\emptyset, \emptyset)$ is the initial object. Indeed, there is a functor that maps objects and morphisms via the empty function and these are the only mappings with the empty set as domain. $\mathbb{1}$ is the terminal object. Indeed, for any locally small category $\mathcal{C}$, there is a functor

$$* : \mathcal{C} \to \mathbb{1}$$
$$X \mapsto *$$
$$f : X \to Y \mapsto \mathbb{1}_*$$

Since $\mathbb{1}$ have only one object and one morphism any such functor must map every object to this one object and every morphism to this one morphism, hence it is equal to $*$.

Example D.0.106. Faithful functors need not preserve epimorphism. Indeed, the ring map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epi but not surjective, but under the forgetful functor Ring $\to$ Set, it is not an epi, since Set is balanced. In Ring^{op}, it is a mono but in Set^{op} it is not.

Proposition D.0.107. *Consider a category with an intial object I and a terminal object T. If there is a map $g : T \to I$, then $I \simeq T$. It follows that T is initial.*

Proof. There is a unique $f : I \to T$. Then the morphism $fg : T \to T$ is the identity on T since T is terminal and $gf : I \to I$ is the identity on I since I is initial. □

Definition D.0.108. An object that is both initial and terminal is called a zero object

Remark D.0.109. Note that initial, terminal and zero objects are unique up to unique isomorphism, so we may aswell say *THE* initial, terminal and zero object. Indeed, take a initial objects I_1, I_2. Then $f : I_1 \to I_2$ and $g : I_2 \to I_1$ define mutual inverses. A dual argument is made for terminal objects

Definition D.0.110. Consider an endofunctor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$. A *coalgebra for* $\mathcal{T}$ is an object X in $\mathcal{C}$ with a morphism $\gamma : X \to \mathcal{T}(X)$. We define a category of coalgebras for $\mathcal{T}$ by letting a morphism of coalgebras $(X, \gamma) \to (X', \gamma')$ be a morphism $f : X \to X'$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ \mathcal{T}(X) & \xrightarrow[\mathcal{T}(f)]{} & \mathcal{T}(X') \end{array}$$

commutes. Composition is given by the composition inherited by $\mathcal{C}$ and the identity morphisms are also inherited by $\mathcal{C}$.

Proposition D.0.111. *For terminal coalgebras (X, γ), we have that $\gamma : X \to \mathcal{T}(X)$ is an isomorphism.*

Proof. There is a unique morphism $\delta : \mathcal{T}(X) \to X$. Note that $X \to X$ has to be the identity morphism. It follows that $\delta\gamma = \mathrm{id}_X$. Additionally, the square

$$\begin{array}{ccc} \mathcal{T}(X) & \xrightarrow{\delta} & X \\ \downarrow{\scriptstyle\mathcal{T}(\gamma)} & & \downarrow{\scriptstyle\gamma} \\ \mathcal{T}\mathcal{T}(X) & \xrightarrow{\mathcal{T}(\delta)} & \mathcal{T}(X) \end{array}$$

commutes. Note that $\mathbb{1}_{\mathcal{T}(X)} = \mathcal{T}(\mathbb{1}_X) = \mathcal{T}(\delta)\mathcal{T}(\gamma)$. Hence we get the commutative triangle

$$\begin{array}{ccc} \mathcal{T}(X) & \xrightarrow{\delta} & X \\ & {\scriptstyle\mathbb{1}}\searrow & \downarrow{\scriptstyle\gamma} \\ & & \mathcal{T}(X) \end{array}$$

We thus conclude that γ is an isomorphism with mutual inverse δ. □

Definition D.0.112. A *concrete category* is a category $\mathcal{C}$ for which there is a faithful functor $\mathcal{C} \to$ Set.

Proposition D.0.113. *Consider a morphism $f : X \to Y$ in a category $\mathcal{C}$. Consider a faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. If $\mathcal{F}(f)$ is mono so in f. Dually if $\mathcal{F}(f)$ is epi so is f.*

Proof. Indeed, suppose $fg = fh$ for a pair of parallel morphisms g, h, composable with f. Then

$$\mathcal{F}(f)\mathcal{F}(g) = \mathcal{F}(fg) = \mathcal{F}(fh) = \mathcal{F}(f)\mathcal{F}(h) \implies \mathcal{F}(g) = \mathcal{F}(h) \implies g = h.$$

To make the dual argument, note the functor $\mathcal{F}^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ which maps a morphism $f \in \text{Hom}(\mathcal{C}^{\text{op}})$ with domain X and codomain Y to $\mathcal{F} : \mathcal{F}(Y) \to \mathcal{F}(X)$, which is a morphism from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ in $\mathcal{D}^{\text{op}}$, is again a faithful functor. Since an epi is the dual concept to mono the result follows by applying the statement for monos to $\mathcal{F}^{\text{op}}$. □

Corollary D.0.114. *If $\mathcal{C}$ is a concrete category, if the image of a morphism $f \in \text{Hom}(\mathcal{C})$ in* Set *is an injection/surjection, then f is mono/epi.*

Example D.0.115. It may not always be that faithful functors preserve epimorphisms. Take any non-zero integer a. Then for any $x, y \in \mathbb{Z}$, if $xa = ya$, then $x = y$. That is in category $\text{B}(\mathbb{Z}, \cdot)$, a is an epimorphism. Consider the injective monoid homomorpism

$$\mathbb{Z} \hookrightarrow \text{Mat}_2(\mathbb{Z}), n \mapsto \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}.$$

This corresponds to a faithful functor

$$\begin{aligned} \text{B}\mathbb{Z} &\to \text{BMat}_2(\mathbb{Z}) \\ \mathbb{Z} &\mapsto \text{Mat}_2(\mathbb{Z}) \\ n &\mapsto \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

So since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we conclude that

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is not an epimorphism in $\text{BMat}_2(\mathbb{Z})$. Dually, a faithful functor does not preserve monos, since these are just epis in $\mathbb{Z}^{\text{op}}$.

Lemma D.0.116. *Consider a faithful functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. If a diagram in $\mathcal{C}$ (a directed graph of morphisms) commutes in the image of $\mathcal{F}$, then it commutes in $\mathcal{C}$*

Proof. Indeed given sequences $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ of composable morphism in $\mathcal{C}$ with the same domain and codomain with $\mathcal{F}(f_n) \cdots \mathcal{F}(f_n) = \mathcal{F}(g_m) \cdots \mathcal{F}(g_1)$, we have that

$$\mathcal{F}(f_n \cdots f_1) = \mathcal{F}(g_m \cdots g_1) \implies f_n \cdots f_1 = g_m \cdots g_1.$$

□

D.0.9 Natural Transformations as Morphisms

Proposition D.0.117. *Consider categories $\mathcal{C}$ and $\mathcal{D}$.* $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$*, i.e. functors between $\mathcal{C}$ and $\mathcal{D}$ define the objects in category where morphisms are natural transformations.*

Proof. Given natural transformations $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ and $\beta : \mathcal{G} \Rightarrow \mathcal{H}$, we define

$$\beta\alpha : \mathcal{F} \Rightarrow \mathcal{H}$$

to be the collection of morphisms

$$\{\beta_X \alpha_X : \mathcal{F}(X) \to \mathcal{H}(X) : X \in \mathrm{Ob}(\mathcal{C})\}.$$

This is seen to be a natural transformation by

$$\begin{array}{ccccc} \mathcal{F}(X) & \xrightarrow{\alpha_X} & \mathcal{G}(X) & \xrightarrow{\beta_X} & \mathcal{H}(X) \\ \downarrow{\scriptstyle \mathcal{F}(f)} & & \downarrow{\scriptstyle \mathcal{G}(f)} & & \downarrow{\scriptstyle \mathcal{H}(f)} \\ \mathcal{F}(X') & \xrightarrow{\alpha_{X'}} & \mathcal{G}(X') & \xrightarrow{\beta_{X'}} & \mathcal{H}(X') \end{array}$$

commuting for each morphism $f : X \to X'$. This composition is clearly associative. The identity morphism is

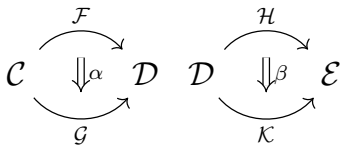
$$\{\mathrm{id}_{\mathcal{F}(X)} : \mathcal{F}(X) \to \mathcal{F}(X) : X \in \mathrm{Ob}(\mathcal{C})\}.$$

□

Remark D.0.118. We call the composition of natural transformations defined above *vertical composition*.

An important instance of the above category is $\mathrm{Hom}(\mathrm{Hom}(X, _), \mathcal{F})$ where X lives in a locally small category $\mathcal{C}$ and $\mathcal{F}$ is another functor from $\mathcal{C}$ to Set. Its importance is due to the Yoneda lemma, which will tell us that this is a set that is isomorphic to the set $\mathcal{F}(X)$.

Definition D.0.119. Given a pair of natural transformations



We define *horizontal composition of* β *and* α denoted $\beta * \alpha$ to be the natural transformations whose components is the diagonal of the commutative square

$$\begin{array}{ccc} \mathcal{HF}(X) & \xrightarrow{\beta_{\mathcal{F}(X)}} & \mathcal{KF}(X) \\ {\scriptstyle \mathcal{H}(\alpha_X)}\downarrow & \overset{(\beta*\alpha)_X}{\searrow} & \downarrow{\scriptstyle \mathcal{K}(\alpha_X)} \\ \mathcal{HG}(X) & \xrightarrow[\beta_{\mathcal{G}(X)}]{} & \mathcal{KG}(X) \end{array}$$

Proposition D.0.120. *horizontal composition is well-defined.*

Proof. The square commutes by naturality of β. We need to check that $\beta * \alpha : \mathcal{HF} \Rightarrow \mathcal{KG}$ is indeed a natural transformation. Indeed, consider the rectangle

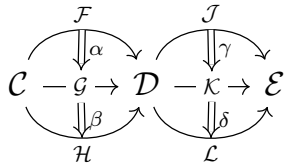
$$\begin{array}{ccccc} \mathcal{HF}(X) & \xrightarrow{\mathcal{H}(\alpha_X)} & \mathcal{HG}(X) & \xrightarrow{\beta_{\mathcal{G}(X)}} & \mathcal{KG}(X) \\ \downarrow{\scriptstyle \mathcal{HF}(f)} & & \downarrow{\scriptstyle \mathcal{HG}(f)} & & \downarrow{\scriptstyle \mathcal{KG}(f)} \\ \mathcal{HF}(X') & \xrightarrow{\mathcal{H}(\alpha_{X'})} & \mathcal{HG}(X') & \xrightarrow{\beta_{\mathcal{G}(X')}} & \mathcal{KG}(X') \end{array}$$

commutes by the naturality of α. This shows that

$$\begin{aligned} \mathcal{KG}(f)(\beta * \alpha)_X &= \mathcal{KG}(f)\beta_{\mathcal{G}(X)}(\beta_{\mathcal{G}(X)}\mathcal{H}(\alpha_X)) \\ &= (\beta_{\mathcal{G}(X')}\mathcal{H}(\alpha_{X'}))\mathcal{HF}(f) = (\beta * \alpha)_{X'}\mathcal{HF}(f) \end{aligned}$$

hence $\beta * \alpha$ is a natural tranformation. Associativity follows by the trivial componentwise associativity □

Lemma D.0.121. *Consider the functors and natural transformations*



Then

$$\delta\gamma * \beta\alpha = (\delta * \beta)(\gamma * \alpha).$$

Proof. By the naturality of δ we get that the square

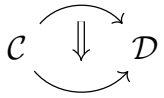
$$\begin{array}{ccc} \mathcal{K}(\mathcal{F}(X)) & \xrightarrow{\delta_{\mathcal{F}(X)}} & \mathcal{L}(\mathcal{F}(X)) \\ \downarrow{\scriptstyle \mathcal{K}(\alpha_X)} & & \downarrow{\scriptstyle \mathcal{L}(\alpha_X)} \\ \mathcal{K}(\mathcal{G}(X)) & \xrightarrow{\delta_{\mathcal{G}(X)}} & \mathcal{L}(\mathcal{G}(X)) \end{array}$$

commutes. It thus follows that

$$\begin{aligned}(\delta\gamma * \beta\alpha)_X &= \mathcal{L}(\beta_X\alpha_X)(\delta\gamma)_{\mathcal{F}(X)} = \mathcal{L}(\beta_X)\mathcal{L}(\alpha_X)\delta_{\mathcal{F}(X)}\gamma_{\mathcal{F}(X)} \\ &= \mathcal{L}(\beta_X)\delta_{\mathcal{G}(X)}\mathcal{K}(\alpha_X)\gamma_{\mathcal{F}(X)} = (\delta * \beta)_X(\gamma * \alpha)_X \\ &= ((\delta * \beta)(\gamma * \alpha))_X.\end{aligned}$$

□

Proposition D.0.122. *The objects of* Cat *form a category with morphisms being* ***cells*** *of the form*

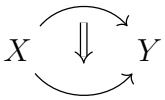


where a composition of cells is horinzontal composition of the natural transformations.

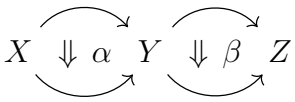
Proof. This follows directly from the prior lemma, i.e. after one checks that horizontal composition is associative, which is readily verifiable. The identity cell is a pair of identity functors together with the identity transformation of these functors. □

Definition D.0.123. A *2-category* $\mathcal{C}$ is a class of objects, a class of 1-morphisms between objects, together with a class of 2-morphisms between parallel morphisms, where for $f, g : X \to Y$, we denote such an arrow α by $\alpha : f \Rightarrow g$. Moreover, this data has to adhere to the following restrictions:

1. objects and 1-morphisms form a category
2. There is an operation of 2-morphisms between 1-morphisms of any fixed pair of objects X, Y called *vertical composition* such that 1-morphisms between X and Y are objects in a category with 2-morphisms form a category.
3. There is a category whose objects are objects in $\mathcal{C}$ where morphisms are *2-cells* of the form

   Morphisms are 2-morphisms and these compose by a binary operation called *horizontal composition*. Given a pair of 2-morphisms

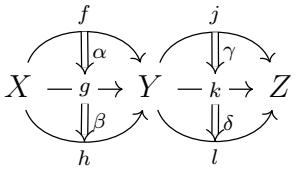
   

   we denote this composition by $\beta * \alpha$.

4. For 1-morphisms $f : X \to Y$ and $g : Y \to Z$,

$$\mathbb{1}_g * \mathbb{1}_f = \mathbb{1}_{gf}.$$

5. For 2-cells,



Then

$$\delta\gamma * \beta\alpha = (\delta * \beta)(\gamma * \alpha).$$

Readily, we see that we have seen one example of a 2-category, namely the 2-category of small categories.

Proposition D.0.124. *Let* $\mathcal{C} \in$ Cat *and* $\mathcal{D} \in$ CAT. *Then* $\mathcal{D}^{\mathcal{C}} \in$ CAT.

Proof. For each pair of functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$, we construct a class function whose image contains all natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$. Consider the set $N(\mathcal{F}, \mathcal{G})$ given by

$$\left\{ (\alpha_X) \in \prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{F}(X), \mathcal{G}(X)) : \mathcal{G}(f)\alpha_Y = \alpha_Z \mathcal{F}(f)\ \forall Y, Z \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Hom}(Y, Z) \right\}.$$

There is a class function taking an element (α_X) of $N(\mathcal{F}, \mathcal{G})$ to the natural transformation whose components are $\{\alpha_X : X \in \mathrm{Ob}(\mathcal{C})\}$. Any natural transformation is obviously contained in the image of this class function, hence $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is a set. □

D.1 The Yoneda Lemma

D.1.1 Representable Functors

We give an alternative definition of initial and terminal objects.

Lemma D.1.1. *1. An object I in a locally small category $\mathcal{C}$ is initial if and only if* $\mathrm{Hom}(I, _) \cong *$, *where* $* : \mathcal{C} \to$ Set, $S \mapsto \{*\}$.

2. An object T in a locally small category $\mathcal{C}$ is terminal if and only if $\mathrm{Hom}(_, T) \cong *^{\mathrm{op}}$, *where* $*^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to$ Set, $S \mapsto \{*\}$.

Proof. 1. Suppose I is initial. For each object X in $\mathcal{C}$, $\mathrm{Hom}(I, X)$ is a singleton $\{f_X\}$. Then

$$\{* : \mathrm{Hom}(I, X) \to *(X), f_X \mapsto to\{*\} : X \in \mathrm{Ob}(\mathcal{C})\}$$

defines a natural isomorphism $\mathrm{Hom}(I, _) \cong *$. Indeed, each of these maps is a bijection with inverse, $\{*\} \mapsto f_X$. Note that any square

$$\begin{array}{ccc} \mathrm{Hom}(I, X) & \xrightarrow{*} & \{*\} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(I, Y) & \xrightarrow{*} & \{*\} \end{array}$$

must have vertical maps that are constant maps, so naturality follows.
Suppose now that $\mathrm{Hom}(I, _) \cong *$. Then there is a collection of bijective maps

$$\{\mathrm{Hom}(I, X) \simeq \{*\} : X \in \mathrm{Ob}(\mathcal{C})\}.$$

In particular, $|\mathrm{Hom}(I, X)| = |\{*\}| = 1$, for each object X in $\mathcal{C}$. It follows that $\mathrm{Hom}(I, X)$ is a singleton for each object and thus that I is the initial object in $\mathcal{C}$.
2. is to dual to 1. □

Definition D.1.2. A co-/contravariant functor $\mathcal{F}$ whose domain is locally small is *representable* if there is an object X such that $\mathrm{Hom}(X, _)$ resp. $\mathrm{Hom}(_, X)$ is naturally isomorphic to $\mathcal{F}$. We then say that $\mathcal{F}$ *is represented by* X

Remark D.1.3. $\mathcal{C}$ has an initial resp. termninal object if and only if $* : \mathcal{C} \to \mathrm{Set}$ resp. $*^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Definition D.1.4. A *representation* of a functor $\mathcal{F}$ whose domain is locally small is a pair of an object X and a natural isomorphism of $\mathcal{F}$ with $Hom(X, _)$ and $\mathrm{Hom}(_, X)$ depending on variance.

The motivation for restating the notion of initial and terminal object is that these are special cases of objects with universal properties. We shall define what this means exactly.

Definition D.1.5. A *universal property* for an object X in a locally small category $\mathcal{C}$ is a co/contravariant representable functor $\mathcal{C} \to \mathrm{Set}$ represented by X.

Example D.1.6. Consider the functor

$$\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set}$$
$$\mathcal{C} \mapsto \mathrm{Ob}(\mathcal{C})$$
$$(\mathcal{F} : \mathcal{C} \to \mathcal{D}) \mapsto (\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}), X \mapsto \mathcal{F}(X))$$

Consider also the functor $\mathrm{Hom}(\mathbb{1}, _)$. For any category $\mathcal{C}$, a functor $\mathbb{1} \to \mathcal{C}$ is of the form

$$\mathbb{1} \to \mathcal{C}$$
$$* \mapsto X$$
$$\mathbb{1}_* \mapsto \mathbb{1}_X$$

for some object X in $\mathcal{C}$. The functor Ob is represented by $\mathbb{1}_X$. Indeed,

$$\{f_{\mathcal{X}} : \mathrm{Hom}(\mathbb{1}, \mathcal{X}) \to \mathrm{Ob}(\mathcal{X}), \mathcal{F}_X \mapsto X \mid \mathcal{X} \in \mathrm{Cat}\}$$

is a family of isomorphisms, where $X \mapsto \mathcal{F}_X$ is the inverse. One should note that this is a collection of well defined maps, since if $\mathcal{F}_X = \mathcal{F}_Y$, then $X = \mathcal{F}_X(*) = \mathcal{F}_Y(*) = Y$. We check that for an arbitrary functor $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$, the square

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{1}, \mathcal{X}) & \xrightarrow{f_{\mathcal{X}}} & \mathrm{Ob}(\mathcal{X}) \\ \downarrow{\scriptstyle \mathcal{G}_*} & & \downarrow{\scriptstyle \mathrm{Ob}(\mathcal{G})} \\ \mathrm{Hom}(\mathbb{1}, \mathcal{Y}) & \xrightarrow{f_{\mathcal{Y}}} & \mathrm{Ob}(\mathcal{Y}) \end{array}$$

commutes. Indeed, given a functor $\mathcal{F}_X : \mathbb{1} \to \mathcal{X}$ for some object $X \in \mathcal{X}$,

$$\mathrm{Ob}(\mathcal{G}) f_{\mathcal{X}}(\mathcal{F}_X) = \mathrm{Ob}(\mathcal{G})(X) = \mathcal{G}(X) = f_{\mathcal{Y}}(\mathcal{H}_{\mathcal{G}(X)}) = f_{\mathcal{Y}}(\mathcal{G}\mathcal{F}_X) = f_{\mathcal{Y}}\mathcal{G}_*(\mathcal{F}_X).$$

Example D.1.7. Consider the functor

$$\begin{gathered} \mathrm{mor} : \mathrm{Cat} \to \mathrm{Set} \\ \mathcal{C} \mapsto \mathrm{mor}(\mathcal{C}) \\ (\mathcal{F} : \mathcal{C} \to \mathcal{D}) \mapsto (\mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D}), f : X \to Y \mapsto \mathcal{F}(f)) \end{gathered}$$

For a small category $\mathcal{C}$ and a morphism $f : X \to Y$ consider the functor

$$\begin{gathered} \mathcal{F}_f : \mathbb{2} \to \mathcal{C} \\ 0 \mapsto X, 1 \mapsto Y \\ (0 \to 1) \mapsto f : X \to Y \end{gathered}$$

Any functor $\mathbb{2} \to \mathcal{C}$ must be of this form. The category $\mathbb{2}$ represents the functor mor. Indeed, we get a natural isomorphism by considering the collection

$$\{g_{\mathcal{X}} : \mathrm{Hom}(\mathbb{2}, \mathcal{X}) \to \mathrm{mor}(\mathcal{X}), \mathcal{F}_f \mapsto f \mid \mathcal{X} \in \mathrm{Cat}\}$$

where each local inverse is given by $f \mapsto \mathcal{F}_f$. Naturality is checked completely analogously to the prior example.

Example D.1.8. We aim to describe natural transformations between Ob and mor. Consider

$$\mathrm{dom} := \{\mathrm{dom}_{\mathcal{X}} : \mathrm{mor}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{X}) f : X \to Y \mapsto X\}.$$

There is an analogous collection, cod, of maps between morphisms of categories and objects of categories that for a given category maps each morphism to its codomain.

$$\begin{array}{ccc} \mathrm{mor}(\mathcal{C}) & \xrightarrow{\mathrm{dom}_{\mathcal{C}}} & \mathrm{Ob}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathrm{mor}(\mathcal{F})} & & \downarrow{\scriptstyle \mathrm{Ob}(\mathcal{F})} \\ \mathrm{mor}(\mathcal{D}) & \xrightarrow{\mathrm{dom}_{\mathcal{D}}} & \mathrm{Ob}(\mathcal{D}) \end{array}$$

commutes. Indeed, let a morphism $f : X \to Y$ in $\mathcal{C}$ be given. Then

$$\mathrm{Ob}(\mathcal{F})\mathrm{dom}_{\mathcal{C}}(f) = \mathrm{Ob}(\mathcal{F})(X) = \mathcal{F}(X) = \mathrm{dom}_{\mathcal{D}}(\mathcal{F}(f)) = \mathrm{dom}_{\mathcal{D}}\mathrm{mor}(\mathcal{F})(f).$$

Proposition D.1.9. *A representable functor $\mathcal{F} : \mathcal{C} \to \mathrm{Set}$ preserves mononomorphisms.*

Proof. Let a monomorphism $f : X \to Y$ in $\mathcal{C}$ be given. We need to show that $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ is injective. Let $x, x' \in \mathcal{F}(X)$ be given with $\mathcal{F}(f)(x) = \mathcal{F}(f)(x')$. Note that $\alpha : \mathcal{F} \cong \mathrm{Hom}(A, _)$ for some A in $\mathcal{C}$. then $x = \alpha_X(g)$ for some $g : A \to X$ and $x' = \alpha_X(g')$ for some $g : A \to X$. By naturality we thus have that

$$\begin{aligned} \alpha_Y(fg) = \alpha_Y f_*(g) = \mathcal{F}(f)\alpha_X(g) = \mathcal{F}(f)(x) = \mathcal{F}(f)(x') = \mathcal{F}(f)\alpha_X(g') \\ = \alpha_Y f_*(g) = \alpha_Y(fg), \end{aligned}$$

hence

$$fg = fg' \implies g = g' \implies x = \alpha_X(g) = \alpha_X(g') = x'.$$

□

Example D.1.10. The forgetful functor $\mathrm{Ring} \hookrightarrow \mathrm{Set}$ is, by the above proposition, not a representable functor.

D.1.2 Currying

Consider the functor

$$\begin{gathered} \mathrm{Hom}(_ \times A, B) : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set} \\ X \mapsto \mathrm{Hom}(X \times A, B) \\ f : X \to Y \mapsto (\mathrm{Hom}(Y \times A, B) \to \mathrm{Hom}(X \times A, B), g \mapsto g \circ f \times \mathrm{id}_A) \end{gathered}$$

Proposition D.1.11. *This functor is represented by* $\mathrm{Hom}(A, B)$*, the set of functions from A to B.*

Proof. Consider the family of *uncurrying* functions

$$\{\text{uncur}_X : \text{Hom}(X, \text{Hom}(A, B)) \to \text{Hom}(X \times A, B)\}$$

where for a function $f : X \to \text{Hom}(A, B)$ and $(x, a) \in X \times A$,

$$\text{uncur}_X(f)(x, a) := (f(x))(a)$$

the inverse component is the function

$$\text{cur}_X : \text{Hom}(X \times A, B) \to \text{Hom}(X, \text{Hom}(A, B))$$

where for a function $g : X \times A \to B$ and an element $x \in X$

$$\text{cur}_X(g)(x) : A \to Ba \mapsto g(x, a)$$

That is, *currying* is the function obtained by fixing the first element in the cartesian product. Let $f \in \text{Hom}(X, \text{Hom}(A, B))$ and $x \in X$, $a \in A$ be given. Then

$$(\text{cur}_X\text{uncur}_X(f))(x)(a) = \text{uncur}_X(f)(x, a) = f(x)(a) = \text{id}_{\text{Hom}(X,\text{Hom}(A,B))}(f)(x)(a)$$
$$\implies \text{cur}_X\text{uncur}_X = \text{id}_{\text{Hom}(X,\text{Hom}(A,B))}.$$

Let $g \in \text{Hom}(X \times A, B)$ and $(x, a) \in X \times A$. Then

$$(\text{uncur}_X\text{cur}_X(f))(x, a) = \text{cur}_X(f)(x)(a) = f(x, a) = \text{id}_{\text{Hom}(X\times A,B)}(f)(x, a)$$
$$\implies \text{uncur}_X\text{cur}_X = \text{id}_{\text{Hom}(X\times A,B)}.$$

We check naturality of uncurrying. Let a function $F : X \to Y$ be given. To show naturality, we need to check that given a function $g \in \text{Hom}(X, \text{Hom}(A, B))$, then $\text{uncur}_Y F^*(g) = (F \times \text{id}_A)^*\text{uncur}_X(g)$. Let $(x, a) \in X \times A$. Then

$$\text{uncur}_Y F^*(g)(x, a) = (gF)(x)(a) = \text{uncur}_X(g)(F(x), a) = (F \times \text{id}_A)^*\text{uncur}_X(x, a).$$

□

D.1.3 Statement & Proof of The Yoneda Lemma

Theorem D.1.12. *(The Yoneda Lemma) For any $\mathcal{C}$ in* CAT*, given a functor $\mathcal{F} : \mathcal{C} \to$ Set and an object $X \in \text{Ob}(\mathcal{C})$*

$$\text{Hom}^{\mathcal{C}^{\text{Set}}}(\text{Hom}(X, _), \mathcal{F}) \simeq \mathcal{F}(X).$$

This bijection maps a natural transformation $\alpha : \text{Hom}(X, _) \Rightarrow \mathcal{F}$ to $\alpha_X(\mathbb{1}_X)$. The assignment of X resp. $\mathcal{F}$ is natural. I.e.

1. ***Naturality in Functors:*** *Fix an object X. For each functor $\mathcal{F}$ we write $\Phi^{\mathcal{F}}$: $\mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) \to \mathcal{F}(X)$ for the bijection. The claim is that*

$$\Phi^{\bullet} : \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\bullet) \Rightarrow \bullet(X)$$

given by the collection of morphisms

$$\left\{\Phi^{\mathcal{F}} : \mathcal{F} \in \mathcal{C}^{\mathrm{Set}}\right\}$$

is natural. Here $\bullet(X)$ denotes the functor

$$\mathcal{C}^{\mathrm{Set}} \to \mathrm{Set}$$
$$\mathcal{F} \mapsto \mathcal{F}(X)$$
$$\beta : \mathcal{F} \Rightarrow \mathcal{G} \mapsto \beta_X$$

2. ***Naturality in Objects:*** *Fix a functor $\mathcal{F} \in \mathcal{C}^{\mathrm{Set}}$. For each object X we write Φ_X : $\mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) \to \mathcal{F}(X)$ for the bijection. The claim is that*

$$\Phi_{\bullet} : \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(\bullet,_),\mathcal{F}) \Rightarrow \mathcal{F}(\bullet)$$

given by the collection of morphisms

$$\{\Phi_X : X \in \mathrm{Ob}(\mathcal{C})\}$$

is natural. Here $\mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(\bullet,_),\mathcal{F})$ is the functor

$$\mathcal{C} \to \mathrm{Set}$$
$$X \mapsto \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F})$$
$$f : X \to Y \mapsto \left((f^*)^* : \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) \Rightarrow \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(Y,_),\mathcal{F}), \alpha \mapsto \alpha f^*\right)$$

Proof. **Explicit construction of bijection:** We define

$$\Phi : \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) \to \mathcal{F}(X)$$
$$(\alpha : \mathrm{Hom}(X,_) \Rightarrow \mathcal{F}) \mapsto \alpha_X(\mathbb{1}_X)$$

We aim to construct an inverse to this (class) function. Let $y \in \mathcal{F}(X)$. We wish to find a natural transformation $\Psi(y) : \mathrm{Hom}(X,_) \Rightarrow \mathcal{F}$. For this to be a natural transformation we must in particular have that for any object Y and any morphism $f : X \to Y$, it must be that the square

$$\begin{array}{ccc} \mathrm{Hom}(X,X) & \xrightarrow{\Psi(y)_X} & \mathcal{F}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathrm{Hom}(X,Y) & \xrightarrow{\Psi(y)_Y} & \mathcal{F}(Y) \end{array}$$

commutes. Note that in particular

$$\mathcal{F}(f)(\Psi(y)_X(\mathbb{1}_X)) = \Psi(y)_Y f_*(\mathbb{1}_X) = \Psi(y)_Y(f)$$

We want

$$X = \Phi(\Psi(y)) = \Psi(y)_X(\mathbb{1}_X).$$

For this to hold we must then have that

$$\mathcal{F}(f)(X) = \Psi(y)_Y(f).$$

That is to say, we must define the inverse of Φ, to be

$$\Psi : \mathcal{F}(X) \to \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X, _), \mathcal{F})$$
$$y \mapsto \Psi(y) := \{\, \Psi(y)_Y : \mathrm{Hom}(X, Y) \to \mathcal{F}(Y), f \mapsto \mathcal{F}(f)(y) |\, Y \in \mathrm{Ob}(\mathcal{C})\}$$

We check that this is indeed maps to natural transformations. Let a morphism $g : Z \to W$ be given. Then for any $f \in \mathrm{Hom}(X, Z)$,

$$\Psi(y)_W g_*(f) = \Psi(y)_W(gf) = \mathcal{F}(gf)(y) = \mathcal{F}(g)\mathcal{F}(f)(y) = \mathcal{F}(g)\Psi(y)_Z(f).$$

hence the square

$$\begin{array}{ccc} \mathrm{Hom}(X, Z) & \xrightarrow{\Psi(y)_Z} & \mathcal{F}(Z) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \mathcal{F}(g)} \\ \mathrm{Hom}(X, W) & \xrightarrow{\Psi(y)_W} & \mathcal{F}(W) \end{array}$$

commutes. Note that we constructed Ψ to be a right inverse of Φ, so it remains to check that it is also a left inverse. Let a natural transformation $\alpha : \mathrm{Hom}(X, _) \Rightarrow \mathcal{F}$ be given. Then for every $f \in \mathrm{Hom}(X, Y)$

$$\begin{array}{ccc} \mathrm{Hom}(X, X) & \xrightarrow{\alpha_X} & \mathcal{F}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathrm{Hom}(X, Y) & \xrightarrow{\alpha_Y} & \mathcal{F}(Y) \end{array}$$

commutes, hence

$$\Psi\Phi(\alpha)(f) = \Psi(\alpha_X(\mathbb{1}_X))(f) = \mathcal{F}(f)(\alpha_X(\mathbb{1}_X)) = \mathcal{F}(f)\alpha_X(\mathbb{1}_X) = \alpha_Y f_*(\mathbb{1}_X) = \alpha_Y(f).$$

We can thus conclude that $\Psi\Phi(\alpha) = \alpha$. Note that we have at this point worked with class functions. Since $\mathcal{F}(X)$ is a set, Ψ is a set function whose image are the natural transformations of $\mathrm{Hom}(X, _)$ to $\mathcal{F}$, which is therefor a set. This upgrades Φ to a set function Foundationally I don't if the functions are functions in the sense I think they are.

Naturality on functors: Let a natural transformation $\beta : \mathcal{F} \Rightarrow \mathcal{G}$ be given. We need to check that the square

$$\begin{array}{ccc} \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) & \xrightarrow{\Phi^{\mathcal{F}}} & \mathcal{F}(X) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_X} \\ \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{G}) & \xrightarrow{\Phi^{\mathcal{G}}} & \mathcal{G}(X) \end{array}$$

Indeed for any natural transformation $\alpha : \mathrm{Hom}(X,_) \Rightarrow \mathcal{F}$,

$$\Phi^{\mathcal{G}}\beta_*(\alpha) = \Phi^{\mathcal{G}}(\beta\alpha) = (\beta\alpha)_X(\mathbb{1}_X) = \beta_X(\alpha_X(\mathbb{1}_X)) = \beta_X\Phi^{\mathcal{F}}(\alpha).$$

Naturality on Objects: Let a morphism be given $f : X \to Y$. We need to check that

$$\begin{array}{ccc} \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(X,_),\mathcal{F}) & \xrightarrow{\Phi_X} & \mathcal{F}(X) \\ \downarrow{\scriptstyle (f^*)^*} & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathrm{Hom}^{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}(Y,_),\mathcal{F}) & \xrightarrow{\Phi_Y} & \mathcal{F}(Y) \end{array}$$

commutes. Indeed, for a natural transformation $\alpha : \mathrm{Hom}(X,_) \Rightarrow \mathcal{F}$,

$$\Phi_Y(f^*)^*(\alpha) = \Phi_Y(\alpha f^*) = \alpha_Y(f^*)_Y(\mathbb{1}_Y) = \alpha_Y(f) = \mathcal{F}(f)(\alpha_X(\mathbb{1}_X)) = \mathcal{F}(f)\Phi_X(\alpha).$$

□

D.1.4 Corollaries of The Yoneda Lemma

We first give an alternative characterization of the Yoneda lemma as a natural isomorphism of a pair of functors.

Definition D.1.13. The first of these functors is the *evaluation functor*: Given a locally small category $\mathcal{C}$, we consider

$$\begin{gathered} \mathrm{ev} : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \to \mathrm{Set} \\ (X,\mathcal{F}) \mapsto \mathcal{F}(X) \\ (f,\alpha) : (X,\mathcal{F}) \to (Y,\mathcal{G}) \mapsto \alpha_X\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{G}(Y) \end{gathered}$$

Before constructing the second functor to consider we prove a lemma to provide a different characterization of bifunctors

Lemma D.1.14. *Consider locally small categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. There is a one-to-one correspondence of bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ onto functors $\mathcal{C} \to \mathcal{E}^{\mathcal{D}}$.*

Proof. Let a bifunctor $\mathcal{F} : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be given. We define a functor

$$\begin{gathered}\text{Cur} : \mathcal{C} \to \mathcal{E}^{\mathcal{D}} \\ X \mapsto \mathcal{F}(X, _) \\ f : X \to Y \mapsto \mathcal{F}(f, _) : \mathcal{F}(X, _) \Rightarrow \mathcal{F}(Y, _)\end{gathered}$$

where

$$\begin{gathered}\mathcal{F}(X, _) : \mathcal{D} \to \mathcal{E} \\ S \mapsto \mathcal{F}(X, S) \\ g : S \to T \mapsto \mathcal{F}(\mathbb{1}_X, g)\end{gathered}$$

and $\mathcal{F}(f, _)$ is the collection

$$\{\mathcal{F}(f, S) \coloneqq \mathcal{F}(f, \mathbb{1}_S) : \mathcal{F}(X, S) \to \mathcal{F}(Y, S)\}.$$

The square

$$\begin{array}{ccc} \mathcal{F}(X,S) & \xrightarrow{\mathcal{F}(f,S)} & \mathcal{F}(Y,S) \\ \downarrow{\scriptstyle \mathcal{F}(X,g)} & & \downarrow{\scriptstyle \mathcal{F}(Y,g)} \\ \mathcal{F}(X,T) & \xrightarrow{\mathcal{F}(f,T)} & \mathcal{F}(Y,T) \end{array}$$

commutes since

$$\begin{aligned}\mathcal{F}(Y, g)\mathcal{F}(f, S) = \mathcal{F}(\mathbb{1}_Y, g)\mathcal{F}(f, \mathbb{1}_S) &= \mathcal{F}(\mathbb{1}_Y f, g\mathbb{1}_S) = \mathcal{F}(f\mathbb{1}_X, \mathbb{1}_T g) \\ &= \mathcal{F}((f, \mathbb{1}_T)(\mathbb{1}_X, g)) = \mathcal{F}(f, \mathbb{1}_T)\mathcal{F}(\mathbb{1}_X, g) = \mathcal{F}(f, T)\mathcal{F}(X, g).\end{aligned}$$

So $\mathcal{F}(f, _)$ is a natural transformation. Given a functor $\mathcal{G} : \mathcal{C} \to \mathcal{E}^{\mathcal{D}}$, the functor

$$\begin{gathered}\mathcal{C} \times \mathcal{D} \to \mathcal{E} \\ (X, S) \mapsto \mathcal{G}(X)(S) \\ (f, g) : (X, S) \to (Y, T) \mapsto (\mathcal{G}(f))_T(\mathcal{G}(X)(g))\end{gathered}$$

is the bifunctor associated with $\mathcal{G}$. Given two bifunctors $\mathcal{F}$ and $\mathcal{F}'$ with the same image under this association, note that

$$\mathcal{F}(X, S) = \mathcal{F}(X, _)(S) = \mathcal{F}'(X, _)(S) = \mathcal{F}'(X, S).$$

Moreover,

$$\mathcal{F}(f, g) = \mathcal{F}(\mathbb{1}_Y, g)\mathcal{F}(f, \mathbb{1}_S) = \mathcal{F}(Y, g)\mathcal{F}(f, S) = \mathcal{F}'(Y, g)\mathcal{F}'(f, S) = \mathcal{F}'(f, g),$$

hence $\mathcal{F} = \mathcal{F}'$. □

Consider the bifunctor

$$\mathrm{Hom}(_,_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}.$$

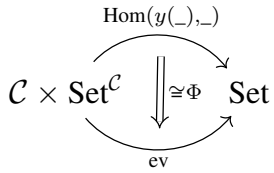
We can then apply the prior lemma to obtain a contravariant functor

$$y : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{C}}$$
$$X \mapsto \mathrm{Hom}(X, _)$$
$$f : X \to Y \mapsto f^* : \mathrm{Hom}(Y, _) \to \mathrm{Hom}(X, _)$$

We may regard this as a functor $\mathcal{C} \to (\mathrm{Set}^{\mathcal{C}})^{\mathrm{op}}$ to obtain a functor

$$\mathrm{Hom}(y(_), _) : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \to (\mathrm{Set}^{\mathcal{C}})^{\mathrm{op}} \times \mathrm{Set}^{\mathcal{C}} \to \mathrm{SET} \tag{13}$$
$$(X, \mathcal{F}) \mapsto (\mathrm{Hom}(X, _), \mathcal{F}) \mapsto \mathrm{Hom}(\mathrm{Hom}(X, _), \mathcal{F}) \tag{14}$$

where SET is the category of sufficient notion of collection of objects to contain the collection of all natural transformations between of objects.
By the Yoneda lemma we are in a position where we know that we can replace SET by Set. We thus more succinctly express the Yoneda Lemma as the existence of a natural isomorphism



Corollary D.1.15. *For a locally small category $\mathcal{C}$ consider the functor*

$$y : \mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$
$$X \mapsto \mathrm{Hom}(_, X)$$
$$f : X \to Y \mapsto f_* : \mathrm{Hom}(_, X) \Rightarrow \mathrm{Hom}(_, Y)$$

which we by Lemma D.1.14 may be identified with the functor

$$y : \mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Set}^{\mathcal{C}}$$
$$X \mapsto \mathrm{Hom}(X, _)$$
$$f : X \to Y \mapsto f^* : \mathrm{Hom}(Y, _) \Rightarrow \mathrm{Hom}(X, _)$$

and $\mathrm{Hom}(_, _)$. *This (or these) functor(s) is a fully faithful embedding called the* ***Yoneda embedding***

Proof. It is clearly an embedding since if $X \not\simeq Y$, then $\mathbb{1}_X \in \mathrm{Hom}(X,X)$ and $\mathbb{1}_X \notin \mathrm{Hom}(X,Y)$, hence $\mathrm{Hom}(_,X) \not\simeq \mathrm{Hom}(_,Y)$. To prove fullness and faithfullness, it is equivalent to show that the mapping

$$\mathrm{Hom}(X,Y) \ni f \mapsto f_* \in \mathrm{Hom}(\mathrm{Hom}(_,X),\mathrm{Hom}(_,Y))$$

is a bijection. By Lemma D.0.76, it is injective. Let a natural transformation $\alpha \in \mathrm{Hom}(\mathrm{Hom}(_,X),\mathrm{Hom}(_,Y$ be given. For an object A, set $f := \alpha_X(\mathbb{1}_X) : X \to Y$. Then f_*, takes a $\mathbb{1}_X$ to

$$f\mathbb{1}_X = f = \alpha_X(\mathbb{1}_X),$$

i.e. $\Phi(f_*) = \alpha_X(\mathbb{1}_X) = \Phi(\alpha)$, hence by the Yoneda Lemma,

$$f_* = \alpha.$$

□

Corollary D.1.16. *Given a ring R and an $n \times m$ matrix A, a row operation on A is uniquely determined by left multiplication by the matrix resulting from the same row operation performed on I_n.*

Proof. Fix a positive integer n. For each $i,j \in \{1,\dots,n\}$ and $a,b \in R$ consider the family of maps

$$\begin{aligned} {}_m\alpha_{ij}^{a,b} : \mathrm{Mat}_{n\times m}(R) \simeq (R^m)^n &\to \mathrm{Mat}_{n\times m}(R) \simeq (R^m)^n \\ (r_1,\dots,r_n) &\mapsto (r_1,\dots,r_{i-1},ar_i + br_j,\dots,r_n) = (r_1,\dots,ar_i,\dots,r_n) + (b\delta_{ip}r_{pq}) \end{aligned}$$

over positive integers m. Clearly every row operation is a special case of such maps. We claim that ${}_\bullet\alpha_{ij}^{a,b}$ defines a natural endomorphism of $\mathrm{Hom}(_,n)$. Indeed, given a matrix $A \in \mathrm{Mat}_{kl} = \mathrm{Hom}(l,k)$, the square

$$\begin{array}{ccc} \mathrm{Mat}_{nk} & \xrightarrow{{}_k\alpha_{ij}^{a,b}} & \mathrm{Mat}_{nk} \\ \downarrow{\scriptstyle A^*} & & \downarrow{\scriptstyle A^*} \\ \mathrm{Mat}_{nl} & \xrightarrow{{}_l\alpha_{ij}^{a,b}} & \mathrm{Mat}_{nl} \end{array}$$

commutes. To see this let $B \in \mathrm{Mat}_{nk}$ be given. We shorten notation and let ${}_\bullet\alpha$ denote horinzontal maps and $r(C)_i$ denote the i'th row of a matrix C. We thus get that

$$\begin{aligned} {}_l\alpha A^*(B) = {}_l\alpha(BA) &= (r(BA)_1,\dots,ar(BA)_i + br(BA)_j,\dots,r(BA)_n) \\ &= (r(B)_1A,\dots,ar(B)_iA + br(B)_jA,\dots,r(B)_nA) \\ &= (r(B)_1A,\dots,(ar(B)_i + br(B)_j)A,\dots,r(B)_nA) \\ &= {}_k\alpha(B)A = A^*{}_k\alpha(B). \end{aligned}$$

This proves naturality. To spell it out, we now have a natural transformation

$$\alpha_{ij}^{a,b} : \text{Hom}(_, n) \Rightarrow \text{Hom}(_, n)$$

This is given by the Yoneda embedding applied to some $n \times n$ matrix. The proof of fullness of the Yoneda embedding or simply applying the Yoneda Lemma tells that this $n \times n$ matrix is ${}_n\alpha_{ij}^{a,b}(I_n)$. Thus in the special case where the map in question is a row operation, ρ say, it is via the Yoneda embedding uniquely determined by $\rho(I_n)$. □

D.2 Universal Properties & Universal Elements

Definition D.2.1. two objects X and Y in the same small category are *representably isomorphic* if $\text{Hom}(X, _) \cong \text{Hom}(Y, _)$ and $\text{Hom}(_, X) \cong \text{Hom}(_, Y)$.

Lemma D.2.2. *If X and Y in the same locally small category are isomorphic, then they are representably isomorphic.*

Proof. This is a consequence functoriality of the Yoneda embeddings. □

Lemma D.2.3. *Let X and Y be a pair of objects in a locally small category $\mathcal{C}$. If* $\text{Hom}(_, X) \cong \text{Hom}(_, Y)$ *or* $\text{Hom}(X, _) \cong \text{Hom}(Y, _)$*, then $X \simeq Y$. As a consequence, if $\mathcal{F} : \mathcal{C} \to$ Set is a functor represented by X and Y, then $X \simeq Y$.*

Proof. Suppose $\alpha : \text{Hom}(_, X) \cong \text{Hom}(_, Y)$ or $\alpha : \text{Hom}(Y, _) \cong \text{Hom}(X, _)$, then since the Yoneda embedding is fully faithful, then $\alpha = y(f)$ for some unique isomorphism $f : X \to Y$. □

Remark D.2.4. With this result in mind, we have motivated the use of the definite article when refering to an object representing a representable functor, since given a pair of representations there is a unique isomorphism between the objects. In this way there is a canonical way of identifying objects in this context.

Definition D.2.5. A *universal property* of an object $X \in \mathcal{C}$, where $\mathcal{C}$ is locally small, is a pair consisting of a representable functor $\mathcal{F}$ together with a *universal element* $x \in \mathcal{F}(X)$ that defines a natural isomorphism of $\text{Hom}(X, _)$ and $\mathcal{F}$ or $\text{Hom}(_, X)$ and $\mathcal{F}$ via Φ (as seen in the Yoneda Lemma).

D.2.1 The Category of Elements

Definition D.2.6. Let $\mathcal{F} : \mathcal{C} \to$ Set, where $\mathcal{C}$ is a locally small category, be a covariant functor. Then $\int \mathcal{F}$ is the category characterised by the following data:

1. Objects are pairs (A, x) where $A \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathcal{F}(A)$.

2. A morphism $(A, x) \to (B, y)$ is a morphism $f : A \to B$ in $\mathcal{C}$ so that $(\mathcal{F}(f))(x) = y$.

This category is called *the category of elements*.

Remark D.2.7. Note that $(\mathcal{F}(\mathbb{1}_A))(x) = \mathrm{id}_{\mathcal{F}(A)}(x) = x$, so $\mathbb{1}_A \in \mathrm{Hom}(\int \mathcal{F})$. Consider morphisms $f \in \mathrm{Hom}((A, x), (B, y))$ and $g \in \mathrm{Hom}((B, y), (C, z))$. Then

$$(\mathcal{F}(gf))(x) = (\mathcal{F}(g) \circ \mathcal{F}(f))(x) = \mathcal{F}(g)(\mathcal{F}(f)(x)) = \mathcal{F}(g)(y) = z,$$

so $gf \in \mathrm{Hom}(\int \mathcal{F})$. Clearly composition is associative and $\mathbb{1}_{(A,x)} := \mathbb{1}_A$ is the identity morphism for each object A. So $\int \mathcal{F}$ indeed does define a category.

We denote the forgetful functor

$$\begin{gathered} \int \mathcal{F} \to \mathcal{C} \\ (A, x) \mapsto A \\ f : (A, x) \to (B, y) \mapsto f : A \to B \end{gathered}$$

by Π.

Lemma D.2.8. $\int _ : \mathrm{Set}^{\mathcal{C}} \to \mathrm{CAT}$ *is functorial.*

Proof. Given a natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ is mapped to the functor

$$\begin{gathered} \int \alpha : \int \mathcal{F} \to \int \mathcal{G} \\ (A, x) \mapsto (A, \alpha_A(x)) \\ f : (A, x) \to (B, y) \mapsto (A, \alpha_A(x)) \to (B, \alpha_B(y)) \end{gathered}$$

By naturality, $\mathcal{G}(f)(\alpha_A(x)) = \alpha_B \mathcal{F}(f)(x) = \alpha_B(y)$, which means the assigment of morphisms is well-defined. Functoriality is trivial. Given $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ and $\beta : \mathcal{G} \Rightarrow \mathcal{H}$, we need to prove that $\int \beta\alpha = \int \beta \int \alpha$. For an object (A, x) in $\mathcal{C}$, we have that

$$\left(\int \beta\alpha\right)(A, x) = (A, (\beta\alpha)_A(X)) = (A, \beta_A \alpha_A(x)) = \int \beta(A, \alpha_A(x)) = \left(\int \beta \int \alpha\right)(A, x).$$

It is similarly easy to verify that

$$\left(\int \beta\alpha\right)(f) = \left(\int \beta \int \alpha\right)(f)$$

for a morphism a $f : (A, x) \to (B, y)$. □

Definition D.2.9. The *category of elements* for a contravariant functor $\mathcal{F} : \mathcal{C} \to \text{Set}$ is just the category of elements of $\mathcal{F}$ regarded as covariant functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Set}$.

Example D.2.10. 1. Let $\mathcal{C}$ be a concrete category associated with a faithful functor $\mathcal{U} : \mathcal{C} \to \text{Set}$. The objects of $\int \mathcal{U}$ are uniquely determined by sets $X \in \mathcal{U}(A)$ where A is an object of $\mathcal{C}$. A morphism $f : (A, x) \to (B, y)$ of $\int \mathcal{U}$ has an underlying function $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ that maps x to y. In this way $\int \mathcal{U}$ defines the category of *based objects*, i.e. objects in $\mathcal{C}$ with a basepoint in the by $\mathcal{U}$ defined underlying set where a morphism is a morphism in $\mathcal{C}$ that as function in the underlying subcategory of sets preserves basepoints. With this in mind, we could choose to define $\mathcal{C}_* := \int \mathcal{U}$ as a generalization of the notion of a based subcategory of Set.

Lemma D.2.11. *For a covariant functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Set}$,*

$$\int \mathcal{F} \cong y \downarrow \mathcal{F}.$$

Here $y : \mathcal{C} \to \text{Set}^{\mathcal{C}^{\text{op}}}$ is the yoneda embedding and $\mathcal{F}$ is regarded as functor $\mathbb{1} \to \text{Set}^{\mathcal{C}^{\text{op}}}$.

Proof. The objects of the comma category $y \downarrow \mathcal{F}$ are objects $(A, *, \alpha : \text{Hom}(_, A) \Rightarrow \mathcal{F})$ and a morphism $(A, *, \alpha) \to (B, *, \beta)$ is a pair of natural transformations $\mu : \text{Hom}(_, A) \Rightarrow \text{Hom}(_, B)$ and $\nu : \mathcal{F} \Rightarrow \mathcal{F}$ such that

$$\begin{array}{ccc} \text{Hom}(_, A) & \xRightarrow{\alpha} & \mathcal{F} \\ {\scriptstyle\mu}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\nu} \\ \text{Hom}(_, B) & \xRightarrow[\beta]{} & \mathcal{F} \end{array}$$

commutes. Note that necessarily $\nu = \text{id}_{\mathcal{F}}$ and that via the Yoneda embedding $\mu = f_*$ for some morphism $f : A \to B$. A morphism $(A, *, \alpha) \to (B, *, \beta)$ is more succinctly described as natural transformation $f_* : \text{Hom}(_, A) \Rightarrow \text{Hom}(_, B)$ such that

$$\begin{array}{ccc} \text{Hom}(_, A) & \xRightarrow{\alpha} & \mathcal{F} \\ {\scriptstyle f_*}\Big\Downarrow & \nearrow_{\beta} & \\ \text{Hom}(_, B) & & \end{array}$$

We now describe the isomorphism $\int \mathcal{F} \cong y \downarrow \mathcal{F}$,

$$\int \mathcal{F} \to y \downarrow \mathcal{F}$$

$$(A, x) \mapsto \Psi(x) : \text{Hom}(_, A) \Rightarrow \mathcal{F}$$

$$(A, x) \to (B, y) \mapsto f_* : \text{Hom}(_, A) \Rightarrow \text{Hom}(_, B)$$

Note that by the Yoneda Lemma, the assignment of objects is a bijection. The assignment of morphisms is just that of the yoneda embedding, hence morphisms in $y \downarrow \mathcal{F}$ are in bijection with morphisms in $\int \mathcal{F}$. We thus conclude that the above functor defines an isomorphism. □

Remark D.2.12. There is of course a dual result for the covariant case. In this case a morphism in the comma category is of the form $f^* : \mathrm{Hom}(B, _) \to \mathrm{Hom}(A, _)$ for a morphism $f : A \to B$.

Lemma D.2.13. *For a locally small category $\mathcal{C}$ and a fixed object X in $\mathcal{C}$,*

$$\int \mathrm{Hom}(X, _) \cong X/\mathcal{C}.$$

Proof. □

Proposition D.2.14. *A covariant functor $\mathcal{F} : \mathcal{C} \to$ Set is representable if and only if $\int \mathcal{F}$ has an initial object. For a contravariant functor, the same but the category of elements has a terminal object.*

Proof. Suppose $\mathcal{F}$ is representable. Then $\mathcal{F} \cong \mathrm{Hom}(X, _)$ for some object X in $\mathcal{C}$. By functoriality and the prior lemma, $\int \mathcal{F} \cong \int \mathrm{Hom}(X, _) \cong X/\mathcal{C}$. The initial object in $X/\mathcal{C}$ is $\mathbb{1}_X$, hence $\int \mathcal{F}$ has an initial object.
Suppose conversely that $\int \mathcal{F}$ has an initial object (A, x). Let an object (B, y) be given. Then there is a unique morphism $f : A \to B$ with

$$\mathcal{F}(f)(x) = y.$$

Recall from the proof of the Yoneda lemma that $\Psi(x)_B(f) = \mathcal{F}(f)$. Then $\psi_B(x)$ is surjective by existence of f and injective by uniqueness of f, hence $\Psi(x) : \mathrm{Hom}(A, _) \Rightarrow \mathcal{F}$ is a natural isomorphism. □

Remark D.2.15. A representation of a set-valued functor on a locally small category $\mathcal{C}$ can thus equivalently be defined as an initial object in $\int \mathcal{C}$.

Definition D.2.16. A category $\mathcal{C}$ is a *contractible groupoid*, if

$$\mathcal{C} \cong \mathbb{1} := (\{*\}, \{\mathbb{1}_*\}).$$

Lemma D.2.17. *Consider a locally small category $\mathcal{C}$. The subcategory of $\mathcal{C}$ consisting of the initial/terminal objects of $\mathcal{C}$,* init()/term($\mathcal{C}$) *is either empty or a contractible groupoid*

Proof. Suppose $\text{init}(\mathcal{C})$ has an object. The functor

$$\begin{gathered}\text{init}(\mathcal{C}) \to \mathbb{1} \\ X \mapsto * \\ f : X \to X' \mapsto \mathbb{1}_*\end{gathered}$$

is clearly full and essentially surjective on objects. and since a morphism from an initial object is unique also faithful. □

Proposition D.2.18. *Consider a set-valued functor $\mathcal{F}$ on a locally small category $\mathcal{C}$. Then the full subcategory of $\int \mathcal{F}$ spanned by its representations,* $\text{rep}(\mathcal{F})$*, is either empty or a contractible groupoid.*

Proof. If $\mathcal{F}$ is not representable, then $\text{rep}(\mathcal{F}) = \emptyset$. If $\mathcal{F}$ is representable, then $\int \mathcal{F}$ has an initial object, hence by the prior lemma and $\text{rep}(\mathcal{F}) = \text{init}\left(\int \mathcal{F}\right) \cong \mathbb{1}$. □

D.3 Limits & Colimits

D.3.1 Universal Cones & Cocones

Definition D.3.1. Consider categories $\mathcal{C}$ and $\mathcal{J}$. Let an object X in $\mathcal{C}$ be given. There is a constant functor

$$\begin{gathered}X := X_{\mathcal{J}} : \mathcal{J} \to \mathcal{C} \\ A \mapsto X \\ f : A \to B \mapsto \mathbb{1}_X\end{gathered}$$

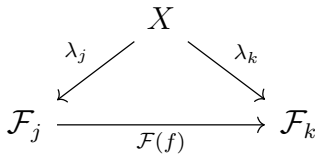
associated with X. We get an embedding

$$\begin{gathered}\Delta : \mathcal{C} \hookrightarrow \mathcal{J}^{\mathcal{C}} \\ X \mapsto X_{\mathcal{J}} \\ f : X \to Y \mapsto \{f : X = X_{\mathcal{J}}(A) \to Y = Y_{\mathcal{J}}(A) : A \in \text{Ob}(\mathcal{J})\}\end{gathered}$$

One readily verifies that given a morphism $f : X \to Y$ in $\mathcal{C}$, $\Delta(f)$ is a natural transformation $X_{\mathcal{J}} \Rightarrow Y_{\mathcal{J}}$, which we refer to as *the constant natural transformation associated with* f.

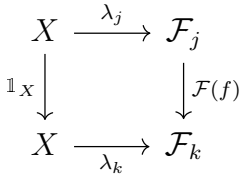
Definition D.3.2. Consider a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ on $\mathcal{C}$. *A cone over* $\mathcal{F}$ with *apex* $X \in \text{Ob}(\mathcal{C})$ is a natural transformation $\lambda : X \Rightarrow \mathcal{C}$

Lemma D.3.3. *Consider a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ on a category $\mathcal{C}$. Let an object X in $\mathcal{C}$ be given. A family of morphisms $\{\lambda_j : X \to \mathcal{F}_j\}_{j \in \mathrm{Ob}(\mathcal{J})}$ is a cone over $\mathcal{F}$ if and only if for each $f : j \to k$,*



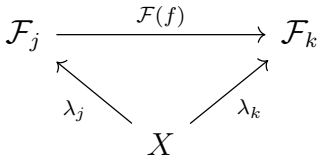
commutes.

Proof. Clearly the family of diagrams



commuting is equivalent to the family of diagrams in question commuting. □

Definition D.3.4. A *cocone under a diagram* $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ with *nadir* $X \in \mathrm{Ob}(\mathcal{C})$ is a natural transformation $\lambda : X_{\mathcal{J}} \Rightarrow \mathcal{F}$, i.e. a family of morphisms $\{\lambda_j : X \to \mathcal{F}_j\}$ indexed by J such that for each morphisms $f : j \to k$,



commutes.

Remark D.3.5. A cocone under $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is just a cone over the associated diagram $\mathcal{F} : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Definition D.3.6. Given a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ we consider the contravariant *cone functor*

$$\mathrm{Cone}(_, \mathcal{F}) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$
$$X \mapsto \mathrm{Hom}(X_{\mathcal{F}}, \mathcal{F})$$
$$(g : X \to Y) \mapsto (\mathrm{Hom}(Y_{\mathcal{F}}, \mathcal{F}) \to \mathrm{Hom}(X_{\mathcal{F}}, \mathcal{F}), \mu \mapsto g^* \mu)$$

A *limit of* $\mathcal{F}$ is representation given by $X \in \mathrm{Ob}(\mathcal{C})$ and $\alpha \in \mathrm{Hom}(\mathrm{Hom}(_, X), \mathrm{Cone}(_, \mathcal{F}))$ of the contravariant cone functor.

Remark D.3.7.

- We denote the object X by $\lim \mathcal{F}$. By Yoneda, such a representation corresponds to an element $\lambda \in \mathrm{Cone}(X, \mathcal{F})$.

- A dual notion of *colimit* similarly from the covariant cone functor

$$\text{Cone}(\mathcal{F}, _) : \mathcal{C} \to \text{Set}$$

that sends an object to the corresponding cone under $\mathcal{F}$ and a morphism $g : X \to Y$ to the function

$$\begin{aligned} \text{Hom}(\mathcal{F}, X_{\mathcal{F}}) &\to \text{Hom}(\mathcal{F}, Y_{\mathcal{F}}) \\ \lambda &\mapsto \lambda g_* \end{aligned}$$

- An equivalent definition of the limit is that it is the terminal object in $\int \text{Cone}(_, \mathcal{F})$. Such an object is a summit $\lim \mathcal{F}$ together with a *limit cone* λ. The colimit is then the initial object in $\int \text{Cone}(\mathcal{F}, _)$, i.e. a pair $\text{colim}\, \mathcal{F}$ and *colimit cone*.

For clarity, consider the category of cones over a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$, namely $\int \text{Cone}(_, \mathcal{F})$. An object in this category is a cone $(X, \lambda : X \Rightarrow \mathcal{F})$ and a morphism of cones $f : (X, \lambda) \to (Y, \mu)$ is a morphism $f : X \to Y$ such that $\lambda = f^* \mu$. With this in mind we present the following proposition.

Proposition D.3.8. *A limit of a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is essentially unique, i.e. limits are unique up to unique isomorphism.*

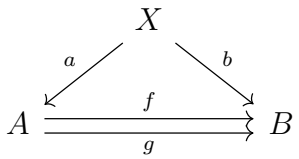
Proof. The subcategory of limit cones is the same as the subcategory of terminal objects in the category cones. These are unique up to unique isomorphism. □

D.3.2 Examples of Limits

Example D.3.9. Consider a diagram indexed by the category with two elements $\{0, 1\}$ with only two distinct non-trivial arrows both from 0 to 1 to a category $\mathcal{C}$. Such a diagram is of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

A cone with summit X of such a diagram is a pair of morphisms $a : X \to A$ and $b : X \to B$ such that the diagram



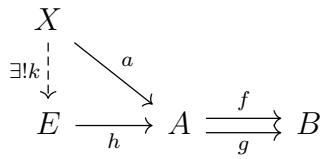
commutes equivalently it is a single morphism $a : X \to A$ such that

$$X \xrightarrow{a} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

commutes. The limit of a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

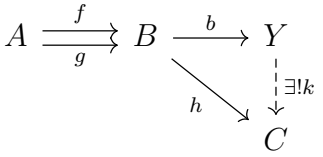
is called an *equaliser* for f and g. This is an object E and a morphism $h : E \to A$ such that for every morphism $a : X \to A$, the diagram



A cocone with nadir Y is a morphism $b : B \to Y$ such that

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{b} Y$$

commutes. A *coequaliser* for f and g is a morphism $h : B \to C$ such that for each $b : B \to Y$



Example D.3.10. Pullbacks

D.3.3 Limits in Set

Definition D.3.11. A diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is *small* if $\mathcal{J}$ is small.

Definition D.3.12. A category $\mathcal{C}$ is *complete*, if it every small diagram has a limit. It is *cocomplete* if every small diagram has a colimit.

Proposition D.3.13. Set *is complete.*

Proof. Given a diagram $\mathcal{F}$, we define

$$\lim\ \mathcal{F} := \text{Cone}(1, \mathcal{F})$$

□

D.3.4 Products & Coproducts

Definition D.3.14. A *product* over a discrete category $\mathcal{J}$ in a category $\mathcal{C}$ is the limit of a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$. We denote this limit by

$$\prod_{j \in \mathcal{J}} \mathcal{F}_j.$$

The components of the limit cone

$$\left\{ \pi_i : \prod_{j \in \mathcal{J}} X_j \to X_i \;\middle|\; i \in \mathcal{J} \right\}$$

are called the *projections* of the product.

Remark D.3.15. Note that any family of maps $\{\lambda_j : X \to \mathcal{F}_j\}$ trivially satisfies the naturality condition and is thus a cone over $\mathcal{F}$.

Example D.3.16. Consider a set I considered as discrete category in the canonical way. Suppose we have a diagram $X : I \to \text{Set}$. The subcategory spanned by this diagram is a family of sets $\{X_i\}$ indexed by I. We will in the next proposition show that the cartesian product, $\prod_{i \in I} X_i$, is in fact the product of X.

Proposition D.3.17. *Consider a set I. The objects of the category of products over I in* Set *are cartesian products over I.*

Proof. Indeed, consider a cartesian product $\prod_{i \in I} X_i$ and the cone over X,

$$\pi := \left\{ \pi_i : \prod_{j \in I} X_j \to X_i, (x_j) \mapsto x_i \;\middle|\; i \in I \right\}.$$

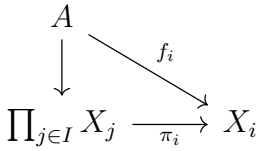
The claim is that this is the terminal object in the category of cones over X. Consider any cone $\phi := \{f_i : A \to X_i\}$ over X indexed by I. We define the function

$$f : A \to \prod_{i \in I} X_i$$
$$a \mapsto (f_i(a))$$

we claim that is the unique morphism of cones from $(A, \{f_i\})$ to $(\prod_{i \in I} X_i, \{\pi_i\})$ such that

$$\phi = f^* \pi.$$

Let $i \in I$. It is sufficient to check that f is the unique map making the diagram



commute. Indeed, for any $a \in A$,

$$\pi_i f(a) = \pi_i((f_j(a))) = f_i(a),$$

hence $pi_i f = f_i$. Suppose $g : A \to \prod_{j \in I} X_j$ is another map such that $\pi_i g = f_i$. Let an arbitrary $a \in A$ be given and set $(x_i) := g(a)$. Then $x_i = \pi_i g(a) = f_i(a)$, hence $f(a) = (f_i(a)) = (x_i) = g(a)$, which means $f = g$. □

Definition D.3.18. A *coproduct* over a discrete category $\mathcal{J}$ in a category $\mathcal{C}$ is the colimit of a diagram $\mathcal{F} : \mathcal{J} \to \mathcal{C}$. We denote this colimit by

$$\coprod_{j \in \mathcal{J}} \mathcal{F}_j.$$

The components of the colimit cocone

$$\left\{ \iota_i : X_i \to \coprod_{j \in \mathcal{J}} X_j \;\middle|\; i \in \mathcal{J} \right\}$$

are called the *injections* of the coproduct.

Example D.3.19. In the category of sets, a coproduct $\coprod_{i \in I} X_i$ index by a set I is simply disjoint union over I, i.e.

$$\coprod_{i \in I} X_i = \bigsqcup_{i \in I} X_i.$$

injection maps simply the inclusion maps

$$\left\{ \iota_i : X_i \to \bigsqcup_{j \in I} X_j, x_i \mapsto (x_i, i) \;\middle|\; i \in I \right\}.$$

Indeed, consider an arbitrary cocone $(A, \{f_i : X_i \to A\})$ over X. We define

$$f : \bigsqcup_{i \in I} X_i \to A$$
$$(x, i) \mapsto f_i(x)$$

We must then check for each $i \in I$ that $f\iota_i = f_i$ and that f is unique with this property. Indeed, for each $x \in X_i$

$$f\iota_i(x) = f(x, i) = f_i(x) \implies f\iota_i = f.$$

Consider another map $g : \bigsqcup_{i \in I} X_i \to A$ such that $g\iota_i = f_i$. Let $(x, i) \in \bigsqcup_{j \in I} X_j$ be given. Then

$$g(x, i) = g\iota_i(x) = f\iota_i(x) = f(x, i) \implies f = g.$$

D.3.5 Direct systems and direct limits

recall the following definition

Definition D.3.20. A preordered set $(A, \leq)$ is a *directed set* if for each pair of elements $a, b \in A$ there is an element $c \in A$ such that $a \leq c$ and $b \leq c$.

Example D.3.21. Consider a topological space X and fix a point $P \in X$. Then the family of subsets in X

$$\{U \in \tau_X : P \in U\}$$

is a directed set. Indeed, a pair of open neighborhoods about P are contained in their union, which is an open neighborhood of P.

Definition D.3.22. Consider a directed set $(I, \leq)$ and a category $\mathcal{C}$. A *direct system* is a diagram

$$X_\bullet : I \to \mathcal{C}$$

We aim to characterise colimits of direct systems. These are called *direct limits*. First we describe cocones of direct systems. Fix a direct system $X_\bullet : (I, \leq) \to \mathcal{C}$ and an object $X \in \text{Ob}(\mathcal{C})$. A cocone over $X_\bullet$ with summit X is then a natural transformation $\lambda : X_\bullet \Rightarrow X$. The subcategory in $\mathcal{C}$ spanned by $X_\bullet$ consists of a collection of objects

$$\{X_i : i \in I\}$$

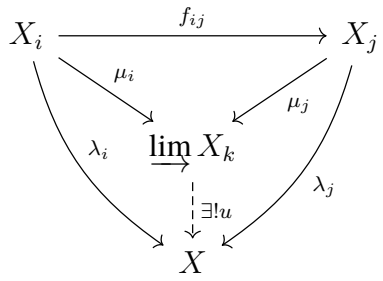
and a collection of morphisms

$$\{f_{ij} : X_i \to X_j \mid i, j \in I, i \leq j\}$$

When the colimit of such a diagram exists (for instance when $\mathcal{C}$ is complete), we denote it by

$$\varinjlim_{i \in I} X_i.$$

This object has the universal property that given another cocone (X, λ) the diagram



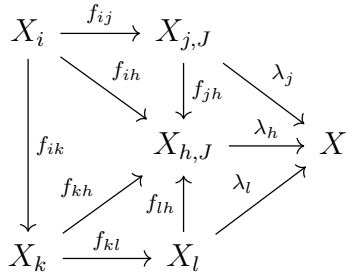
commutes for each $i, j \in I$ with $i \leq j$. When we get to describing sheafs valued in the category of groups, we will give an explicit construction of such an object in the category of groups.

Definition D.3.23. Consider a preordered set $(I, \leq)$. A subset $J \subset I$ is *cofinal (in I)*, if for each $i \in I$ there is a $j \in J$ with $i \leq j$. If $J^{\mathrm{op}} \subset I^{\mathrm{op}}$ is cofinal, we say that J *is coinitial in* I.

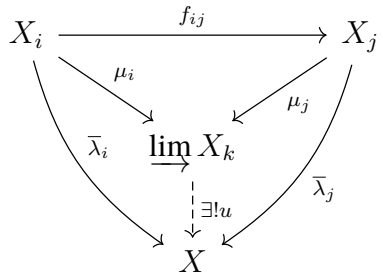
Remark D.3.24. If I is a directed set, then J is also a directed set. Indeed, given $a, b \in J$, there is an $x \in I$ with $a \leq x$ and $b \leq x$. Since J is cofinal in I there is a $c \in J$ with $x \leq c$. It thus follows that $a \leq c$ and $b \leq c$.

Lemma D.3.25. *Consider a directed set $(I, \leq)$ and subset J cofinal in I. Let $X_\bullet : I \to \mathcal{C}$ be a direct system and suppose $\mathcal{C}$ admits colimits. Let $X_{\bullet,J}$ denote the restriction of $X_\bullet$ to J. Then $\varinjlim X_{k,J} \simeq \varinjlim X_k$.*

Proof. Consider a cocone (X, λ) of the direct system $X_{\bullet,J}$. We claim that this can be extended to a cocone $(X, \overline{\lambda})$ of the direct system $(X_\bullet)$. Indeed, given $i \in I$ pick $j \in J$ and factor through X_j to get a morphism $\overline{\lambda}_i : X_i \to X$ making sure to factor through X_i when $i \in I$. To check naturality, note that given $i \leq k$, with $i \leq j$ for $j \in J$ and $k \leq l$ for $l \in J$ we may find an $h \in J$ with $j, l \leq h$, giving us the commutative diagram

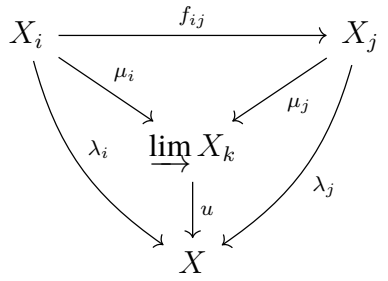


shows the naturality of $\overline{\lambda}$.
Now let an arbitrary cocone (X, λ) of $X_{\bullet,J}$ be given. Then



commutes for each $i, j \in I$ with $i \leq j$. It thus follows that $u : \varinjlim X_k \to X$ is the unique

morphism such that



commutes for each $i, j \in J$ with $i \leq j$, thus $\varinjlim X_k$ is a direct limit of $X_{\bullet,J}$. In other words $\varinjlim X_k$ is the initial object in the category of cocones of $X_{\bullet,J}$, hence it must be isomorphic to $\varinjlim X_{\bullet,J}$. □

D.4 Abelian Categories